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ON ORDINALS

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Abstract. The aim of this paper is two fold: One of them is to introduce a formal definition of ordinals which is equivalent to Neumann's definition without assuming the axiom of regularity. The other is to introduce the weak transfinite set and show that the weak transfinite set is a transfinite limit ordinal.

1. Introduction

The concept of ordinals was introduced by Cantor in 1897, who also introduced transfinite induction ([6]). Frege defined an ordinal as an equivalence class of well-ordered sets([3]). The idea of identifying an ordinal with the set of smaller ordinals is due to Zermelo and von Neumann([6]). *Zermelo* has (1915) set up the formal definition of ordinals having the three properties ([11])

Z_1) $m = \emptyset$ or $\emptyset \in m$.

Z_2) For every element $a \in m$ we have either $a \cup \{a\} = m$ or $a \cup \{a\} \in m$.

Z_3) For every $n \subseteq m$ we have either $sn = m$ or $sn \in m$.

v. Neumann has defined (1923) a set m as an ordinal as follows([5], [6], [8], [9]):

A set is an ordinal, if

N_1) m is full.

N_2) m is well ordered by $\in$.

R. M. Robinson has defined (1937) a set m as an ordinal as follows([3], [7], [10], [11]):

A set m is an ordinal, if

R_1) m is full.

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R_2) If $\emptyset \subset n \subseteq m$, n is basic, which means that it is disjoint to one of its elements.

R_3) $\in$ is trichotomous on m, which means that $s, t \in m$ imply either $s \in t$ or $s = t$ or $t \in s$.

Gödel has defined (1939) a set m as an ordinal as follows([11]):

A set m is an ordinal, if

G_1) m is full.

G_2) If $\emptyset \subset n \subseteq m$, n is basic.

G_3) Every element of m is full.

Bernays has defined (1941) a set m as an ordinal as follows ([11]):

A set m is an ordinal, if

B_1) m is full.

B_2) Every full proper subset of m is a member.

In [11], Skolem proved that assuming the axiom of regularity, they are all the same sets.

J. R. Isbell has defined (1960) a set m as an ordinal as follows([1]): A set m is an ordinal if each of its full proper subsets is a member.

In the present paper, we have two goals. One of them is to introduce a defintion of ordinals which is equivalent to the Neumann's definition without assuming the axiom of regularity. The other is to classify limit ordinals into two classes: the usual limit ordinals and the transfinite limit ordinals and then introduce the transfinite set and show that the weak transfinite set is a transfinite limit ordinal.

This paper is organized as follows. In section 2, we first modify the definition of full sets and then we introduce a concept of bi-transitive sets and a definition of ordinals which is equivalent to Neumann's defintion without assuming the axiom of regularity. Eventually, we show that our definition is equivalent to Robinson's definition. In section 3, we classify limit ordinals into two classes: the usual limit ordinals and the transfinite limit ordinals. We characterize the usual limit ordinals. In section 4, we introduce a concept of weak transfinite inductive sets and the weak transfinite set. Moreover, we present an induction principle and then using the induction principle, we show that the weak transfinite set is a transfinite limit ordinal. In sections 2 and 3, we also present a few further results, as well as new proofs of known ones that connected with ordinals. For basic results on ordinals and set theory we refer to [6], [7] and [11].

2. A definition of ordinals

The concept of full sets is central to the study of ordinals. We first recall that a set x is *full* (or transitive) if every element of x is a subset of x(see [1], [5], [6], [7], [8]). To attain our purpose, we need to modify the definition as follows: A set x is *transitive* if $x \subseteq pr(x)$, where $pr(x)$ denotes the set of proper subsets of x. It is very important to note that every transitive set is full and that assuming the axiom of regularity, every full set is transitive. From now on, we do not assume the axiom of regularity. One has the following:

PROPOSITION 2.1. (1) *The empty set $\emptyset$ is transitive.*

(2) *If x is a transitive set, then $x \notin x$.*

(3) *If s and t are transitive sets, then $s \in t$ and $t \in s$ cannot both hold.*

(4) *For a set x, $\{x\}$ is transitive if and only if $x = \emptyset$.*

(5) *If x is a transitive set whose elements are transitive, then $\cap x = \emptyset$, where $\cap x$ is the intersection of elements of x.*

(6) *A set x is transitive if and only if $s \in t \in x$ imply $s \in x$ and $t \neq x$.*

(7) *If x is a set of transitive sets, then $\cup x$ is a transitive set, where $\cup x$ is the union of elements of x.*

(8) *If x is a set of transitive sets and $z \subseteq x$, then $x \cap p(z)$ is transitive, where $p(x)$ denotes the power set of x.*

Proof. (1) it is vacuously true.

