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EXISTENCE OF POSITIVE SOLUTIONS FOR BVPS TO INFINITE DIFFERENCE EQUATIONS WITH ONE-DIMENSIONAL p-LAPLACIAN

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Abstract. Motivated by Agarwal and O'Regan (Boundary value problems for general discrete systems on infinite intervals, Comput. Math. Appl. 33(1997)85-99), this article deals with the discrete type BVP of the infinite difference equations. The sufficient conditions to guarantee the existence of at least three positive solutions are established. An example is presented to illustrate the main results. It is the purpose of this paper to show that the approach to get positive solutions of BVPs by using multi-fixed-point theorems can be extended to treat BVPs for infinite difference equations. The strong Caratheodory (S-Caratheodory) function is defined in this paper.

1. Introduction

Let $N_0 = \{0, 1, 2, 3, \cdots\}$ and $N = \{1, 2, 3, \cdots\}$. Denote $\sum_{i=a}^{b} x(i) = x(a) + x(a+1) + \cdots + x(b)$ for $a, b \in N_0$ with $a \leq b$ and $\sum_{i=a}^{b} x(i) = 0$ if $a, b \in N_0$ and $b < a$. In recent years, there have been many papers discussed with the solvability of boundary value problems for finite difference equations, see [1-21], we know except [22] no other paper concerns with the boundary value problems for infinite difference equations.

The purpose of this paper is to investigate the following boundary value problem (BVP for short) for the second order p-Laplacian infinite

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difference equation

$$(1.1)\qquad \begin{cases} \Delta[\phi(\Delta x(n))] + f(n, x(n), \Delta x(n)) = 0, \quad n \in N_0, \\ x(0) - \sum_{n=1}^{\infty} \alpha_n x(n) = 0, \\ \lim_{n\to\infty} \Delta y(n) - \sum_{n=1}^{\infty} \beta_n \Delta y(n) = 0, , \end{cases}$$

where $\alpha_n, \beta_n \geq 0$ for all $n \in N$ with

$$\sum_{n\in N} \alpha_n < 1, \quad \sum_{N\in N} n\alpha_n < \infty, \quad \sum_{n\in N} \beta_n < 1,$$

$f : N_0 \times [0, \infty)^2 \to [0, \infty)$ is a S-Caratheodory function (strong Caratheodory function), i.e., for each $n \in N_0$ $f(n, \cdot, \cdot)$ is continuous, and for each $r > 0$ there exists a nonnegative sequence $\{\psi_r(n)\}$ with

$$\sum_{n\in N_0} \psi_r(n) < \infty, \quad \sum_{n=0}^{\infty}\sum_{j=n}^{\infty} \phi_r(j) < +\infty$$

such that

$$|f(n, (1+n)x, y)| \leq \psi_r(n) \text{ for all } n \in N_0, \quad |x|, |y| \leq r,$$

$f(n, 0, 0) \not\equiv 0$ for all $n \in N_0$, ϕ is called $p-$Laplacian, $\phi(x) = |x|^{p-2}x$ with $p > 1$, its inverse function is denoted by $\phi^{-1}(x)$ with $\phi^{-1}(x) = |x|^{q-2}x$ with $1/p + 1/q = 1$.

We establish sufficient conditions for the existence of at least three positive solutions of BVP(1).

The remainder of this paper is organized as follows: to get the main results, in Section 2, we first give seven lemmas and then construct an operator in cones in a suitable Banach space, then the proof of Theorem L is presented at the end of this section. An example is given in Section 3 to illustrate the main results.

2. Main results

Choose

$$X = \{\{x(n)\} : x(n) \in R, n \in N_0 \\ \text{there exist the limits } \lim_{n\to\infty} \frac{x(n)}{n+1}, \lim_{n\to\infty} \Delta x(n)\}.$$

Define the norm

$$||x|| = \max\left\{ \sup_{n\in N_0} \frac{|x(n)|}{1+n}, \sup_{n\in N_0} |\Delta x(n)| \right\}.$$

It is easy to see that X is a real Banach space.

Let $k_1, k_2 \in N$ with $k_1 < k_2$. Choose

$$P = \left\{ x \in X : \begin{array}{l} x(n) \geq 0 \text{ for all } n \in N_0, \\ \Delta x(n) \geq 0 \text{ for all } n \in N_0, \\ \min_{n\in[k_1,k_2]} \frac{x(n)}{1+n} \geq \frac{1}{2(1+k_2)} \sup_{n\in N_0} \frac{x(n)}{1+n}, \\ x(0) - \sum_{n=1}^{\infty} \alpha_i x(n) = 0, \\ \lim_{n\to\infty} \Delta x(n) - \sum_{n=1}^{\infty} \beta_n \Delta x(n) = 0 \end{array} \right\}, \tag{2.1}$$

Suppose $\lambda > 0$ and $\mu = \lambda \sum_{n\in N} \alpha_n$. Set

$$x_0(n) = \begin{cases} \mu, n = 0, \\ \lambda, n \in N. \end{cases}$$

It is easy to see that $x_0 \in P$. Then P is a nontrivial cone in X.

Let $h(n)(n \in N_0)$ be a nonnegative sequence with $\sum_{n\in N_0} h(n)$ converging, consider the following BVP

$$\begin{cases} \Delta[\phi(\Delta y(n))] + h(n) = 0, \quad n \in N_0, \\ y(0) - \sum_{n=1}^{\infty} \alpha_i y(n) = 0, \\ \lim_{n\to\infty} \Delta y(n) - \sum_{n=1}^{\infty} \beta_i \Delta y(n) = 0, \end{cases} \tag{2.2}$$

Lemma 2.1. *If y is a solution of BVP(2.2), then $y(n) \geq 0$ and $\Delta y(n) \geq 0$ for all $n \in N_0$, $\Delta y(n)$ is decreasing.*

Proof. Since $\Delta[\phi(\Delta y(n))] = -h(n) \leq 0$ for all $n \in N_0$, we get that $\phi(\Delta y(n))$ is decreasing. Then $\Delta y(n)$ is decreasing. It follows from the boundary conditions that

$$\lim_{n\to\infty} \Delta y(n) = \sum_{n=1}^{\infty} \beta_i \Delta y(n) \geq \sum_{n=1}^{\infty} \beta_i \lim_{n\to\infty} \Delta y(n).$$

Then

$$\left(1 - \sum_{n=1}^{\infty} \beta_i\right) \lim_{n\to\infty} \Delta y(n) \geq 0.$$

Since $\sum_{n=1}^{\infty} \beta_i < 1$,, we get $\lim_{n\to\infty} \Delta y(n) \geq 0$. Together with the decreasing property of $\Delta y(n)$, we get $\Delta y(n) \geq 0$ for all $n \in N_0$. Thus

$$y(0) = \sum_{n=1}^{\infty} \alpha_i y(n) \geq \sum_{n=1}^{\infty} \alpha_i y(0).$$

Since $\sum_{n=1}^{\infty} \alpha_i < 1$, we get $y(0) \geq 0$. Together with the increasing property of $y(n)$, we get $y(n) \geq 0$ for all $n \in N_0$. The proof is complete. □

LEMMA 2.2. *Suppose y is a solution of BVP(2.2). Then*

$$y(n) = B_h + \sum_{k=0}^{n-1} \phi^{-1}\left(\phi(A_h) + \sum_{s=k}^{\infty} h(s)\right), \quad n \in N_0, \tag{2.3}$$

where A_h satisfies

$$A_h = \sum_{n=1}^{\infty} \beta_n \phi^{-1}\left(\phi(A_h) + \sum_{s=n}^{\infty} h(s)\right), \tag{2.4}$$

and

$$B_h = \frac{1}{1-\sum_{n=1}^{\infty}\alpha_n} \sum_{n=0}^{\infty} \alpha_n \sum_{k=0}^{n-1} \phi^{-1}\left(\phi(A_h) + \sum_{s=k}^{\infty} h(s)\right). \tag{2.5}$$

