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h-STABILITY OF THE NONLINEAR PERTURBED DIFFERENTIAL SYSTEMS VIA t_∞-SIMILARITY

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Abstract. In this paper, we investigate h-stability of the nonlinear perturbed differential systems using the the notion of t_∞-similarity

1. Introduction

As is traditional in a pertubation theory of nonlinear differential equations, the behavior of solutions of a perturbed equation is determined in terms of the behavior of solutions of an unperturbed equation. There are three useful methods for studying the qualitative behavior of the solutions of perturbed nonlinear system of differential equations: the method of variation of constants formula, the second method of Lyapunov and the use of integral inequalities.

Pinto[13,15] introduced h-stability(hS) which is an important extention of the notions of exponential asymptotic stability and uniform Lipschitz stability. Also, he obtained some properties about asymptotic behavior of solutions of perturbed differential systems, some general results about asymptotic integration and gave some important examples in [14].

Choi and Ryu [3] dealt with hS of the solutions of the various differential systems and Volterra integro-differential systems. Recently, Choi et al. [4] and Goo and Ry [7,8] obtained results for hS of nonlinear differential systems via t_∞-similarity. Goo et al.[9,10] investigated hS for the nonlinear Volterra integro-differential system.

In this paper, we investigate h-stability of the nonlinear perturbed differential systems using the the notion of t_∞-similarity.

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2. Preliminaries

We consider the nonlinear nonautonomous differential system

$$x'(t) = f(t, x(t)), \quad x(t_0) = x_0, \tag{2.1}$$

where $f \in C[\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n]$, $\mathbb{R}^+ = [0, \infty)$ and $\mathbb{R}^n$ is the Euclidean n-space. We assume that the Jacobian matrix $f_x = \partial f / \partial x$ exists and is continuous on $\mathbb{R}^+ \times \mathbb{R}^n$ and $f(t, 0) = 0$.

Let $x(t) = x(t, t_0, x_0)$ denote the unique solution of (2.1) through (t_0, x_0) in $\mathbb{R}^+ \times \mathbb{R}^n$ such that $x(t_0, t_0, x_0) = x_0$. Also, we consider the associated variational systems around the zero solution of (2.1) and around $x(t)$, respectively,

$$v'(t) = f_x(t, 0)v(t), \; v(t_0) = v_0 \tag{2.2}$$

and

$$z'(t) = f_x(t, x(t, t_0, x_0))z(t), \; z(t_0) = z_0. \tag{2.3}$$

The fundamental matrix solution $\Phi(t, t_0, x_0)$ of (2.3) is given by

$$\Phi(t, t_0, x_0) = \frac{\partial}{\partial x_0} x(t, t_0, x_0),$$

and $\Phi(t, t_0, 0)$ is the fundamental matrix solution of (2.2). The symbol $|.|$ denotes arbitrary vector norm in $\mathbb{R}^n$.

We recall some notions of h-stability [13] and the notion of t_∞-similarity [5].

Definition 2.1. The system (2.1) (the zero solution $x = 0$ of (2.1)) is called an *h-system* if there exist a constant $c \geq 1$, and a positive continuous function h on $\mathbb{R}^+$ such that

$$|x(t)| \leq c\,|x_0|\,h(t)\,h(t_0)^{-1}$$

for $t \geq t_0 \geq 0$ and $|x_0|$ small enough (here $h(t)^{-1} = \frac{1}{h(t)}$).

Definition 2.2. The system (2.1) (the zero solution $x = 0$ of (2.1)) is called *h-stable (hS)* if there exist $\delta > 0$ such that (2.1) is an h-system for $|x_0| \leq \delta$ and h is bounded.

Let $\mathcal{M}$ denote the set of all $n \times n$ continuous matrices $A(t)$ defined on $\mathbb{R}^+$ and $\mathcal{N}$ be the subset of $\mathcal{M}$ consisting of those nonsingular matrices $S(t)$ that are of class C^1 with the property that $S(t)$ and $S^{-1}(t)$ are bounded. The notion of t_∞-similarity in $\mathcal{M}$ was introduced by Conti [5].

