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DIFFEOMORPHISMS WITH ROBUSTLY AVERAGE SHADOWING

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Abstract. In this paper, we prove that for C^1 generically, if every hyperbolic periodic point in a chain component is uniformly far away from being nonhyperbolic, and it is C^1-robustly average shadowable, then the chain component is hyperbolic.

1. Introduction

Let M be a closed manifold, and let $\text{Diff}(M)$ be the space of diffeomorphisms of M endowed with the C^1-topology. Denote by d the distance on M induced from a Riemannian metric $\|\cdot\|$ on the tangent bundle TM. Let $f \in \text{Diff}(M)$.

For $\delta > 0$ a sequence $\{x_i\}_{i=-\infty}^{\infty}$ in M is called a *δ-average pseudo orbit* of $f \in \text{Diff}(M)$ if there is a natural number $N = N(\delta) > 0$ such that for all $n \geq N$, and $k \in \mathbb{Z}$,

$$\frac{1}{n}\sum_{i=1}^{n-1} d(f(x_{i+k}), x_{i+k+1}) < \delta.$$

We say that f has the *average- shadowing property* if for every $\epsilon > 0$ there is a $\delta > 0$ such that every δ-average pseudo orbit $\{x_i\}_{i=-\infty}^{\infty}$ is ϵ-shadowed in average by some $z \in M$, that is,

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$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} d(f^i(z), x_i) < \epsilon.$$

Let $\Lambda \subset M$ be a closed f-invariant set. We say that $f|_\Lambda$ has the average shadowing property if for every $\epsilon > 0$ there is $\delta > 0$ such that for any δ-average pseudo orbit $\{x_i\}_{i=a}^{b} \subset \Lambda(-\infty \leq a < b \leq \infty)$ of f, ϵ-shadowed in average by some $z \in \Lambda$.
For $\delta > 0$, a sequence of points $\{x_i\}_{i=a}^{b}(-\infty \leq a < b \leq \infty)$ in M is called a *δ-pseudo orbit* of f if $d(f(x_i), x_{i+1}) < \delta$ for all $a \leq i \leq b-1$. We say that f has the shadowing property if for any $\epsilon > 0$ there is $\delta > 0$ such that for any δ-pseudo orbit $\{x_i\}_{i\in\mathbb{Z}}$ of f there is $y \in M$ such that $d(f^i(y), x_i) < \epsilon$, for $i \in \mathbb{Z}$. For given $x, y \in M$, we write $x \rightsquigarrow y$ if for any $\delta > 0$, there is a δ-pseudo orbit $\{x_i\}_{i=a}^{b}(a < b)$ of f such that $x_a = x$ and $x_b = y$. The set of points $\{x \in M : x \leftrightsquigarrow x\}$ is called the *chain recurrent set* of f and is denoted by $\mathcal{R}(f)$. The relation $\leftrightsquigarrow$ induces on $\mathcal{R}(f)$ an equivalence relation, whose classes are called *chain components* of f. Denote by $C_f(p) = \{x \in M : x \rightsquigarrow p \text{ and } p \rightsquigarrow x\}$, where p is a hyperbolic saddle. Note that if p is not hyperbolic saddle then $C_f(p) = p$. Thus in this paper, we consider that p is a hyperbolic saddle.

It is well known that if p is a hyperbolic periodic point f with period k then the sets

$$W^s(p) = \{x \in M : f^{kn}(x) \to p \text{ as } n \to \infty\} \text{ and}$$

$$W^u(p) = \{x \in M : f^{-kn}(x) \to p \text{ as } n \to \infty\}$$

are C^1-injectively immersed submanifolds of M. Every point $x \in W^s(p)$ $\overline{\pitchfork} W^u(p)$ is called a *homoclinic point* of f. The closure of the homoclinic points of f associated to p is called the *homoclinic class* of f and it is denoted by $H_f(p)$.