Further more, we have

$$A_h \in \left[0, \phi^{-1}\left(\frac{\sum_{n=1}^{\infty}\beta_n}{1-\sum_{n=1}^{\infty}\beta_n} \sum_{n=0}^{\infty} h(n)\right)\right]. \tag{2.6}$$

Proof. Since $\sum_{n=0}^{\infty} h(n)$ converges, we get from (3) that

$$\phi(\Delta y(\infty)) - \phi(\Delta y(n)) = -\sum_{s=n}^{\infty} h(s).$$

So

$$\Delta y(n) = \phi^{-1}\left(\phi(\Delta y(\infty)) + \sum_{s=n}^{\infty} h(s)\right).$$

It follows that

$$y(n) = y(0) + \sum_{k=0}^{n-1} \phi^{-1}\left(\phi(\Delta y(\infty)) + \sum_{s=k}^{\infty} h(s)\right), \quad n \in N_0.$$

It follows from the boundary conditions that

$$y(0) = y(0)\sum_{n=1}^{\infty}\alpha_n + \sum_{n=1}^{\infty}\alpha_n \sum_{k=0}^{n-1} \phi^{-1}\left(\phi(\Delta y(\infty)) + \sum_{s=k}^{\infty} h(s)\right)$$

and

$$\Delta y(\infty) = \sum_{n=1}^{\infty} \beta_n \phi^{-1}\left(\phi(\Delta y(\infty)) + \sum_{s=n}^{\infty} h(s)\right).$$

Let $\Delta y(\infty) = A_h$ and $B_h = y(0)$. Then we get (2.3), (2.4), and (2.5). Now, from Lemma 2.1, we see $\Delta y(\infty) = A_h \geq 0$. On the other hand, let

$$G(c) = 1 - \sum_{n=1}^{\infty} \beta_n \phi^{-1}\left(1 + \frac{\sum_{s=n}^{\infty} h(s)}{\phi(c)}\right).$$

It is easy to see that $G(c)$ is continuous on $(0, \infty)$ and is strictly increasing on $(0, \infty)$. Since

$$\lim_{c \to 0^+} G(c) = -\infty,$$

and

$$\begin{aligned} &G\left(\phi^{-1}\left(\frac{\sum_{n=1}^{\infty}\beta_n}{1-\sum_{n=1}^{\infty}\beta_n}\sum_{n=0}^{\infty}h(n)\right)\right) \\ &= 1 - \sum_{n=1}^{\infty}\beta_n\phi^{-1}\left(1 + \frac{1-\sum_{n=1}^{\infty}\beta_n}{\sum_{n=1}^{\infty}\beta_n}\frac{\sum_{s=n}^{\infty}h(s)}{\sum_{n=0}^{\infty}h(n)}\right) \geq 0. \end{aligned}$$

It follows that

$$\Delta y(\infty) = A_h \in \left[0, \phi^{-1}\left(\frac{\sum_{n=1}^{\infty}\beta_n}{1-\sum_{n=1}^{\infty}\beta_n}\sum_{n=0}^{\infty}h(n)\right)\right].$$

The proof is complete. □

LEMMA 2.3. *If y is a solution of BVP(2.2), then*

$$\min_{n \in [k_1, k_2]} \frac{y(n)}{1+n} \geq \frac{1}{2(1+k_2)} \sup_{n \in N_0} \frac{y(n)}{1+n}. \tag{2.7}$$

Proof. It follow from Lemma 2.1 that $y(n) \geq 0$ and $\Delta y(n) \geq 0$ for $n \in N_0$, $\Delta y(n)$ is decreasing. Since there exists the limit $\lim_{n\to\infty} \Delta y(n)$, we can prove that there exists the limit $\lim_{n\to\infty} \frac{y(n)}{1+n}$. In fact, suppose that $\lim_{n\to\infty} \Delta y(n) = c$. If $c = 0$, then for any $\epsilon > 0$ there exists $H > 0$ such that

$$|\Delta y(n)| < \frac{\epsilon}{2}, \quad n \geq H.$$

It follows that

$$|y(n)| \leq |y(H)| + \sum_{s=H}^{n-1} |\Delta y(s)| \leq |y(H)| + \frac{\epsilon}{2}(n - H), \quad n \geq H.$$

Then

$$\frac{|y(n)|}{1+n} \leq \frac{|y(H)|}{1+n} + \frac{n-H}{1+n}\frac{\epsilon}{2} < \frac{|y(H)|}{1+n} + \frac{\epsilon}{2}, \quad n \geq H.$$

Choose $H' > H$ large enough so that

$$\frac{|y(n)|}{1+n} \le \frac{|y(H)|}{1+n} + \frac{\epsilon}{2} < \epsilon, \quad n \ge H',$$

which implies that

$$\lim_{n\to\infty} \frac{y(n)}{1+n} = 0.$$

If $c \neq 0$, then $\lim_{t\to\infty} \big(\Delta y(n) - c \big) = 0$. It follows that

$$\lim_{t\to\infty} \Delta \left[y(n) - cn \right] = 0.$$

Then we get similarly that

$$\lim_{t\to\infty} \frac{y(n) - cn}{1+n} = 0.$$

It follows that $\lim_{n\to\infty} \frac{y(n)}{1+n} = c$. It follows that there exists the number

$$\sup_{n\in N_0} \frac{y(n)}{1+n}.$$

To complete the proof, we consider two cases:

Case 1. there is $n_0 \in N_0$ such that $\sup_{n\in N_0} \frac{y(n)}{1+n} = \frac{y(n_0)}{1+n_0}$.

For $n_1, n, n_2 \in N_0$ with $n_1 < n < n_2$, we have

$$\begin{aligned}
&(n - n_1)\frac{y(n_2) - y(n)}{n_2 - n} + y(n_1) - y(n) \\
&= \frac{(n - n_1)(y(n_2) - y(n)) + (n_2 - n)(y(n_1) - y(n))}{n_2 - n} \\
&= \frac{(n - n_1)\sum_{s=n}^{n_2-1} \Delta y(s) - (n_2 - n)\sum_{s=n_1}^{n-1} \Delta y(s)}{n_2 - n} \\
&= \frac{-(n_2 - n)\sum_{s=n_1}^{n-1} \Delta y(s) + (n - n_1)\sum_{s=n}^{n_2-1} \Delta y(s)}{n_2 - n}.
\end{aligned}$$

Since $\Delta y(n)$ is decreasing, we get $\Delta y(s) \le \Delta y(k)$ for all $s \ge k$. Then we get

$$(n_2 - n)\sum_{s=n_1}^{n-1} \Delta y(s) \ge (n - n_1)\sum_{s=n}^{n_2-1} \Delta y(s).$$

So

$$(n - n_1)\frac{y(n_2) - y(n)}{n_2 - n} + y(n_1) - y(n) \le 0.$$

It follows that

$$y(n) \geq \frac{n_2 - n}{n_2 - n_1} y(n_1) + \frac{n - n_1}{n_2 - n_1} y(n_2). \tag{2.8}$$

If $n_0 = k_1$, we get

$$\min_{n \in [k_1,k_2]} \frac{y(n)}{1+n} \geq \frac{y(k_1)}{1+k_2} = \frac{y(n_0)}{1+n_0}\frac{1+k_1}{1+k_2} \geq \frac{1}{2(1+k_2)} \sup_{n \in N_0} \frac{y(n)}{1+n}.$$