(2) It is immediate from the axiom of extensionality.

(3) It is immediate from (2).

(4) ($\Rightarrow$) Suppose $x \neq \emptyset$ and let $t \in x$. Since $\{x\}$ is transitive, $t \in \{x\}$ and so $x \in x$. This is impossible because $x \subset \{x\}$. Thus $x = \emptyset$.

($\Leftarrow$) Since $\emptyset \subset \{\emptyset\}$, $\{\emptyset\}$ is transitive.

(5) Suppose $t \in \cap x$. Since x is transitive, $t \in x$; hence $t \in t$. By (2), this is impossible. Thus $\cap x = \emptyset$.

The proofs of (6) and (7) are routine.

(8) Suppose $s \in t \in x \cap p(z)$, then $t \in x$ and $t \in p(z)$. Since $s \in t$ and t is transitive, $s \in p(t)$; hence $s \in p(z)$. Since $t \in p(z)$ and $z \subseteq x$, $t \in p(x)$. Since $s \in t$, $s \in x$. Thus $s \in x \cap p(z)$. Suppose $t = x \cap p(z)$, then by the assumption, $t \in t$, contradicting the fact that t is transitive. Thus $t \neq x \cap p(z)$. Therefore $x \cap p(z)$ is transitive. □

It is clear that, by (4), $\{\{\emptyset\}\}$ is not transitive.

NOTATION. For a set x, $tr(x)$ denotes the set of proper transitive subsets of x. That is, $tr(x) = \{t \in pr(x) \mid t \subseteq pr(t)\}$.

Definition 2.2. A set x is called:

(1) a *bi-transitive set* if $tr(x) \subseteq x \subseteq pr(x)$.

(2) an *ordinal* if it is a transitive set whose elements are bi-transitive sets.

Proposition 2.3. *(1) The empty set $\emptyset$ is an ordinal.*

(2) If x is a bi-transitive set, then $x = \emptyset$ or $\emptyset \in x$.

Proof. (1) It is vacuously true.

(2) Suppose $x \neq \emptyset$, then $\emptyset \subset x$. Since $\emptyset$ is transitive, $\emptyset \in x$. □

Lemma 2.4. *If x is a set of bi-transitive sets and $z \subseteq x$, then $x \cap p(z)$ is an ordinal.*

Proof. Clearly every element of $x \cap p(z)$ is bi-transitive. By Proposition 2.1 (2), $x \cap p(z)$ is transitive; hence $x \cap p(z)$ is an ordinal. □

For two sets x and y, $x \leq y$ means that $x \in y$ or $x = y$.

Theorem 2.5. (1) *Every element of an ordinal is also an ordinal.*

(2) *If x is a nonempty set of ordinals, then it is well-ordered by $\leq$.*

(3) *If x is a nonempty ordinal, then it is well-ordered by $\leq$.*

(4) *If x is a nonempty ordinal, then $tr(x) \subseteq x$ and hence it is bi-transitive.*

(5) *If x is a nonempty ordinal and $s \in tr(x)$, then s is the least element of $x - s$.*

Proof. (1) Let x be an ordinal and $t \in x$. Clearly t is transitive. Since x is transitive, $t \subset x$; hence every element of t is a bi-transitive set. Thus t is an ordinal.

(2) Let $\cap x$ be the intersection of all elements of x. Firstly, we show that $\cap x$ is transitive. Take any $y \in \cap x$, then $y \in t$ for all $t \in x$. Since t is transitive, $y \subset t$; hence $y \subseteq \cap x$. If $y = \cap x$, then $y \in y$ for $y \in \cap x$, contradicting that y is transitive. Thus $y \subset \cap x$. Hence $\cap x$ is transitive. Since every element t of x is bi-transitive, $\cap x \leq t$ for all $t \in x$. Now we show that $\cap x \in x$. Suppose $\cap x \neq t$ for all $t \in x$, then $\cap x \in t$ for all $t \in x$; hence $\cap x \in \cap x$, contradicting that $\cap x$ is transitive. Thus $\cap x = t$ for some $t \in x$ and hence $\cap x \in x$. Thus by Proposition 2.1 (3) and (6), x is well-orderd by $\leq$.