Let $\Lambda \subset M$ be a f-invariant closed set. We say that Λ admits a *dominated splitting* if the tangent bundle $T_\Lambda M$ has a continuous Df-invariant splitting $E \oplus F$ and there exist constants $C > 0$ and $0 < \lambda < 1$ such that

$$\|D_x f^n|_{E(x)}\| \cdot \|D_x f^{-n}|_{F(f^n(x))}\| \leq C\lambda^n$$

for all $x \in \Lambda$ and $n \geq 0$.

We say that Λ is *hyperbolic* if the tangent bundle $T_\Lambda M$ has a Df-invariant splitting $E^s \oplus E^u$ and there exists constants $C > 0$ and $0 < \lambda < 1$ such that

$$\|D_x f^n|_{E^s_x}\| \leq C\lambda^n \ \text{ and } \ \|D_x f^{-n}|_{E^u_x}\| \leq C\lambda^{-n}$$

for all $x \in \Lambda$ and $n \geq 0$.

We say that a subset $\mathcal{G} \subset \mathrm{Diff}(M)$ is *residual* if $\mathcal{G}$ contains the intersection of a countable family of open and dense subsets of $\mathrm{Diff}(M)$, and so $\mathcal{G}$ is dense in $\mathrm{Diff}(M)$. A property $\mathcal{P}$ is said to be *generic* (C^1) if $\mathcal{P}$ holds for all diffeomorphisms which belong to some residual subset of $\mathrm{Diff}(M)$.

In this paper, we introduce the notion of C^1-robustly average shadowing property, and study the cases when the chain component $C_f(p)$ of f associated to a hyperbolic periodic point p has the C^1-robustly average shadowing property.

Now, we introduce the C^1-interior average shadowing property.

DEFINITION 1.1. Let $C_f(p)$ be the chain component having hyperbolic a hyperbolic periodic point p. We say that $C_f(p)$ has the C^1-*robustly average shadowing property* if there exists a C^1-neighborhood $\mathcal{U}(f)$ of f such that for any $g \in \mathcal{U}(f)$, $g|_{C_g(p_g)}$ has the average shadowing property, where p_g is called the *continuation* of p.

Let p be a hyperbolic periodic point of f. In [1], Bonatti and Crovisier proved that C^1 generically, a chain component $C_f(p)$ containing a periodic point p is the homoclinic class $H_f(p)$. Thus we get the the following Theorem.

The following fact is the main Theorem in this paper.

Theorem A. *For C^1-generic f, assume the chain component $C_f(p)$ is locally maximal and satisfies the following properties:*

(a) *for any $g \in \mathcal{U}_0(f)$, if $q \in C_g(p_g) \cap P(g)$ has minimum period $\pi(q) \geq m$, then*

$$\prod_{i=0}^{k-1} \|D_{g^{im}(q)} g^m|_{E^s|_{g^{im}(q)}}\| < K\lambda^k \text{ and}$$

$$\prod_{i=0}^{k-1} \|D_{g^{-im}(q)} g^m|_{E^u|_{g^{-im}(q)}}\| < K\lambda^k,$$

where $k = [\pi(q)/m]$, and p_g is the continuation of p.

(b) *$C_f(p)$ admits a dominated splitting $T_{C_f(p)}M = E \oplus F$ with $\dim E = \mathrm{index}(p)$, for any $x \in C_f(p)$,*

$$\|Df|_{E(x)}\|/m(Df|_{F(x)}) < \lambda^2.$$

(c) *f has the average shadowing property on $C_f(p)$.*

Then $C_f(p)$ is hyperbolic for f.

2. Proof of the Theorem A

Let M be as before, and let $f \in \text{Diff}(M)$.