If $n_0 > k_1$, by using (2.8) we have

$$\begin{aligned} y(k_1) &= y\left(\frac{n_0 - k_1}{n_0 - (k_1 - 1)}(k_1 - 1) + \frac{k_1 - (k_1 - 1)}{n_0 - (k_1 - 1)} n_0\right) \\ &\geq \frac{n_0 - k_1}{n_0 - (k_1 - 1)} y(k_1 - 1) + \frac{k_1 - (k_1 - 1)}{n_0 - (k_1 - 1)} y(n_0) \\ &\geq \frac{1 + n_0}{n_0 - (k_1 - 1)} \frac{y(n_0)}{1 + n_0}. \end{aligned}$$

Then

$$\begin{aligned} &\min_{n \in [k_1,k_2]} \frac{y(n)}{1+n} \geq \frac{y(k_1)}{1+k_2} \\ &\quad \geq \frac{1}{1+k_2}\frac{1+n_0}{n_0 - (k_1 - 1)}\frac{y(n_0)}{1+n_0} \geq \frac{1}{2(1+k_2)} \sup_{n \in N_0} \frac{y(n)}{1+n}. \end{aligned}$$

If $n_0 < k_1$, we have

$$\begin{aligned} &\min_{n \in [k_1,k_2]} \frac{y_n)}{1+n} \\ \geq\ & \frac{y(k_1)}{1+k_2} \\ =\ & \frac{1}{1+k_2} y\left(\frac{(2k_1 + 1 - n_0) - k_1}{(2k_1 + 1 - n_0) - n_0} n_0 \right. \\ & \qquad \left. + \frac{k_1 - n_0}{(2k_1 + 1 - n_0) - n_0}(2k_1 + 1 - n_0)\right) \\ \geq\ & \frac{1}{1+k_2}\left[\frac{(2k_1 + 1 - n_0) - k_1}{(2k_1 + 1 - n_0) - n_0} y(n_0) \right. \\ & \qquad \left. + \frac{k_1 - n_0}{(2k_1 + 1 - n_0) - n_0} y(2k_1 + 1 - n_0)\right] \end{aligned}$$

$$\begin{aligned}
&\geq \frac{1}{1+k_2}\frac{(k_1+1-n_0)(1+n_0)}{2k_1+1-2n_0}\frac{y(n_0)}{1+n_0}\\
&\geq \frac{1}{2(1+k_2)}\sup_{n\in N_0}\frac{y(n)}{1+n}.
\end{aligned}$$

Case 2. $\sup_{n\in N_0}\frac{y(n)}{1+n}=\lim_{n\to\infty}\frac{y(n)}{1+n}$.
Choose $n'>k_2$, similarly to Case 1 we can prove that

$$\min_{n\in[k_1,k_2]}\frac{y(n))}{1+n}\geq \frac{1}{2(1+k_2)}\frac{y(n')}{1+n'}.$$

Let $n'\to\infty$, one sees

$$\min_{n\in[k_1,k_2]}\frac{y(n))}{1+n}\geq \frac{1}{2(1+k_2)}\sup_{n\in N_0}\frac{y(n)}{1+n}.$$

From Cases 1 and 2, we get (2.7). The proof is complete. □

Now, we state some definitions and a very novel fixed point theorem called five functional fixed point theorem, whose proof can be found in [15].

Definition 2.4. [15] A map $\psi : P\to[0,+\infty)$ is a nonnegative continuous concave or convex functional map provided ψ is nonnegative, continuous and satisfies

$$\psi(tx+(1-t)y)\geq t\psi(x)+(1-t)\psi(y),$$

or

$$\psi(tx+(1-t)y)\leq t\psi(x)+(1-t)\psi(y),$$

for all $x,y\in P$ and $t\in[0,1]$.

Definition 2.5. [15] An operator $T;X\to X$ is completely continuous if it is continuous and maps bounded sets into pre-compact sets.

Definition 2.6. [15] Let $a,b,c,d,h>0$ be positive constants, α,ψ be two nonnegative continuous concave functionals on the cone P, γ,β,θ be three nonnegative continuous convex functionals on the cone P. Define the convex sets as follows:

$$\begin{aligned}
&P_c=\{x\in P:||x||<c\},\\
&P(\gamma,\alpha;a,c)=\{x\in P:\alpha(x)\geq a,\ \gamma(x)\leq c\},\\
&P(\gamma,\theta,\alpha;a,b,c)=\{x\in P:\alpha(x)\geq a,\ \theta(x)\leq b,\ \gamma(x)\leq c\},\\
&Q(\gamma,\beta;,d,c)=\{x\in P:\beta(x)\leq d,\ \gamma(x)\leq c\},\\
&Q(\gamma,\beta,\psi;h,d,c)=\{x\in P:\psi(x)\geq h,\ \beta(x)\leq d,\ \gamma(x)\leq c\}.
\end{aligned}$$

LEMMA 2.7. [15] *Let X be a real Banach space, P be a cone in X, α, ψ be two nonnegative continuous concave functionals on the cone P, γ, β, θ be three nonnegative continuous convex functionals on the cone P. There exist constant $M > 0$ such that*

$$\alpha(x) \leq \beta(x),\ ||x|| \leq M\gamma(x) \text{ for all } x \in P.$$

Furthermore, Suppose that $h, d, a, b, c > 0$ are constants with $d < a$. Let $T : \overline{P_c} \to \overline{P_c}$ be a completely continuous operator. If

(C1) $\{y \in P(\gamma, \theta, \alpha; a, b, c) | \alpha(x) > a\} \neq \emptyset$ *and*

$$\alpha(Tx) > a \text{ for every } x \in P(\gamma, \theta, \alpha; a, b, c);$$

(C2) $\{y \in Q(\gamma, \theta, \psi; h, d, c) | \beta(x) < d\} \neq \emptyset$ *and*

$$\beta(Tx) < d \text{ for every } x \in Q(\gamma, \theta, \psi; h, d, c);$$

(C3) $\alpha(Ty) > a$ *for* $y \in P(\gamma, \alpha; a, c)$ *with* $\theta(Ty) > b$;

(C4) $\beta(Tx) < d$ *for each* $x \in Q(\gamma, \beta; , d, c)$ *with* $\psi(Tx) < h$,

then T has at least three fixed points y_1, y_2 and y_3 such that

$$\beta(y_1) < d,\ \alpha(y_2) > a,\ \beta(y_3) > d,\ \alpha(y_3) < a.$$

Define the functionals on $P : P \to R$ by

$$\begin{aligned}
\gamma(x) &= \sup_{n \in N_0} |\Delta x(n)|,\ x \in P, \\
\beta(x) &= \sup_{n \in N_0} \frac{x(n)}{1+n},\ x \in P, \\
\theta(x) &= \sup_{n \in N_0} \frac{x(n)}{1+n},\ x \in P, \\
\alpha(x) &= \min_{n \in [k_1, k_2]} \frac{x(n)}{1+n},\ x \in P, \\
\psi(x) &= \min_{n \in [k_1, k_2]} \frac{x(n)}{1+n},\ x \in P.
\end{aligned}$$

LEMMA 2.8. *If y is a solution of BVP(2.2), we have $||y||| \leq M\gamma(y)$ for all $y \in P$, where*

$$M = \max\left\{1,\ \frac{\sum_{n=1}^{\infty} n\alpha_n}{1 - \sum_{n=1}^{\infty} \alpha_n}\right\}. \tag{2.9}$$

Proof. Since y is the solution of BVP(3), we get

$$|y(n)| = |y(n) - y(0) + y(0)| = \left|\sum_{i=0}^{n-1} \Delta y(i)\right| + \left|\sum_{n \in N} \alpha_n y(n)\right|$$

$$\begin{aligned}
&\leq && n\sup_{n\in N_0}|\Delta y(n)|+\left|\frac{\sum_{n\in N}\alpha_n[y(n)-y(0)]}{1-\sum_{i=1}^{\infty}\alpha_i}\right|\\
&\leq && n\sup_{n\in N_0}|\Delta y(n)|+\frac{\sum_{n=1}^{\infty}\alpha_n\sum_{s=0}^{n-1}|\Delta y(s)|}{1-\sum_{i=1}^{\infty}\alpha_i}\\
&\leq && \left(n+\frac{\sum_{n=1}^{\infty}n\alpha_n}{1-\sum_{i=1}^{\infty}\alpha_i}\right)\sup_{n\in N_0}|\Delta y(n)|.
\end{aligned}$$