(3) It is immediate from the above statements (1) and (2).

(4) Let $s \in tr(x)$ and $t \in x - s$. Now we show that $s \subseteq t$. Take any $u \in s$, then $u \in x$. Since x is linearly ordered by $\leq$ and $t, u \in x$, $u \in t$ or $t \leq u$. Suppose $t \leq u$, then $t \in s$ because s is transitive. Thus $u \in t$.

Hence $s \subseteq t$. Since $tr(t) \subseteq t$, $s \leq t$; hence $s \in x$ because x is transitive and $t \in x$. Therefore $tr(x) \subseteq x$ and hence x is bi-transitive.

(5) In the proof of (4), it was proved that $s \leq t$ for all $t \in x - s$ and $s \in x$. Since s is transitive, $s \in x - s$. Hence s is the least element of $x - s$. □

Remark 2.6. (1) By Proposition 2.1 (2) and Theorem 2.5 (5), if x is an ordinal, then $x = \emptyset$ or $\emptyset \in x$.

(2) If x is the set of bi-transitive sets, then by Lemma 2.4, $z = x \cap p(x)$ is an ordinal; hence $z \in z$ and $z \notin z$. Moreover, by Theorem 2.5 (5), z is the set of all ordinals. Thus we have the following: (a) There does not exist a set of all bi-transitive sets, and (b) There does not exist a set of all ordinals. (b) is essentially the statement of the Burali-Forti pardox-historically the first of the pardoxes of intuitive set theory([7]).

Theorem 2.7 ([5], [7]). *If x is a transitive set and well ordered by $\leq$, then $tr(x) \subseteq x$ and hence x is bi-transitive.*

The ordinal defined by Neumann is called an N-ordinal. In [5], without assuming the axiom of regularity, it was proved that $x \notin x$ for all N-ordinals x. In fact, the proof relies on the fact that $\in$ is an asymmetry relation. Combining this fact with Theorems 2.5 and 2.7, we have the following:

Theorem 2.8. *A set x is an ordinal if and only if it is an N-ordinal.*

Proof. Since both ordinal and N-ordinal are transitive, it is enough to show that every element of x is bi-transitive if and only if it is well-ordered by $\leq$.

($\Rightarrow$) It is immediate from Theorem 2.5 (3).

($\Leftarrow$) Since x is transitive, it is enough to show that every element of x is bi-transitive. Let $y \in x$ and take any $z \in y$. Since x is well-ordered by $\leq$, $z \subseteq y$. Suppose $z = y$, then $z \in z$; hence $y \in z$, contrary to asymmetry of $\in$. Thus $z \in pr(y)$; hence $y \subseteq p(y)$. Let $t \in tr(y)$. Then $t \in pr(y)$. Since $x \subseteq pr(x)$ and $y \in x$, $t \in pr(x)$. Since $t \subseteq pr(t)$, $t \in tr(x)$; hence, by Theorem 2.7, $t \in x$. Since $\in$ linearly orders x and $t \in tr(y)$, $t \in y$; hence $tr(y) \subseteq y$. Thus every element of x is bi-transitive. □

The ordinal defined by Robinson is called an R-ordinal.

Theorem 2.9 ([7], [11]). *If x is a R-ordinal, then $x = tr(x)$ and $t = tr(t)$ for all $t \in x$.*

Theorem 2.10. *A set is an ordinal if and only if it is an R-ordinal.*

Proof. It is immediate from Theorems 2.5 and 2.9. □

THEOREM 2.11. *Let x be a set such that $x \notin x$. Then x is bi-transitive if and only if $x \cup \{x\}$ is bi-transitive.*