DEFINITION 2.3. A matrix $A(t) \in \mathcal{M}$ is *t_∞-similar* to a matrix $B(t) \in \mathcal{M}$ if there exists an $n \times n$ matrix $F(t)$ absolutely integrable over $\mathbb{R}^+$, i.e.,

$$\int_0^\infty |F(t)|dt < \infty$$

such that

$$\dot{S}(t) + S(t)B(t) - A(t)S(t) = F(t) \tag{2.4}$$

for some $S(t) \in \mathcal{N}$.

The notion of t_∞-similarity is an equivalence relation in the set of all $n \times n$ continuous matrices on $\mathbb{R}^+$, and it preserves some stability concepts [4, 11].

We give some related properties that we need in the sequal.

LEMMA 2.4. [15] *The linear system*

$$x' = A(t)x, \; x(t_0) = x_0, \tag{2.5}$$

where $A(t)$ is an $n \times n$ continuous matrix, is an h-system (respectively h-stable) if and only if there exist $c \geq 1$ and a positive and continuous (respectively bounded) function h defined on $\mathbb{R}^+$ such that

$$|\phi(t, t_0)| \leq c\, h(t)\, h(t_0)^{-1} \tag{2.6}$$

for $t \geq t_0 \geq 0$, where $\phi(t, t_0)$ is a fundamental matrix of (2.5).

THEOREM 2.5. [3] *If the zero solution of (2.1) is hS, then the zero solution of (2.2) is hS.*

THEOREM 2.6. [7] *Suppose that $f_x(t, 0)$ is t_∞-similar to $f_x(t, x(t, t_0, x_0))$ for $t \geq t_0 \geq 0$ and $|x_0| \leq \delta$ for some constant $\delta > 0$. Then the solution $v = 0$ of (2.2) is hS if and only if the solution $z = 0$ of (2.3) is hS.*

3. Main results

In this section, we investigate hS for the nonlinear perturbed differential systems.

Now, we examine the properties of hS for the perturbed system

$$y' = f(t, y) + \int_{t_0}^t g(s, y(s), Ty(s))ds, \; y(t_0) = y_0, \tag{3.1}$$

where $g \in C[\mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n]$, $g(t, 0, 0) = 0$ and $T : C(\mathbb{R}^+, \mathbb{R}^n) \to C(\mathbb{R}^+, \mathbb{R}^n)$ is a continuous operater .

Theorem 3.1. *Suppose that $f_x(t,0)$ is t_∞-similar to $f_x(t,x(t,t_0,x_0))$ for $t \geq t_0 \geq 0$ and $|x_0| \leq \delta$ for some constant $\delta > 0$, the solution $x = 0$ of (2.1) is hS with the increasing function h, and g in (3.1) satisfies*

$$\left| \int_{t_0}^{s} g(\tau, y(\tau), Ty(\tau))d\tau \right| \leq a(s)|y(s)| + b(s) \int_{t_0}^{s} c(\tau)|y(\tau)|d\tau,\ t \geq t_0 \geq 0,$$

where $a, b, c \in C(\mathbb{R}^+, \mathbb{R}^+)$ and $\int_{t_0}^{\infty}[a(s) + b(s)\int_{t_0}^{s} c(\tau)d\tau]ds < \infty$. Then, the solution $y = 0$ of (3.1) is hS.

Proof. Using a slight variant of the nonlinear variation of constants formula of Alekseev[1], any solution $y(t) = y(t, t_0, x_0)$ of (3.1) passing through (t_0, x_0) is given by

(3.2)

$$y(t, t_0, y_0) = x(t, t_0, y_0) + \int_{t_0}^{t} \Phi(t, s, y(s)) \int_{t_0}^{s} g(\tau, y(\tau), Ty(\tau))d\tau ds,$$

where $x(t) = x(t, t_0, x_0)$ is a solution of (2.1) passing through (t_0, x_0). By Theorem 2.5, since the solution $x = 0$ of (2.1) is hS, the solution $v = 0$ of (2.2) is hS. Therefore, by Theorem 2.6, the solution $z = 0$ of (2.3) is hS. Applying Lemma 2.4 and the increasing property of the function h, we have

$$\begin{aligned} |y(t)| &\leq |x(t)| + \int_{t_0}^{t} |\Phi(t, s, y(s))| \left| \int_{t_0}^{s} g(\tau, y(\tau), Ty(\tau))d\tau \right| ds \\ &\leq c_1|y_0|h(t)\, h(t_0)^{-1} \\ &+ \int_{t_0}^{t} c_2 h(t)\, [a(s)|y(s)|h(s)^{-1} + b(s) \int_{t_0}^{s} c(\tau)|y(\tau)|h(\tau)^{-1}d\tau]ds. \end{aligned}$$