It follows that

$$\begin{aligned}
\frac{y(n)}{1+n} &\leq \frac{n+\frac{\sum_{n=1}^{\infty}n\alpha_n}{1-\sum_{i=1}^{\infty}\alpha_i}}{1+n}\sup_{n\in N_0}|\Delta y(n)|\\
&\leq \max\left\{1,\ \frac{\sum_{n=1}^{\infty}n\alpha_n}{1-\sum_{i=1}^{\infty}\alpha_i}\right\}\sup_{n\in N_0}|\Delta y(n)|.
\end{aligned}$$

we get that

$$\begin{aligned}
||y|| &= \max\left\{\sup_{n\in N_0}\frac{|y(n)|}{1+n},\ \sup_{n\in N_0}|\Delta y(n)|\right\}\\
&\leq \max\left\{1,\ \frac{\sum_{n=1}^{\infty}n\alpha_n}{1-\sum_{i=1}^{\infty}\alpha_i}\right\}\sup_{n\in N_0}|\Delta y(n)|\\
&= \max\left\{1,\ \frac{\sum_{n=1}^{\infty}n\alpha_n}{1-\sum_{i=1}^{\infty}\alpha_i}\right\}\gamma(y).
\end{aligned}$$

Then $||y||\leq M\gamma(y)$ for all $y\in P$. The proof is complete.

For $x\in P$, define $(Tx)(n)$ by

$$(Tx)(n)=B_x+\sum_{k=0}^{n-1}\phi^{-1}\left(\phi(A_x)+\sum_{s=k}^{\infty}f(s,x(s),\Delta x(s))\right),\quad n\in N_0,$$

where A_x satisfies

$$A_x=\sum_{n=1}^{\infty}\beta_n\phi^{-1}\left(\phi(A_x)+\sum_{s=n}^{\infty}f(s,x(s),\Delta x(s))\right), \tag{2.10}$$

and

$$B_x=\frac{1}{1-\sum_{n=1}^{\infty}\alpha_n}\sum_{n=1}^{\infty}\alpha_n\sum_{k=0}^{n-1}\phi^{-1}\left(\phi(A_x)+\sum_{s=k}^{\infty}f(s,x(s),\Delta x(s))\right). \tag{2.11}$$

One sees easily that

$$(2.12)\qquad \begin{cases} \Delta[\phi(\Delta(Tx)(n))] + f(n, x(n), \Delta x(n)) = 0, \quad n \in N_0, \\ (Tx)(0) - \sum_{i=1}^{\infty} \alpha_i (Tx)(i) = 0, \\ \lim_{n\to\infty} \Delta(Tx)(n) - \sum_{i=1}^{\infty} \beta_i \Delta(Tx)(i) = 0. \end{cases}$$

By Lemma 2.2, we have

$$(2.13)\qquad A_x \in \left[0, \phi^{-1}\left(\frac{\sum_{n=1}^{\infty}\beta_n}{1-\sum_{n=1}^{\infty}\beta_n}\sum_{n=0}^{\infty} f(n, x(n), \Delta x(n))\right)\right].$$

□

LEMMA 2.9. *Let $V = \{x \in X : ||x|| < l\}(l > 0)$. If $\left\{\frac{x(n)}{1+n} : x \in V\right\}$ and $\{\Delta x(n) : x \in V\}$ are both equiconvergent at infinity, where*

$$V_1 =: \left\{\frac{x(n)}{1+n} : x \in V\right\} \bigcup \{\Delta x(n) : x \in V\}$$

is called equiconvergent at infinity if and only if for all $\epsilon > 0$, there exists $N = N(\epsilon) > 0$ such that for all $x \in V$, it holds that

$$\left|\frac{x(n_1)}{1+n_1} - \frac{x(n_2)}{1+n_2}\right| < \epsilon, \ |\Delta x(n_1) - \Delta x(n_2)| < \epsilon \ \ n_1, n_2 > N.$$

Then V is pre-compact on X.

Proof. The proof is similar to that of the proof of Lemma in [22] and is omitted. □

LEMMA 2.10. *It holds that*

(i) *$Tx \in P$ for each $x \in P$;*

(ii) *x is a solution of BVP(1) if and only if x is a solution of the operator equation $x = Tx$;*

(iii) *$T : P \to P$ is completely continuous;*

Proof. (i) Note the definition of P. For $x \in P$, Lemma 2.1, Lemma 2.2 and Lemma 2.3 imply that $y(n) \geq 0$, $\Delta y(n) \geq 0$ for all $n \in N_0$, $\Delta(Tx)(n)$ is decreasing and $\min_{n\in[k_1,k_2]} \frac{(Tx)(n)}{1+n} \geq \frac{1}{2(1+k_2)} \sup_{n\in N_0} \frac{(Tx)(n)}{1+n}$. Together with (2.12), it follows that $Tx \in P$.

(ii) It is easy to see from (13) that x is a solution of BVP(1) if and only if x is a solution of the operator equation $x = Tx$.

(iv) It suffices to prove that T is continuous on P and T maps bounded subsets into pre-compact sets. We divide the proof into four steps:

Step 1. For each bounded subset $D \subset P$, prove that $\{(A_x, B_x) : x \in \overline{D}\}$ is bounded in R^2, where A_x and B_x are given in (2.10) and (2.11).

Denote

$$L_1 = \sup\left\{\max_{n\in N_0}\frac{|x(n)|}{1+n},\ \sup_{n\in N_0}|\Delta x(n)|:\ \ x\in\overline{D}\right\}$$

and

$$B_{L_1}(j) = \max_{|x|,|y|\le L_1}|f(j,(1+j)x,y)|.$$

Since f is a S-Caratheodory function, it follows from (2.13) that

$$0\le A_x\le\phi\left(\frac{\sum_{n=1}^{\infty}\beta_n}{1-\sum_{n=1}^{\infty}\beta_n}\sum_{j=0}^{\infty}B_{L_1}(j)\right)<\infty,$$

and B_x satisfies that

$$\begin{aligned}0 &\le |B_x|\\ &= \frac{1}{1-\sum_{i=1}^{\infty}\alpha_i}\sum_{j=1}^{\infty}\alpha_j\sum_{i=0}^{j-1}\phi^{-1}\left(\phi(A_x)+\sum_{s=i}^{\infty}f(s,x(s),\Delta x(s))\right)\\ &\le \frac{1}{1-\sum_{i=1}^{\infty}\alpha_i}\sum_{j=1}^{\infty}j\alpha_j\phi^{-1}\left(\frac{1}{1-\sum_{n=1}^{\infty}\beta_n}\sum_{s=0}^{\infty}f(s,x(s),\Delta x(s))\right)\\ &\le \frac{1}{1-\sum_{i=1}^{\infty}\alpha_i}\sum_{j=1}^{\infty}j\alpha_j\phi^{-1}\left(\frac{1}{1-\sum_{n=0}^{\infty}\beta_n}\sum_{j=0}^{\infty}B_{L_1}(j)\right)\\ &< \infty.\end{aligned}$$

Hence $\{(A_x,B_x):x\in\overline{D}\}$ is bounded in R^2.

Step 2. For each bounded subset $D\subset P$, and each $x_0\in D$, prove that T is continuous at x_0.