Proof. Suppose x is bi-transitive, then it is clear that $x \cup \{x\}$ is transitive. Take any $t \in tr(x \cup \{x\})$, then $t \in tr(x)$. Otherwise $\{x\} \subseteq t$; hence $x \in t$. Since t is transitive, $x \subset t$. Thus $x \cup \{x\} \subseteq t$, contradicting $t \subset x \cup \{x\}$. Since x is bi-transitive and $t \in tr(x)$, $t \in x \cup \{x\}$. Thus $x \cup \{x\}$ is bi-transitive. Suppose $x \cup \{x\}$ is bi-transitive. Take any $t \in x$, then $t \subset x \cup \{x\}$. Suppose $t \cap \{x\} \neq \emptyset$, then $x \in t$; hence $x \in x$, contradicting $x \notin x$. Thus $t \subset x$; hence x is transitive. Take any $t \in x$, then $t \in tr(x \cup \{x\})$; hence $t \in x \cup \{x\}$. Thus $t \in x$ or $t = x$. Since $t \subset x$, $t \in x$. Therefore x is bitransitive. □

The following is due to Cesare Burali-Forti([5]).

COROLLARY 2.12. *A set x is an ordinal if and only if $x \cup \{x\}$ is an ordinal.*

Proof. It is immediate from the above theorem. □

3. Two types of limit ordinals

In this section, we classify limit ordinals into two types: usual limit ordinals and transfinite limit ordinals and we characterize the usual limit ordinals.

Let x be an ordinal, then $\cup x \leq x$; hence for any $s \in x$, there is $t \in x$ such that $s \in t$. The set $\{t \in x \mid s \in t\}$ has the least element, say l_s. Then $l_s = s \cup \{s\}$. Thus if x is an ordinal, then $x = S_x \cup L_x$, where $S_x = \{l_s \mid s \in x\}$ and $L_x = x - S_x$. In particular, $\emptyset \in L_x$. Now we recall (see [5], [6], [8] or [9]) that an ordinal x is called:
(1) a *successor ordinal* if $x \in S_y$ for some ordinal y.
(2) a *limit ordinal* if $x \in L_y$ for some ordinal y.

REMARK 3.1. For every ordinal x, $x \cup \{x\}$ is a successor ordinal; hence every ordinal is an element of a successor ordinal. If there is the set x of successor ordinals, then $\cup x$ is the set of all the ordinals. Hence, by Cesare Burali-Forti's pardox, there does not exist the set of successor ordinals.

We recall that a set i is *inductive* if $\emptyset \in i$, and $x \in i$ implies $x \cup \{x\} \in i$.

THEOREM 3.2. *Let x be an inductive set such that $x \notin x$. Then x is transitive if and only if $x = \cup x$.*

Proof. ($\Rightarrow$) Since x is transitive, $\cup x \subseteq x$. It remains to show that $x \subseteq \cup x$. If $t \in x$, then $t \cup \{t\} \in x$ because x is inductive. Hence $t \in \cup x$. Thus $x \subseteq \cup x$.

($\Leftarrow$) It is clear that $t \subseteq x$ for all $t \in x$. Since $x \neq x$, $x \neq t$. Thus x is transitive. □

Using the above theorem, we have the following:

Corollary 3.3. *Let x be an ordinal. Then one has the following:*
(1) *x is inductive if and only if $\cup x = x$.*
(2) *x is a successor ordinal if and only if $\cup x \in x$.*

Proof. (1) ($\Rightarrow$) It is immediate from the above lemma.

($\Leftarrow$) Since x is an ordinal, $\emptyset \in x$. Suppose $t \in x$. Since $x \subseteq pr(x)$, $t \subset x$. Hence $t \cup \{t\} \subseteq x$. Suppose $t \cup \{t\} = x$. Then $\cup x = t$ for $t \subseteq pr(t)$. Hence $x \in x$, contradicting $x \notin x$. Thus $t \cup \{t\} \subset x$. By Corollary 2.12, $t \cup \{t\} \in tr(x)$. Since $tr(x) \subseteq x$, $t \cup \{t\} \in x$. Therefore, x is inductive.

(2) If $x \in \cup x$, then $t \in x \in t$ for some $t \in x$; hence $t \in t$, contradicting $t \notin t$. Thus $x \notin \cup x$. Hence it is immediate from (1). □

Remark 3.4. It is well known that an ordinal x is a limit ordinal if and only if $x = \cup x$. Thus, by the above corollary, the following are equivalent for an ordinal x:
(1) x is a limit ordinal.
(2) x is inductive.
(3) $x = \cup y$ for some subset y of x.