PROPOSITION 2.1 ([5]). *Let $\lambda \in (0,1)$ be given. Let Λ be a compact f-invariant set with a continuous Df-invariant spitting $T_\Lambda M = E \oplus F$ such that $\|Df|_{E(x)}\|/m(Df|_{F(x)}) < \lambda^2$, for any $x \in \Lambda$. Suppose there is a point $y \in \Lambda$ satisfying*

$$\log \lambda < \log \lambda_1 = \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log(\|Df|_{E(f^i(y))}\|) < 0,$$

and

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log(\|Df|_{E(f^i(y))}\|) < \log \lambda_1.$$

Then for any λ_2 and λ_3 with $\lambda < \lambda_2 < \lambda_1 < \lambda_3 < 1$, and any neighborhood U of Λ, there exists a hyperbolic periodic point q of index $\dim(E)$ such that its orbit $\mathcal{O}(q)$ is entirely contained in U and the derivatives along $\mathcal{O}(q)$ satisfy

$$\prod_{i=0}^{k-1} \|Df|_{E^s(f^i(q))}\| < \lambda_3^k,$$
$$\prod_{i=k-1}^{\pi(q)-1} \|Df|_{E^s(f^i(q))}\| > \lambda_2^{\pi(q)-k+1}$$

for all $k = 1, 2, \cdots, \pi(q)$. Furthermore, q can be chosen such that $\pi(q)$ is arbitrarily large.

LEMMA 2.2 ([3]). *Let $\Lambda \subset M$ be a closed f-invariant set and $\varphi(x)$ be a continuous function defined on Λ. For $\epsilon > 0$, there is $\delta > 0$ such that for sequences $\{x_i\}_{i\in\mathbb{N}}, \{y_i\}_{i\in\mathbb{N}}$ if*

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n-1} d(x_i, y_i) < \delta,$$

then

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n-1} \|\varphi(x_i) - \varphi(y_i)\| < \epsilon.$$

Lemma 2.3 ([3]). *Let $C_f(p)$ satisfy (a)-(c) of Theorem A. Suppose that E is not contracting. Then for any $\lambda < \gamma_1 < \gamma_2 < 1$, there exists $x \in C_f(p)$ with $\mathcal{O}(x)$ such that*

$$\liminf_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \log(\|Df|_{E(f^i(x))}\| < \log \gamma_1$$
$$< \limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \log(\|Df|_{E(f^i(x))}\|) < \log \gamma_2.$$

The following lemmas are well known results [5].

Lemma 2.4. *Assume Λ has a dominated splitting $E \oplus F$, and let $x \in U$ be a point such that $\mathcal{O}_f(x) \subset U$, where U is an admissible neighborhood of Λ. There is $\epsilon_1 > 0$ such that the local center unstable manifold $W^{cu}_{\epsilon_1}(x)$ is defined and is transverse to F. Such manifolds are of class C^1. Moreover, there is a $N > 0$ such that for any $0 < \epsilon_2 < \epsilon_1$, there exists $\delta > 0$ such that $f^N(W^{cs}_\delta(x)) \subset W^{cs}_{\epsilon_2}(f^N(x))$, and $f^{-N}(W^{cu}_\delta(x)) \subset W^{cu}_{\epsilon_2}(f^{-N}(x))$.*

Lemma 2.5. *Let $W^{cs}_{\epsilon_1}(x)$ and $W^{cu}_{\epsilon_1}(x)$ be as in Lemma2.4. For any $0 < \lambda < 1$, there is $\epsilon > 0$ such that:*

(a) *If $\prod_{i=0}^{n-1} \|Df|_{E(f^i(x))}\| \le \lambda^n$ for any $n \ge 1$, then for any $y \in W^{cs}_\epsilon(x)$, $d(f^n(x), f^n(y)) \to 0$ as $n \to \infty$.*
(b) *If $\prod_{i=0}^{n-1} \|Df^{-1}|_{E(f^{-i}(x))}\| \le \lambda^n$ for any $n \ge 1$, then for any $y \in W^{cu}_\epsilon(x)$, $d(f^n(x), f^n(y)) \to 0$ as $n \to -\infty$.*

Note that let Λ be a closed f-invariant set. Then if Λ is not hyperbolic then either E is not contracting or F is not expanding.

Lemma 2.6. *Let Λ be a closed f-invariant set and let $T_\Lambda M = E \oplus F$. Suppose that E is not contracting. Then there is $y \in H_f(p)$ such that*

$$\prod_{i=0}^{n-1} \|Df|_{E(f^i y)}\| \ge 1$$

for all $n \ge 1$.