For $x_0\in D$ and $x_n\in D$ with $x_n\to x_0(n\to+\infty)$ in D. Denote $u_n(k)=(Tx_n)(k), u_0(k)=(Tx_0)(k)$ for all $k\in N_0$. We prove that T is continuous at x_0, i.e., $u_n\to u_0(n\to+\infty)$. Let A_{x_0},B_{x_0} be defined by

$$A_{x_0}=\sum_{n=1}^{\infty}\beta_n\phi^{-1}\left(\phi(A_{x_0})+\sum_{s=n}^{\infty}f(s,x(s),\Delta x(s))\right), \tag{2.14}$$

and

(2.15)

$$B_{x_0}=\frac{1}{1-\sum_{n=1}^{\infty}\alpha_n}\sum_{n=1}^{\infty}\alpha_n\sum_{k=0}^{n-1}\phi^{-1}\left(\phi(A_{x_0})+\sum_{s=k}^{\infty}f(s,x(s),\Delta x(s))\right).$$

First, we prove that A_x,B_x are continuous in x, i.e.,

$$(A_{x_n},B_{x_n})\to(A_{x_0},B_{x_0}),\ \ n\to+\infty.$$

It follows from Step 1 that (A_{x_n}, B_{x_n}) is bounded. Without loss of generality, suppose that $(A_{x_n}, B_{x_n}) \to (\overline{A}, \overline{B}) \neq (A_{x_0}, B_{x_0})$.

It is easy to see that

$$\begin{aligned}
& \lim_{n\to+\infty} u_n(k) \\
= & \lim_{n\to+\infty} \left[B_{x_n} + \sum_{i=0}^{k-1} \phi^{-1} \left(\phi(A_{x_n}) + \sum_{j=i}^{\infty} f(j, x_n(j), \Delta x_n(j)) \right) \right] \\
= & \overline{B} + \sum_{i=0}^{k-1} \phi^{-1} \left(\phi(\overline{A}) + \lim_{n\to+\infty} \sum_{j=i}^{\infty} f(j, x_n(j), \Delta x_n(j)) \right) \\
= & \overline{B} + \sum_{i=0}^{k-1} \phi^{-1} \left(\phi(\overline{A}) + \lim_{n\to+\infty} \sum_{j=i}^{\infty} f(j, x_0(j), \Delta x_0(j)) \right) \\
= & \overline{u}(k).
\end{aligned}$$

One sees that $\overline{B} = \overline{u}(0), \overline{A} = \lim_{n\to\infty} \Delta\overline{u}(n)$ and $\overline{u}$ satisfies

$$\overline{u}(0) - \sum_{i=1}^{\infty} \alpha_i \overline{u}(i) = 0, \quad \lim_{n\to\infty} \Delta\overline{u}(n) - \sum_{i=1}^{\infty} \beta_i \Delta\overline{u}(i) = 0.$$

So

$$\overline{A} = \sum_{n=1}^{\infty} \beta_n \phi^{-1} \left(\phi(\overline{A}) + \sum_{s=n}^{\infty} f(s, x(s), \Delta x(s)) \right),$$

and

$$\overline{B} = \frac{1}{1 - \sum_{n=1}^{\infty} \alpha_n} \sum_{n=1}^{\infty} \alpha_n \sum_{k=0}^{n-1} \phi^{-1} \left(\phi(\overline{A}) + \sum_{s=k}^{\infty} f(s, x(s), \Delta x(s)) \right).$$

It follows from Lemma 2.2, (2.14) and (2.15) that $\overline{A} = A_{x_0}$, then $\overline{B} = B_{x_0}$.

Hence

$$(A_{x_n}, B_{x_n}) \to (\overline{A}, \overline{B}) = (A_{x_0}, B_{x_0}), \quad n \to +\infty.$$

This together with the continuous property of f implies that T is continuous at x_0.

Step 3. For each bounded subset $\Omega \subset P$, prove that $T\Omega$ is bounded.

In fact, for each bounded subset $\Omega \subseteq D$, and $x \in \Omega$. Suppose

$$||x|| = \max\left\{ \sup_{n\in N_0} \frac{|x(n)|}{1+n}, \max_{n\in[0,N+1]} |\Delta x(n)| \right\} \le M_1$$

and Step 1 implies that there exist constants $M_2 > 0$ such that $|A_x|, |B_x| < M_2$ for all $x \in \Omega$. Then

$$\begin{aligned}
&\frac{|(Tx)(n)|}{1+n} \\
= \quad & \frac{1}{1+n}\left|B_x + \sum_{i=0}^{n-1}\phi^{-1}\left(\phi(A_x) + \sum_{j=i}^{\infty} f(j, x(j), \Delta x(j))\right)\right| \\
\leq \quad & \frac{M_2}{1+n} + \frac{1}{1+n}\sum_{i=0}^{n-1}\phi^{-1}\left(\phi(M_2) + \sum_{j=i}^{\infty}|f(j, x(j), \Delta x(j))|\right) \\
\leq \quad & M_2 + \frac{1}{1+n}\sum_{i=0}^{n-1}\phi^{-1}\left(\phi(M_2) + \sum_{j=0}^{\infty} f_{M_1}(j)\right) \\
\leq \quad & M_2 + \phi^{-1}\left(\phi(M_2) + \sum_{j=0}^{\infty} f_{M_1}(j)\right) \\
=: \quad & M_3,
\end{aligned}$$

where $f_{M_1}(j) = \max_{|x| \leq M_1, |y| \leq M_1 \leq M_1} |f(j, (1+j)x, y)|$. Similarly, one has that

$$\begin{aligned}
|\Delta(Tx)(n)| \quad &= \quad \left|\phi^{-1}\left(\phi(A_x) + \sum_{j=n}^{\infty} f(j, x(j), \Delta x(j))\right)\right| \\
&\leq \quad \phi^{-1}\left(\phi(M_2) + \sum_{j=0}^{\infty} f_{M_1}(j)\right) =: M_4.
\end{aligned}$$

It follows that $T\Omega$ is bounded.

Step 4. For each bounded subset $\Omega \subset P$, prove that $T\Omega$ is pre-compact.

Since $|A_x| \leq M_2$, we get

$$\phi(A_x) + \sum_{n=0}^{\infty} f(n, x(n), \Delta x(n)) \leq \phi(M_4).$$

Then there $\xi \in \left[\phi(A_x), \phi(A_x) + \sum_{j=n}^{\infty} f(j, x(j), \Delta x(j))\right]$ such that

$$\begin{aligned}
|\Delta(Tx)(n) - A_x| &= \left|\phi^{-1}\left(\phi(A_x) + \sum_{j=n}^{\infty} f(j, x(j), \Delta x(j))\right) - A_x\right| \\
&= (q-1)\xi^{q-2} \sum_{j=n}^{\infty} f(j, x(j), \Delta x(j)) \\
&\leq (q-1)\phi(M_4)^{q-2} \sum_{j=n}^{\infty} f(j, x(j), \Delta x(j)) \\
&\rightarrow 0 \text{ uniformly as } n \rightarrow \infty.
\end{aligned}$$

For any $\epsilon > 0$, there exists $N_{1,\epsilon} > 0$ such that

$$|\Delta(Tx)(n_1) - \Delta(Tx)(n_2)| < \epsilon, \quad n > N_{1,\epsilon}. \tag{2.16}$$

