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CAUCHY–RASSIAS STABILITY OF A GENERALIZED ADDITIVE MAPPING IN BANACH MODULES AND ISOMORPHISMS IN C^*–ALGEBRAS

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Abstract. Let X, Y be vector spaces, and let r be 2 or 4. It is shown that if an odd mapping $f : X \to Y$ satisfies the functional equation

$$(\ddagger) \quad rf(\frac{\sum_{j=1}^d x_j}{r}) + \sum_{\substack{\iota(j)=0,1 \\ \sum_{j=1}^d \iota(j)=l}} rf(\frac{\sum_{j=1}^d (-1)^{\iota(j)} x_j}{r}) = ({}_{d-1}C_l - {}_{d-1}C_{l-1} + 1) \sum_{j=1}^d f(x_j)$$

then the odd mapping $f : X \to Y$ is additive, and we prove the Cauchy-Rassias stability of the functional equation (‡) in Banach modules over a unital C^*-algebra. As an application, we show that every almost linear bijection $h : \mathcal{A} \to \mathcal{B}$ of a unital C^*-algebra $\mathcal{A}$ onto a unital C^*-algebra $\mathcal{B}$ is a C^*-algebra isomorphism when $h(2^n uy) = h(2^n u)h(y)$ for all unitaries $u \in \mathcal{A}$, all $y \in \mathcal{A}$, and $n = 0, 1, 2, \cdots$.

1. Introduction

Let X and Y be Banach spaces with norms $||\cdot||$ and $\|\cdot\|$, respectively. Consider $f : X \to Y$ to be a mapping such that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$. Assume that there exist constants $\theta \geq 0$ and $p \in [0, 1)$ such that

$$\|f(x+y) - f(x) - f(y)\| \leq \theta(||x||^p + ||y||^p)$$

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for all $x, y \in X$. Rassias [5] showed that there exists a unique $\mathbb{R}$-linear mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \leq \frac{2\theta}{2 - 2^p}||x||^p$$

for all $x \in X$. Găvruta [1] generalized the Rassias' result: Let G be an abelian group and Y a Banach space. Denote by $\varphi : G \times G \to [0, \infty)$ a function such that

$$\widetilde{\varphi}(x, y) = \sum_{j=0}^{\infty} 2^{-j} \varphi(2^j x, 2^j y) < \infty$$

for all $x, y \in G$. Suppose that $f : G \to Y$ is a mapping satisfying

$$\|f(x + y) - f(x) - f(y)\| \leq \varphi(x, y)$$

for all $x, y \in G$. Then there exists a unique additive mapping $T : G \to Y$ such that

$$\|f(x) - T(x)\| \leq \frac{1}{2}\widetilde{\varphi}(x, x)$$

for all $x \in G$. C. Park [4] applied the Găvruta's result to linear functional equations in Banach modules over a C^*-algebra. Several functional equations have been investigated in [6]–[11].

Throughout this paper, assume that r is 2 or 4, and that d and l are integers with $d > 1$ and $1 < l < \frac{d}{2}$.

In this paper, we solve the following functional equation

$$rf\left(\frac{\sum_{j=1}^{d} x_j}{r}\right) + \sum_{\substack{\iota(j)=0,1 \\ \sum_{j=1}^{d} \iota(j)=l}} rf\left(\frac{\sum_{j=1}^{d} (-1)^{\iota(j)} x_j}{r}\right)$$

$$= ({}_{d-1}C_l - {}_{d-1}C_{l-1} + 1) \sum_{j=1}^{d} f(x_j). \tag{1.i}$$

We moreover prove the Cauchy-Rassias stability of the functional equation (1.i) in Banach modules over a unital C^*-algebra. These results are applied to investigate C^*-algebra isomorphisms in unital C^*-algebras.

2. An odd functional equation in d variables

Throughout this section, assume that X and Y are real linear spaces.

LEMMA 2.1. *If an odd mapping $f : X \to Y$ satisfies (1.i) for all $x_1, x_2, \cdots, x_d \in X$, then f is additive.*

Proof. Note that $f(0) = 0$ and $f(-x) = -f(x)$ for all $x \in X$ since f is an odd mapping. Putting $x_1 = x, x_2 = y$ and $x_3 = \cdots = x_d = 0$ in (1.i), we get

$$({}_{d-2}C_l - {}_{d-2}C_{l-2} + 1) r f(\frac{x+y}{r}) = ({}_{d-1}C_l - {}_{d-1}C_{l-1} + 1)(f(x) + f(y)) \tag{2.1}$$

for all $x, y \in X$. Since ${}_{d-2}C_l - {}_{d-2}C_{l-2} + 1 = {}_{d-1}C_l - {}_{d-1}C_{l-1} + 1$,

$$r f(\frac{x+y}{r}) = f(x) + f(y)$$

for all $x, y \in X$. Letting $y = 0$ in (2.1), we get $rf(\frac{x}{r}) = f(x)$ for all $x \in X$. Hence

$$f(x+y) = r f(\frac{x+y}{r}) = f(x) + f(y)$$

for all $x, y, \in X$. Thus f is additive. □

3. Stability of an odd functional equation in Banach modules over a C^*-algebra

Throughout this section, assume that A is a unital C^*-algebra with norm $|\cdot|$ and unitary group $\mathcal{U}(A)$, and that X and Y are left Banach modules over a unital C^*-algebra A with norms $||\cdot||$ and $\|\cdot\|$, respectively.