Theorem 3.5. *If ω is the least inductive set whose elements are bi-transitive sets, then one has the following:*
(a) *$\omega = \omega \cap p(\omega)$.*
(b) *ω is a limit ordinal.*
(c) *Every element of ω is either a successor ordinal or the empty set.*
(d) *If p is an inductive subset of ω, then $p = \omega$.*

Proof. (a) Let $x = \omega \cap p(\omega)$. Clearly x is a set of bi-transitive sets. Now we show that x is inductive. Clearly $\emptyset \in x$. Suppose $s \in x$, then $s \in \omega$ and $s \subset \omega$; hence $s \cup \{s\} \subseteq \omega$. Since ω is inductive and $s \in \omega$, $s \cup \{s\} \in x$. Therefore x is an inductive set whose elements are bi-transitive sets. Since $x \subseteq \omega$, by the assumption, $x = \omega$.

(b) It is immediate from (a), Lemma 2.4 and the above remark.

(c) Suppose $t \in \omega$ and t is a nonempty limit ordinal, then $t \subset \omega$ because ω is transitive. Since t is a limit ordinal, t is an inductive set whose elements are bi-transitive sets; hence $\omega \subseteq t$, which is a contradiction to $t \subset \omega$. Thus every element of ω is a successor ordinal or the empty set.

(d) Suppose $\omega - p \neq \emptyset$ and let l be the least element of $\omega - p$. Then $l \neq \emptyset$ for $\emptyset \in p \cap \omega$; hence by (c), l is a successor ordinal; hence $l = y \cup \{y\}$ for some ordinal y. Since $l \in \omega$ and ω is transitive, $l \subset \omega$; hence $y \in \omega$. Since l is the least element of $\omega - p$, $y \in p$. Since p is inductive, $l \in p$, which is a contradiction to $l \notin p$. Thus $p = \omega$. □

Remark 3.6. The limit ordinal ω is exactly the set of all the naturals.

For any limit ordinal x, $\cup L_x \leq x$ because $\cup L_x$ is a limit ordinal. Motivated by this observation, we can classify limit ordinals into two types as follows:

Definition 3.7. A limit ordinal x is called:
(a) a *usual limit ordinal* if $\cup L_x \in x$.
(b) a *transfinite limit ordinal* if $\cup L_x = x$.

Remark 3.8. Let x be a transfinite limit ordinal, then there is $t \in L_x$ such that $s \in t$ for every $s \in x$. For any $s \in x$, let $k = \{t \in L_x \mid s \in t\}$. Then there exists the least element of k, say l_s. Therefore $x = \cup\{l_s \in L_x \mid s \in x\}$.

Now we characterize usual limit ordinals.

Theorem 3.9. *For a nonempty ordinal x, x is a usual limit ordinal if and only if there is a limit ordinal y such that*
(1) $x = y \cup [y, x)$, *where* $[y, x) = \{z \in x \mid y \leq z\}$,
(2) *every element of* $[y, x) - \{y\}$ *is a successor ordinal, and*
(3) *if p is a subset of $[y, x)$ such that $y \in p$ and $y \in p$ implies $y \cup \{y\} \in p$, then $p = [y, x)$.*

Proof. ($\Rightarrow$) Let $y = \cup L_x$. Then y is a limit ordinal, $y \in x$ and $z \leq y$ for all $z \in L_x$; hence y is the largest element of L_x. Take any $z \in [y, x) - \{y\}$. Then it is clear that $z \neq \emptyset$. Suppose z is a limit ordinal. Since $z \in x$, $z \leq y$. However, this is impossible because $y \in z$. Thus z is a successor ordinal. Since every nonempty ordinal is either a successor ordinal or a limit ordinal, $x = y \cup [y, x)$. Since $[y, x)$ is a set of ordinals, by Theorem 2.5 (2), $[y, x)$ is a well-ordered set. Suppose $[y, x) - p \neq \emptyset$ and let l be the least element of $[y, x) - p$. Then $l \in [y, x)$ and $l \neq y$; hence by (2), $l = u \cup \{u\}$ for some ordinal u. Since $y \in u \cup \{u\} \in x$, $u \in [y, x)$. Since l is the least element of $[y, x) - p$, $u \in p$. By the assumption, $l \in p$, which is a contradiction. Thus $p = [y, x)$.