Now, noting that f is S-Caratheodory function, one sees that there exists

$$\xi_i \in \left[\phi(A_x), \phi(A_x) + \sum_{j=i}^{\infty} |f(j, x(j), \Delta x(j))|\right]$$

such that

$$\begin{aligned}
&\left|\frac{(Tx)(n)}{1+n} - A_x\right| \\
= & \left|\frac{B_x}{1+n} + \frac{\sum_{i=0}^{n-1} \phi^{-1}\left(\phi(A_x) + \sum_{j=i}^{\infty} |f(j, x(j), \Delta x(j))|\right)}{1+n} - A_x\right| \\
\leq & \frac{|B_x|}{1+n} + \frac{|A_x|}{1+n} \\
& + \frac{\left|\sum_{i=0}^{n-1} \phi^{-1}\left(\phi(A_x) + \sum_{j=i}^{\infty} |f(j, x(j), \Delta x(j))|\right) - A_x n\right|}{1+n} \\
\leq & \frac{2M_2}{1+n} + \frac{\left|\sum_{i=0}^{n-1} \left[\phi^{-1}\left(\phi(A_x) + \sum_{j=i}^{\infty} |f(j, x(j), \Delta x(j))|\right) - A_x\right]\right|}{1+n} \\
= & \frac{2M_2}{1+n} + \frac{(q-1) \sum_{i=0}^{n-1} \xi_i^{q-2} \sum_{j=i}^{\infty} f(j, x(j), \Delta x(j))}{1+n} \\
\leq & \frac{2M_2}{1+n} + \frac{(q-1)\phi(M_4)^{q-2} \sum_{i=0}^{n-1} \sum_{j=i}^{\infty} f_{M_1}(j)}{1+n}
\end{aligned}$$

$$\begin{aligned} &\leq \frac{2M_2}{1+n} + \frac{(q-1)\phi(M_4)^{q-2}\sum_{i=0}^{\infty}\sum_{j=i}^{\infty} f_{M_1}(j)}{1+n} \\ &\to 0 \text{ uniformly as } n \to \infty. \end{aligned}$$

So there exists $N_{2,\epsilon} > 0$ such that

$$\left|\frac{(Tx)(n_1)}{1+n_1} - \frac{(Tx)(n_2)}{1+n_2}\right| < \epsilon, \quad n > N_{2,\epsilon}. \tag{2.17}$$

Choose $N_\epsilon = \max\{N_{1,\epsilon}, N_{2,\epsilon}\}$. Then

$$|\Delta(Tx)(n_1) - \Delta(Tx)(n_2)| < \epsilon, \left|\frac{(Tx)(n_1)}{1+n_1} - \frac{(Tx)(n_2)}{1+n_2}\right| < \epsilon, \quad n > N_\epsilon.$$

One knows that $T\Omega$ is pre-compact. Lemma 2.6 with Steps 1, 2, 3 and 4 imply that T is completely continuous. □

Theorem L. Suppose that there exist positive constants e_1, e_2, c such that

$$c \geq 2(1+k_2)e_2 > e_2 > e_1 > 0.$$

Let

$$\begin{aligned} Q &= \phi\left(\frac{c}{M}\right)\left(1 - \sum_{n=1}^{\infty}\beta_n\right); \\ W &= \frac{1}{2^{k_1+2} - 2^{2k_1-k_2-1}}\phi\left(\frac{(1+k_2)e_2}{k_1}\right); \\ E &= \left(1 - \sum_{n=1}^{\infty}\beta_n\right)\phi\left(\frac{e_1\left(1 - \sum_{n=1}^{\infty}\alpha_n\right)}{1 - \sum_{n=1}^{\infty}\alpha_n + \sum_{n=1}^{\infty}n\alpha_n}\right). \end{aligned}$$

If $Q > W$ and

(A1) $f(n, (1+n)u, v) \leq \frac{Q}{2^{n+1}}$ for all $n \in N_0, u \in [0, c], v \in [0, c]$;

(A2) $f(n, (1+n)u, v) \geq \frac{w}{2^{n+1}}$ for all $n \in [k_1, k_2], u \in [e_2, 2(1+k_2)e_2], v \in [0, c]$;

(A3) $f(n, (1+n)u, v) \leq \frac{E}{2^{n+1}}$ for all $n \in N_0, u \in [0, e_1], v \in [0, c]$;

then BVP(1.1) has at least three positive solutions x_1, x_2, x_3 such that

$$\sup_{n \in N_0} \frac{x_1(n)}{1+n} < e_1, \quad \min_{n \in [k_1,k_2]} \frac{x_2(n)}{1+n} > e_2,$$

and

$$\sup_{n \in N_0} \frac{x_3(n)}{1+n} > e_1, \quad \min_{n \in [k_1,k_2]} \frac{x_3(n)}{1+n} < e_2.$$

Proof. To apply Lemma 2.4, we prove that all conditions in Lemma 2.4 are satisfied. By the definitions, it is easy to see that α, ψ are two nonnegative continuous concave functionals on the cone P, γ, β, θ are three nonnegative continuous convex functionals on the cone P.

One sees $\alpha(x) \leq \beta(x)$ for all $x \in P$. From Lemma 2.5, we have $||x|| \leq M\gamma(x)$ for all $x \in P$.

Lemma 2.7 implies that $x = x(n)$ is a solution of BVP(1) if and only if x is a solution of the operator equation $x = Tx$ and $T : P \to P$ is completely continuous.

Corresponding to Lemma 2.4,

$$h = \frac{e_1}{2(1+k_2)}, \quad d = e_1, \quad a = e_2, \quad b = 2(1+k_2)e_2, \quad c = c.$$

Now, we prove that all conditions of Lemma 2.4 hold. One sees that $0 < d < a$. The remainder is divided into four steps.

Step 1. Prove that $T : \overline{P_c} \to \overline{P_c}$;

For $x \in \overline{P_c}$, we have $||x|| \leq c$. Then $0 \leq \frac{x(n)}{1+n} \leq c$ and $0 \leq \Delta x(n) \leq c$ for $n \in N_0$. So (A1) implies that

$$f(n, x(n), \Delta x(n)) = f\left(n, (1+n)\frac{x(n)}{1+n}, \Delta x(n)\right) \leq \frac{Q}{2^{n+1}}, \ n \in N_0.$$

It follows from Lemma 2.7 that $Tx \in P$. One sees from Lemma 2.2 that

$$0 \leq A_x \leq \phi^{-1}\left(\frac{\sum_{n=1}^{\infty}\beta_n}{1-\sum_{n=1}^{\infty}\beta_n}\sum_{j=0}^{\infty} f(j, x(j), \Delta x(j))\right). \tag{2.18}$$

We have that

$$\begin{aligned} |\Delta(Tx)(n)| &= \left|\phi^{-1}\left(\phi(A_x) + \sum_{j=n}^{\infty} f(j, x(j), \Delta x(j))\right)\right| \\ &\leq \phi^{-1}\left(\phi(A_x) + \sum_{j=n}^{\infty} f(j, x(j), \Delta x(j))\right) \end{aligned}$$

$$
\begin{aligned}
&\leq \phi^{-1}\left(\frac{\sum_{n=1}^{\infty}\beta_n}{1-\sum_{n=1}^{\infty}\beta_n}\sum_{j=0}^{\infty}f(j,x(j),\Delta x(j))\right.\\
&\qquad\qquad\left.+\sum_{j=n}^{\infty}f(j,x(j),\Delta x(j))\right)\\
&\leq \phi^{-1}\left(\frac{1}{1-\sum_{n=1}^{\infty}\beta_n}\sum_{j=0}^{\infty}\frac{Q}{2^{j+1}}\right)\\
&\leq \phi^{-1}\left(\frac{Q}{1-\sum_{n=0}^{\infty}\beta_n}\right)\leq c.
\end{aligned}
$$

From Lemma 2.5, we have

$$
\frac{|(Tx)(n)}{1+n}\leq M\sup_{n\in N_0}|\Delta(Tx)(n)|\leq M\phi^{-1}\left(\frac{Q}{1-\sum_{n=0}^{\infty}\beta_n}\right)\leq c.
$$

It follows that $||Tx|| = \max\left\{\max_{n\in N_0}\frac{|(Tx)(n)|}{1+n},\max_{n\in N_0}|\Delta(Tx)(n)|\right\}$ $\leq c$. Then $T:\ \overline{P_c}\to\overline{P_c}$.