Set $u(t) = |y(t)|h(t)^{-1}$. Then, by Gronwall's inequality, we obtain

$$\begin{aligned} |y(t)| &\leq c_1|y_0|h(t)\, h(t_0)^{-1} e^{c_2 \int_{t_0}^{t}[a(s)+b(s)\int_{t_0}^{s} c(\tau)]ds} \\ &\leq c|y_0|h(t)\, h(t_0)^{-1}, \quad c = c_1 e^{c_2 \int_{t_0}^{\infty}[a(s)+b(s)\int_{t_0}^{s} c(\tau)]ds} \end{aligned}$$

It follows that $y = 0$ of (3.1) is hS. Hence, the proof is complete. □

Remark 3.1. Letting $g(s, y(s), Ty(s)) = g(s, y(s)), b(s) = 0$ in Theorem 3.1, we obtain the same result as that of Theorem 3.3 in [8].

Remark 3.2. In the linear case, we can obtain that if the zero solution $x = 0$ of (2.5) is hS, then the perturbed system

$$y' = A(t)y + \int_{t_0}^{t} g(s, y(s), Ty(s))ds, y(t_0) = y_0,$$

is also hS under the same hypotheses in Theorem 3.1 except the condition of t_∞-similarity.

We need the following lemma for hS of (3.1).

LEMMA 3.2. *Let $u, p, q, w, r \in C(\mathbb{R}^+, \mathbb{R}^+)$ and suppose that, for some $c \geq 0$, we have*

(3.3)
$$u(t) \leq c + \int_{t_0}^{t} p(s) \int_{t_0}^{s} [q(\tau)u(\tau) + w(\tau) \int_{t_0}^{\tau} r(a)u(a)da]d\tau ds, \; t \geq t_0.$$

Then

$$u(t) \leq c\exp(\int_{t_0}^{t} p(s) \int_{t_0}^{s} [q(\tau) + w(\tau) \int_{t_0}^{\tau} r(a)da]d\tau ds), \; t \geq t_0. \tag{3.4}$$

Proof. Setting $v(t) = c + \int_{t_0}^{t} p(s) \int_{t_0}^{s} [q(\tau)u(\tau) + w(\tau) \int_{t_0}^{\tau} r(a)u(a)da]d\tau ds$, we have $v(t_0) = c$ and

$$\begin{aligned} v'(t) &= p(t) \int_{t_0}^{t} [q(s)u(s) + w(s) \int_{t_0}^{s} r(a)u(a)da]ds \\ &\leq p(t) \int_{t_0}^{t} [q(s) + w(s) \int_{t_0}^{s} r(a)da]v(s)ds \\ &\leq [p(t) \int_{t_0}^{t} [q(s) + w(s) \int_{t_0}^{s} r(a)da]ds]v(t), \; t \geq t_0, \end{aligned} \tag{3.5}$$

since $v(t)$ is nondecreasing and $u(t) \leq v(t)$. It follows from the Gronwall inequality that (3.5) yields the estimate (3.4). □

THEOREM 3.3. *Suppose that $f_x(t, 0)$ is t_∞-similar to $f_x(t, x(t, t_0, x_0))$ for $t \geq t_0 \geq 0$ and $|x_0| \leq \delta$ for some constant $\delta > 0$. If the solution $x = 0$ of (2.1) is an h-system with a positive continuous function h and g in (3.1) satisfies*

$$|g(t, y, Ty)| \leq \lambda(t)|y| + \beta(t) \int_{t_0}^{t} \gamma(s)|y(s)|ds, \; t \geq t_0, \; y \in \mathbb{R}^n,$$

where $\lambda, \beta, \gamma : \mathbb{R}^+ \to \mathbb{R}^+$ are continuous with

$$\int_{t_0}^{\infty} \frac{1}{h(s)} \int_{t_0}^{s} [h(\tau)\lambda(\tau) + \beta(\tau) \int_{t_0}^{\tau} h(r)\gamma(r)dr]d\tau ds < \infty, \tag{3.6}$$

for all $t_0 \geq 0$, then the solution $y = 0$ of (3.1) is an h-system.