($\Leftarrow$) By (3), x is inductive because y is limit ordinal and by (1) and (2), $y \in x$ and $y = \cup Ł_x$. Since x is an ordinal, x is a usual limit ordinal. □

Remark 3.10. Since $L_\omega = \{\emptyset\}$, $\cup L_\omega = \emptyset$.

4. Transfinite set

In this section, we introduce a concept of y-inductive sets and using this, we also introduce the weak transfinite set and present an induction principle. Using the induction principle, we show that the weak transfinite set is a transfinite limit ordinal.

Definition 4.1. Let x and y be sets. Then x is called a *y-inductive set* if it has the following properties:
(1) $y \in x$,
(2) $t \in x$ implies $t \cup \{t\} \in x$, and
(3) $t \in x$ implies $t \notin y$.

Remark 4.2. (1) Every $\emptyset$-inductive set is an inductive set.
(2) It is clear that the intersection of all the y-inductive sets is also a y-inductive set.

Definition 4.3. (1) The intersection of all the y-inductive sets is called the *y-set*.
(2) An inductive set y is said to be a *super predeccessor* of an inductive set x if the following properties hold:
(a) $\{t \in x \mid y \leq t\}$ is a y-set.
(b) $y \cap \{t \in x \mid y \leq t\} = \emptyset$.
(c) $x = y \cup \{t \in x \mid y \leq t\}$.

Remark 4.4. (1) The $\emptyset$-set is exactly the set ω of all the naturals.
(2) If y is a super predeccessor of x, then $y \in x$ and $y \subset x$.
(3) If y and z are super predeccessors of x, then $y = z$.

If y is the super predeccessor of x, then we write $y = x_{sp}$.

In the rest of this paper, we assume $\emptyset$ is the super predecessor of the $\emptyset$-set.

Definition 4.5. A set x is said to be *weak transfinite inductive* if it satisfies the following properties:
(I_1) $\emptyset \in x$.
(I_2) $t \in x$ implies $t \cup \{t\} \in x$, and

(I_3) $t_{sp} \in x$ implies $t \in x$ for all inductive sets t.

The following is immediate from the above definition:

THEOREM 4.6. (1) *Every weak transfinite inductive set contains the $\emptyset$-set.*

(2) *The intersection of all the weak transfinite inductive sets is also a weak transfinite inductive set.*

DEFINITION 4.7. The intersection of all the weak transfinite inductive sets is called the *weak transfinite set* and is denoted by π.

COROLLARY 4.8. *If p is a subset of π such that*
(T_1) $\emptyset \in p$,
(T_2) $t \in p$ *implies* $t \cup \{t\} \in p$, *and*
(T_3) $t_{sp} \in p$ *implies* $t \in p$ *for all inductive sets* t, *then* $p = \pi$.

Proof. It is immediate from Theorem 4.6. □

Using the above corollary, we have the following theorem:

THEOREM 4.9. *π is an ordinal.*

Proof. First we show that π is the set of ordinals. Let o be the set of all ordinals in π. It is clear that $\emptyset \in o$ because $\emptyset$ is an ordinal and $\emptyset \in \pi$. Suppose $s \in o$, then by corollary 2.13 $s \cup \{s\} \in o$ because π is inductive. Suppose t is an inductive set such that $t_{sp} \in o$ and let $u = \{s \in t \mid t_{sp} \leq s\}$. That is, $t = t_{sp} \cup u$. Since $t_{sp} \in o$, by Theorem 2.5 (1), every element of t_{sp} is an ordinal. Now we show that every element of u is an ordinal. Let k be the set of all $s \in u$ such that s is an ordinal. Clearly $t_{sp} \in k$. Suppose $s \in k$, then s is an ordinal; hence by corollary 2.13, $s \cup \{s\} \in k$ because u is inductive. Thus $k = u$. Hence every element of t is an ordinal. It remains to show that t is transitive. First we show that $t_{sp} \subseteq pr(t)$. If $s \in t_{sp}$ and $y \in s$, then $y \in t_{sp}$ and $s \neq t_{sp}$ because t_{sp} is transitive. Since $t_{sp} \subset t$, $y \in t$ and $s \neq t$. Hence $t_{sp} \subseteq pr(t)$. Now we show that $u \subseteq pr(t)$. Let k be the set of all $s \in u$ such that $s \in pr(t)$. It is clear that $t_{sp} \in k$ because $t_{sp} \subset t$ and $t_{sp} \in t$. Suppose $s \in k$, then $s \in u$. Since u is inductive, $s \cup \{s\} \in u$. Since $s \subset t$ and $s \in t$, $s \cup \{s\} \subset t$ because t is inductive. That is, $s \cup \{s\} \in pr(t)$. Thus $s \cup \{s\} \in k$. Hence by the definition of u, $k = u$. In all, $t \in o$. By corollary 4.8, $o = \pi$.