Step 2. Prove that

$$
\begin{aligned}
&\{y\in P(\gamma,\theta,\alpha;a,b,c)|\alpha(x)>a\}\\
&=\{y\in P\left(\gamma,\theta,\alpha;e_2,2(1+k_2)e_2,c\right)|\alpha(x)>e_2\}\neq\emptyset
\end{aligned}
$$

and $\alpha(Tx)>e_2$ for every $x\in P\left(\gamma,\theta,\alpha;e_2,2(1+k_2)e_2,c\right)$;

Choose

$$
x(n)=\begin{cases} 2(1+k_2)\sum_{n=1}^{\infty}\alpha_n, n=0,\\ 2(1+k_2), n\geq 1.\end{cases}
$$

Then $x\in P$ and

$$
\alpha(x)=\min_{n\in[k_1,k_2]}\frac{x(n)}{1+n}=2e_2>e_2,
$$

$$
\theta(x)=\sup_{n\in N_0}\frac{x(n)}{1+n}\leq 2(1+k_2)e_2\leq b,
$$

and

$$
\gamma(x)=\sup_{n\in N_0}|\Delta x(n)|<c.
$$

It follows that $\{y\in P(\gamma,\theta,\alpha;a,b,c)|\alpha(x)>a\}\neq\emptyset$.

For $x\in P(\gamma,\theta,\alpha;a,b,c)$, one has that

$$
\alpha(x)=\min_{n\in[k_1,k_2]}\frac{x(n)}{1+n}\geq e_2,
$$

$$\theta(x) = \sup_{n\in N_0} \frac{x(n)}{1+n} \leq 2(1+k_2)e_2,$$

and

$$\gamma(x) = \sup_{n\in N_0} |\Delta x(n)| \leq c.$$

Then

$$e_2 \leq \frac{x(n)}{1+n} \leq 2(1+k_2)e_2,\ n\in[k_1,k_2],\ \ 0\leq \Delta x(n)\leq c.$$

Thus (A2) implies that

$$f(n, x(n+1), \Delta x(n), \Delta x(n+1)) \geq \frac{W}{2^{n+1}},\ \ n\in[k_1,k_2].$$

We get

$$\begin{aligned}
\alpha(Tx) &= \min_{n\in[k_1,k_2]} \frac{(Tx)(n)}{1+n} \\
&\geq \frac{(Tx)(k_1)}{1+k_2} \\
&= \frac{1}{1+k_2}\left(B_x + \sum_{k=0}^{k_1-1}\phi^{-1}\left(\phi(A_x)+\sum_{s=k}^{\infty} f(s,x(s),\Delta x(s))\right)\right) \\
&\geq \frac{1}{1+k_2}\sum_{k=0}^{k_1-1}\phi^{-1}\left(\sum_{s=k_1}^{k_2} f(s,x(s),\Delta x(s))\right) \\
&\geq \frac{1}{1+k_2}\sum_{k=0}^{k_1-1}\phi^{-1}\left(\sum_{s=k_1}^{k_2} \frac{W}{2^{s+1}}\right) \\
&\geq \frac{k_1}{1+k_2}\phi^{-1}\left(2^{k_1+2}-2^{2k_1-k_2-1}\right)\phi^{-1}(W) \\
&= e_2.
\end{aligned}$$

This completes Step 2.

Step 3. Prove that

$$\begin{aligned}
&\{y \in Q(\gamma,\theta,\psi;h,d,c) | \beta(x) < d\} \\
&\quad = \left\{y\in Q\left(\gamma,\theta,\psi;\frac{e_1}{2(1+k_2)},e_1,c\right) |\beta(x) < e_1\right\} \neq \emptyset
\end{aligned}$$

and

$$\beta(Tx) < e_1$$

$$\text{for every } x \in Q(\gamma,\theta,\psi;h,d,c) = Q\left(\gamma,\theta,\psi;\frac{e_1}{2(1+k_2)},e_1,c\right);$$

Choose

$$x(n) = \begin{cases} e_1\sum_{n=1}^{\infty}\alpha_n, n = 0, \\ e_1, n \geq 1. \end{cases}$$

Then $x \in P$, and

$$\psi(x) = \min_{n\in[k_1,k_2]}\frac{x(n}{1+n} = \frac{e_1}{1+k_2} \geq h,$$

$$\beta(x) = \theta(x) = \sup_{n\in N_0}\frac{x(n)}{1+n} < e_1 = d,$$

and

$$\gamma(x) = \sup_{n\in N_0}|\Delta x(n)| \leq c.$$

It follows that $\{y \in Q(\gamma,\theta,\psi;h,d,c)|\beta(x) < d\} \neq \emptyset$.

For $x \in Q(\gamma,\theta,\psi;h,d,c)$, one has that

$$\psi(x) = \min_{n\in[k_1,k_2]}\frac{x(n)}{1+n} \geq \frac{e_1}{2(1+k_2},$$

$$\theta(x) = \sup_{n\in N_0}\frac{x(n)}{1+n} \leq d = e_1,$$

and

$$\gamma(x) = \sup_{n\in N_0}|\Delta x(n)| \leq c.$$

Hence we get that

$$0 \leq \frac{x(n)}{1+n} \leq e_1,\ n \in N_0;\ \ 0 \leq \Delta x(n) \leq c,\ n \in N_0.$$

Then (A3) implies that

$$f(n,x(n+1),\Delta x(n),\Delta x(n+1)) \leq \frac{E}{2^{n+1}},\ \ n \in N_0.$$

So (19) implies that

$$\begin{aligned}
&\beta(Tx)\\
= &\sup_{n\in N_0}\frac{(Tx)(n)}{1+n}\\
= &\sup_{n\in N_0}\left(\frac{B_x}{1+n}+\frac{1}{1+n}+\sum_{i=0}^{n-1}\phi^{-1}\left(\phi(A_x)+\sum_{j=i}^{\infty}f(j,x(j),\Delta x(j))\right)\right)\\
\le &\sup_{n\in N_0}\left[\frac{\frac{1}{1-\sum_{n=1}^{\infty}\alpha_n}\sum_{n=1}^{\infty}\alpha_n\sum_{k=0}^{n-1}\phi^{-1}\left(\phi(A_x)+\sum_{s=k}^{\infty}f(s,x(s),\Delta x(s))\right)}{1+n}\right.\\
&\left.+\frac{1}{1+n}\sum_{k=0}^{n-1}\phi^{-1}\left(\phi(A_x)+\sum_{s=k}^{\infty}f(s,x(s),\Delta x(s))\right)\right]\\
\le &\sup_{n\in N_0}\left[\frac{\frac{1}{1-\sum_{n=1}^{\infty}\alpha_n}\sum_{n=1}^{\infty}\alpha_n\sum_{k=0}^{n-1}\phi^{-1}\left(\frac{1}{1-\sum_{n=1}^{\infty}\beta_n}\sum_{j=0}^{\infty}f(j,x(j),\Delta x(j))\right)}{1+n}\right.\\
&\left.+\frac{1}{1+n}\sum_{k=0}^{n-1}\phi^{-1}\left(\frac{1}{1-\sum_{n=1}^{\infty}\beta_n}\sum_{j=0}^{\infty}f(j,x(j),\Delta x(j))\right)\right]\\
\le &\sup_{n\in N_0}\left[\frac{\frac{1}{1-\sum_{n=1}^{\infty}\alpha_n}\sum_{n=1}^{\infty}\alpha_n\sum_{k=0}^{n-1}\phi^{-1}\left(\frac{1}{1-\sum_{n=1}^{\infty}\beta_n}\sum_{j=0}^{\infty}\frac{E}{2^{j+1}}\right)}{1+n}\right.\\
&\left.+\frac{1}{1+n}\sum_{k=0}^{n-1}\phi^{-1}\left(\frac{1}{1-\sum_{n=1}^{\infty}\beta_n}\sum_{j=0}^{\infty}\frac{E}{2^{j+1}}\right)\right]\\
< &\left[1+\frac{\sum_{n=1}^{\infty}n\alpha_n}{1-\sum_{n=1}^{\infty}\alpha_n}\right]\phi^{-1}\left(\frac{1}{1-\sum_{n=1}^{\infty}}\right)\phi^{-1}(E)\\
\le &e_1=d.
\end{aligned}$$

This completes Step 3.