Proof. It is known that the solutions of (2.1) and (3.1) with the same initial values are represented by the integral equation (3.2). By Theorem 2.5, since the solution $x = 0$ of (2.1) is an h-system, the solution $v = 0$

of (2.2) is an h-system. Therefore, by Theorem 2.6, the solution $z = 0$ of (2.3) is an h-system. Using (2.6) and (3.2), we obtain

$$\begin{aligned} |y(t)| &\le |x(t)| + \int_{t_0}^{t} |\Phi(t,s,y(s))| \int_{t_0}^{s} |g(\tau, y(\tau), Ty(\tau))| d\tau ds \\ &\le c_1 |y_0| h(t)\, h(t_0)^{-1} \\ &+ \int_{t_0}^{t} c_2 \frac{h(t)}{h(s)} [\int_{t_0}^{s} h(\tau)\lambda(\tau) \frac{|y(\tau)|}{h(\tau)} + \beta(\tau) \int_{t_0}^{\tau} h(r)\gamma(r) \frac{|y(r)|}{h(r)} dr d\tau] ds. \end{aligned}$$

Applying Lemma 3.2 with $u(t) = |y(t)| h(t)^{-1}$ and using (3.6), we obtain

$$\begin{aligned} |y(t)| &\le c_1 |y_0| h(t)\, h(t_0)^{-1} e^{c_2 \int_{t_0}^{t} \frac{1}{h(s)} \int_{t_0}^{s} [h(\tau)\lambda(\tau) + \beta(\tau) \int_{t_0}^{\tau} h(r)\gamma(r) dr] d\tau ds} \\ &\le c |y_0| h(t)\, h(t_0)^{-1},\ t \ge t_0, \end{aligned}$$

where $c = c_1 e^{c_2 \int_{t_0}^{\infty} \frac{1}{h(s)} \int_{t_0}^{s} [h(\tau)\lambda(\tau) + \beta(\tau) \int_{t_0}^{\tau} h(r)\gamma(r) dr] d\tau ds}$. It follows that $y = 0$ of (3.1) is an h-system. Hence, the proof is complete. □

Remark 3.3. In the linear case, we can obtain that if the zero solution $x = 0$ of (2.5) is an h-system, then the perturbed system

$$y' = A(t)y + \int_{t_0}^{t} g(s, y(s), Ty(s)) ds, y(t_0) = y_0,$$

is also an h-system under the same hypotheses in Theorem 3.3 except the condition of t_∞-similarity.

Theorem 3.4. *For the system (3.1), suppose that*

$$\left| \int_{t_0}^{t} g(\tau, y(\tau), Ty(\tau)) d\tau \right| \le r(t, |y|, |Ty|),$$

where $r \in C[\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+]$ is strictly increasing in u,v for each fixed $t \ge t_0 \ge 0$ with $r(t,0,0) = 0$. Assume also that $z = 0$ of (2.3) is hS with the nonincreasing function h. Consider the scalar differential equation

$$u' = cr(t, u, Tu), u(t_0) = u_0 = c|y_0|. \tag{3.7}$$

If $u = 0$ of (3.7) is hS, then $y = 0$ of (3.1) is also hS whenever $u_0 = c|y_0|$.

Proof. Using the nonlinear variation of constants formula of Alekseev[1], the solutions of (2.1) and (3.1) with the same initial values are related by

$$y(t, t_0, y_0) = x(t, t_0, y_0) + \int_{t_0}^{t} \Phi(t, s, y(s)) \int_{t_0}^{s} g(\tau, y(\tau), Ty(\tau)) d\tau ds.$$

Then, we have

$$|y(t)| \leq |x(t)| + \int_{t_0}^{t} |\Phi(t,s,y(s))| \left| \int_{t_0}^{s} g(\tau, y(\tau), Ty(\tau))d\tau \right| ds,$$

where $\Phi(t,s,y(s))$ is the fundamental matrix of (2.3). Then, the rest of proof can be proved in the similar manner as that of Theorem 3.4 of [8], so we omit the details. □

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