Proof. Assume there is no $y \in H_f(p)$ such that

$$\prod_{i=0}^{n-1} \|Df|_{E(f^i y)}\| \ge 1$$

for all $n \geq 1$. Then for any $y \in H_f(p)$ there is $n(y) \in \mathbb{N}$ such that

$$\prod_{i=0}^{n(y)-1} \|Df|_{E(f^i y)}\| < 1.$$

Denote

$$\lambda(y) = \prod_{i=0}^{n(y)-1} \|Df|_{E(f^i y)}\|.$$

We can get a neighborhood $U(y)$ of y such that

$$\prod_{i=0}^{n(y)-1} \|Df|_{E(f^i x)}\| < \frac{1+\lambda(y)}{2} < 1.$$

for any $x \in U(y)$. By the compactness of $H_f(p)$, we can find $y_1, y_2, \cdots, y_k$ contained in $H_f(p)$ such that

$$H_f(p) \subset \bigcup_{i=1}^{k} U(y_k).$$

Let $N = \max\{n(y_1), n(y_2), \cdots, n(y_k)\}$, $\lambda = \max\{(\frac{1+\lambda(y_i)}{2})^{1/N} : i = 1, 2, \cdots, k\}$, $C = \lambda^{-N} \max\{\|Df^i\| : i = 0, 1, 2, \cdots, N-1\}$. Then one can check that

$$\|Df^n|_{E(x)}\| < C\lambda^n$$

for any $x \in H_f(p)$ and any $n \geq 1$. □

Proof of Proposition 2.3: Suppose that $C_f(p)$ is not hyperbolic. Then we may assume that E is not contracting. Choose a neighborhood $\{W_k\}_{k\geq 1}$ of $C_f(p)$ such that for $k \geq 1$, W_k is decreasing, $W_k \subset U$ and $\bigcap_{k\geq 1} W_k = C_f(p)$. By [1], C^1 generically $C_f(p) = H_f(p)$. Let $\varphi(x) = \log \|Df|_{E(f^i(x))}\|$, for $x \in M$. Then by Lemma 2.2 and Lemma 2.3, for any $\lambda < \gamma_1 < \gamma_2 < 1$, there is a point $x \in H_f(p)$ such that

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i(x)) < \log \gamma_1 < \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i(x)) < \log \gamma_2,$$

and for any k, we can choose a hyperbolic periodic point $q_k \in W_k$. Then we use Proposition 2.1, for any $\lambda < \gamma_1 < \gamma_2 < 1$, there is a periodic point q arbitrarily closed to $H_f(p)$ such that

$$\prod_{i=0}^{k-1} \|Df|_{E^s(f^i(q))}\| < \gamma_2^k, \tag{2.1}$$

$$\prod_{i=k-1}^{\pi(q)-1} \|Df|_{E^s(f^i(q))}\| > \gamma_1^{\pi(q)-k+1}. \tag{2.2}$$

Since a periodic point q is arbitrarily close to $H_f(p)$, we may assume q_k converge to a point $z \in H_f(p)$. Thus for every q_k, it satisfies the inequalities (2.1) and (2.2) By Lemma 2.4, there is $\epsilon > 0$ such that $W^{cs}_\epsilon(q_k) \subset W^s(q_k)$ and $W^{cu}_\epsilon(q_k) \subset W^u(q_k)$. This means that there exists $N \geq 1$ such that for any $k, m \geq N$, q_k and q_m are homoclinic related, i.e., $W^s(q_k) \pitchfork W^u(q_m)$ and $W^u(q_k) \pitchfork W^s(q_m)$. Therefore, $z \in H_f(\mathcal{O}(q_k))$ for all $k \geq N$. Thus $q_k \in H_f(p) = C_f(p)$ for all $k \geq N$. Thus, due to (2.2),

$$\prod_{i=k-1}^{\pi(q)-1} \|Df|_{E^s(f^i(q))}\| > \gamma_1^{\pi(q)-k+1}$$

of q which contradicts the hypothesis (a). This ends the proof of proposition. □

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