Step 4. Prove that $\alpha(Ty)>a$ for $y\in P(\gamma,\alpha;a,c)$ with $\theta(Ty)>b$;

For $x\in P(\gamma,\alpha;a,c)=P(\gamma,\alpha;e_2,c)$ with $\theta(Tx)=\beta(Tx)>b=2(1+k_2)e_2$, we have that $\alpha(x)=\min_{n\in[k_1,k_2]}\frac{x(n)}{1+n}\ge e_2$ and $\gamma(x)=\sup_{n\in N_0}|\Delta x(n)|\le c$ and $\sup_{n\in N_0}\frac{(Tx)(n)}{1+n}>2(1+k_2)e_2$. Then

$$\alpha(Tx)=\min_{n\in[k_1,k_2]}\frac{(Tx)(n)}{1+n}\ge\frac{1}{2(1+k_2)}\beta(Tx)>e_2=a.$$

This completes Step 4.

Step 5. Prove that $\beta(Tx)<d$ for each $x\in Q(\gamma,\beta;d,c)$ with $\psi(Tx)<h$.

For $x\in Q(\gamma,\beta;d,c)$ with $\psi(Tx)<d$, we have $\gamma(x)=\sup_{n\in N_0}|\Delta x(n)|\le c$ and $\beta(x)=\sup_{n\in N_0}\frac{x(n)}{1+n}\le d=e_1$ and $\psi(Tx)=\min_{n\in[k_1,k_2]}\frac{(Tx)(n)}{1+n}$

$< h = \frac{e_1}{2(1+k_2)}$. Then

$$\beta(Tx) = \sup_{n\in N_0} \frac{(Tx)(n)}{1+n} \le 2(1+k_2) \min_{n\in[k_1,k_2]} \frac{(Tx)(n)}{1+n} < e_1 = d.$$

This completes the Step 5.

Then Lemma 2.4 implies that T has at least three fixed points y_1, y_2 and y_3 such that

$$\beta(y_1) < e_1,\ \alpha(y_2) > e_2,\ \beta(y_3) > e_1,\ \alpha(y_3) < e_2.$$

Hence BVP(1.1) has three positive solutions y_1, y_2 and y_3 such that

$$\sup_{n\in N_0} \frac{x_1(n)}{1+n} < e_1,\quad \min_{n\in[k_1,k_2]} \frac{x_2(n)}{1+n} > e_2,$$

and

$$\sup_{n\in N_0} \frac{x_3(n)}{1+n} > e_1,\quad \min_{n\in[k_1,k_2]} \frac{x_3(n)}{1+n} < e_2.$$

The proof is complete. $\square$

3. An example

In this section, we present an example to illustrate the main result.

Example 3.1. Consider the following BVP

$$\left\{\begin{array}{l} \Delta^2 x(n) + f(n, x(n), \Delta x(n),) = 0, \quad n \in N_0, \\ x(0) = \sum_{n=1}^{\infty} \frac{1}{n2^{n+1}} x(n), \\ \lim_{n\to\infty} \Delta x(n) = \sum_{n=1}^{\infty} \frac{1}{2^{n+1}} \Delta x(n), \end{array}\right. \tag{3.1}$$

where $f(n,x,y)$ is a nonnegative S-Caratheodory function which is defined by

$$f(n,x,y) = \frac{1}{2^{n+1}} f_0\left(\frac{x}{1+n}\right) + \frac{y}{10^{30000n}}$$

with $f_0(x)$ satisfying

$$f_0(x) = \left\{\begin{array}{ll} \frac{\ln 2}{12-4\ln 2} x, & x \in [0,100], \\ \frac{872100 + \frac{52\times 1350}{2^6-2^{-96}} - \frac{25\ln 2}{3-\ln 2}}{5400-100}(x-100) + \frac{25\ln 2}{3-\ln 2}, & x \in [100, 5400], \\ 872100 + \frac{52\times 1350}{2^6-2^{-96}}, & x \in [5400, 3488400], \\ 872100 + \frac{52\times 1350}{2^6-2^{-96}} e^{x-3488400}, & x \ge 3488400. \end{array}\right.$$

Then

$$f(n,(1+n)x,u) = \frac{1}{2^{n+1}} f_0(x) + \frac{y}{10^{30000n}}.$$

It is easy to show that f is a S-Caratheodory function.

Corresponding to BVP(1.1), we have $\phi(x) = \phi^{-1}(x) = x$, $\alpha_n = \frac{1}{n2^{n+1}}$ and $\beta_n = \frac{1}{2^{n+1}}$. One sees

$$0 < \sum_{n=1}^{\infty} \alpha_n < 1, \ \ 0 < \sum_{n=1}^{\infty} n\alpha_n < \infty, \ \ 0 < \sum_{n=1}^{\infty} \beta_n < 1.$$

Choose the constant $k_1 = 4$ and $k_2 = 103$, $e_1 = 100, e_2 = 5400, c = 3488400$, then

$$c \geq 2(1+k_2)e_2 > e_2 > e_1 > 0.$$

One sees that

$$\begin{aligned}
M &= \max\left\{1, \frac{\sum_{n=1}^{\infty} n\alpha_n}{1 - \sum_{n=1}^{\infty} \alpha_n}\right\} = 1,\\
Q &= \phi\left(\frac{c}{M}\right)\left(1 - \sum_{n=1}^{\infty} \beta_n\right) = 1744200;\\
W &= \frac{1}{2^{k_1+2} - 2^{2k_1-k_2-1}}\phi\left(\frac{(1+k_2)e_2}{k_1}\right) = \frac{104 \times 1350}{2^6 - 2^{-96}};\\
E &= \left(1 - \sum_{n=1}^{\infty} \beta_n\right)\phi\left(\frac{e_1\left(1 - \sum_{n=1}^{\infty} \alpha_n\right)}{1 - \sum_{n=1}^{\infty} \alpha_n + \sum_{n=1}^{\infty} n\alpha_n}\right) = \frac{50 \ln 2}{3 - \ln 2}.
\end{aligned}$$

Hence $Q > W$. If

(A_1) $f(n, (1+n)x, u) \leq \frac{1744200}{2^{n+1}}$ for all $n \in N_0, x \in [0, 3488400], u \in [0, 3488400]$;

(A_2) $f(n, (1+n)x, u) \geq \frac{104 \times 1350}{2^6 - 2^{-96}} \frac{1}{2^{n+1}}$ for all $n \in [4, 103], x \in [5400, 208 \times 5400], u \in [0, 3488400]$;

(A_3) $f(n, (1+n)x, u) \leq \frac{50 \ln 2}{3 - \ln 2} \frac{1}{2^{n+1}}$ for all $n \in N_0, x \in [0, 100], u \in [0, 3488400]$;

then Theorem L implies that BVP(3.1) has at least three positive solutions x_1, x_2, x_3 such that

$$\sup_{n \in N_)} \frac{x_1(n)}{1+n} < 100, \ \ \min_{n \in [4,103]} \frac{x_2(n)}{1+n} > 5400,$$

and

$$\sup_{n \in N_0} \frac{x_3(n)}{1+n} > 100, \ \ \min_{n \in [4,103]} \frac{x_3(n)}{1+n} < 5400.$$

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