Now we show that π is transitive. Let p be the set of all $t \in \pi$ such that $t \in pr(\pi)$. Clearly $\emptyset \in p$. Suppose $t \in p$, then $t \cup \{t\} \in \pi$ for $t \in \pi$. Since $t \in pr(\pi)$ and $t \in \pi$, $t \cup \{t\} \subset \pi$ because π is inductive. Thus $t \cup \{t\} \in p$. Suppose $t_{sp} \in p$ and t is an inductive set, then $t_{sp} \in \pi$; hence by (I_3), $t \in \pi$. Now we show that $t \subset \pi$. Since $t_{sp} \in p$, $t_{sp} \subset \pi$. Note that $t = t_{sp} \cup u$, where u denotes $\{s \in t \mid t_{sp} \leq s\}$. It remains to show that u is a subset of x. Let k be the set of all $s \in u$ such that $s \in \pi$. By the definition of the t_{sp}, $t_{sp} \in u$. Since $t_{sp} \in \pi$, $t_{sp} \in k$. Suppose $s \in k$, then $s \in u$ and $s \in \pi$. Since u and π are both inductive sets, $s \cup \{s\} \in u$ and $s \cup \{s\} \in \pi$; hence $s \cup \{s\} \in k$. By the definition of u, $k = u$ and hence $u \subseteq \pi$. Since $t_{sp} \cap u = \{t_{sp}\}$ and $t_{sp} \subset \pi$, $t \subset \pi$ because $\pi \notin \pi$ and $t \in \pi$. Thus $t \in p$. By Corollary 4.8, $p = \pi$. In all, π is transitive. In all, π is an ordinal. □

THEOREM 4.10. *π is a transfinite limit ordinal.*

Proof. Clearly $\cup L_\pi = \pi_{sp} \subseteq \pi$. Let k be the set of all $s \in \pi$ such that $s \in \pi$ implies $s \in \pi_{sp}$. Clearly $\emptyset \in k$. Suppose $s \in k$. Since π and π_{sp} are inductive, $s \cup \{s\} \in k$ for all successor ordinals s. Suppose t is a limit ordinal and $t_{sp} \in k$. Since π is the transfinite set, $t \in \pi$; hence $t \leq \pi_{sp}$. If $t = \pi_{sp}$, then $\pi_{sp} \in \pi$; by (I_3), $\pi \in \pi$. However, this is impossible because π is an ordinal. Hence $t \in \pi_{sp}$. By Corollary 4.8, $k = \pi$. In all, $\pi_{sp} = \pi$. □

DEFINITION 4.11. Every element of π is called an *ordinary ordinal*.

Using Corollary 4.8, we now introduce the weak transfinite induction principle which resembles closely the usual formulation of the induction principle for the set ω of natural s.

COROLLARY 4.12. *Let $P(t)$ be a property involving the ordinary ordinal t such that*
(*a*) *$P(\emptyset)$,*
(*b*) *$P(t)$ implies $P(t \cup \{t\})$ for all $t \in x$, and*
(*c*) *$P(t_{sp})$ implies $P(t)$ for all usual limit ordinals t.*
Then $P(t)$ for all ordinary ordinals t.

Proof. Let $p = \{t \in \pi \mid P(t)\}$. Then it is clear that p is a subset of π which satisfies conditions T_1, T_2 and T_3 of Corollary 4.8. Thus $P(t)$ for all ordinary ordinals t. □

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