Given a mapping $f : X \to Y$, we set

$$\begin{aligned} D_u f(x_1, \cdots, x_d) :=& r f(\frac{\sum_{j=1}^d u x_j}{r}) + \sum_{\substack{\iota(j)=0,1 \\ \sum_{j=1}^d \iota(j)=l}} r f(\frac{\sum_{j=1}^d (-1)^{\iota(j)} u x_j}{r}) \\ & - ({}_{d-1}C_l - {}_{d-1}C_{l-1} + 1) \sum_{j=1}^d u f(x_j) \end{aligned}$$

for all $u \in \mathcal{U}(A)$ and all $x_1, \cdots, x_d \in X$.

THEOREM 3.1. *Let $r = 2$ and $d > 4$. Let $f : X \to Y$ be an odd mapping for which there is a function $\varphi : X^d \to [0, \infty)$ such that*

$$\widetilde{\varphi}(x_1, \cdots, x_d) := \sum_{j=0}^{\infty} \frac{1}{2^j} \varphi(2^j x_1, \cdots, 2^j x_d) < \infty, \tag{3.i}$$

$$\|D_u f(x_1, \cdots, x_d)\| \le \varphi(x_1, \cdots, x_d) \tag{3.ii}$$

for all $u \in \mathcal{U}(A)$ and all $x_1, \cdots, x_d \in X$. Then there exists a unique A-linear generalized additive mapping $L : X \to Y$ such that

$$\|f(x) - L(x)\| \le \frac{1}{4({}_{d-4}C_l - {}_{d-4}C_{l-4} + 1)} \widetilde{\varphi}(x, x, x, x, \underbrace{0, \cdots, 0}_{d-4 \text{ times}}) \tag{3.iii}$$

for all $x \in X$.

Proof. Note that $f(0) = 0$ and $f(-x) = -f(x)$ for all $x \in X$ since f is an odd mapping. Let $u = 1 \in \mathcal{U}(A)$. Putting $x_1 = x_2 = x_3 = x_4 = x$ and $x_5 = \cdots = x_d = 0$ in (3.ii), we have

$$\|f(2x) - 2f(x)\| \le \frac{1}{2({}_{d-4}C_l - {}_{d-4}C_{l-4} + 1)} \varphi(x, x, x, x, \underbrace{0, \cdots, 0}_{d-4 \text{ times}})$$

for all $x \in X$. So we get

$$\|f(x) - \frac{1}{2} f(2x)\| \le \frac{1}{4({}_{d-4}C_l - {}_{d-4}C_{l-4} + 1)} \varphi(x, x, x, x, \underbrace{0, \cdots, 0}_{d-4 \text{ times}})$$

for all $x \in X$. Hence

$$\|\frac{1}{2^n} f(2^n x) - \frac{1}{2^{n+1}} f(2^{n+1} x)\| \tag{3.1}$$

$$\le \frac{1}{2^{n+2}({}_{d-4}C_l - {}_{d-4}C_{l-4} + 1)} \varphi(2^n x, 2^n x, 2^n x, 2^n x, \underbrace{0, \cdots, 0}_{d-4 \text{ times}})$$

for all $x \in X$ and all positive integers n. By (3.1), we have

$$\|\frac{1}{2^m} f(2^m x) - \frac{1}{2^n} f(2^n x)\| \tag{3.2}$$

$$\leq \sum_{k=m}^{n-1} \frac{1}{2^{k+2}({}_{d-4}C_l - {}_{d-4}C_{l-4} + 1)} \varphi(2^k x, 2^k x, 2^k x, 2^k x, \underbrace{0, \cdots, 0}_{d-4 \text{ times}})$$

for all $x \in X$ and all positive integers m and n with $m < n$. This shows that the sequence $\{\frac{1}{2^n} f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{2^n} f(2^n x)\}$ converges for all $x \in X$. So we can define a mapping $L : X \to Y$ by

$$L(x) := \lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$$

for all $x \in X$. Since $f(-x) = -f(x)$ for all $x \in X$, we have $L(-x) = L(x)$ for all $x \in X$. Also, we get

$$\begin{aligned} \|D_1 L(x_1, \cdots, x_d)\| &= \lim_{n\to\infty} \frac{1}{2^n} \|D_1 f(2^n x_1, \cdots, 2^n x_d)\| \\ &\leq \lim_{n\to\infty} \frac{1}{2^n} \varphi(2^n x_1, \cdots, 2^n x_d) = 0 \end{aligned}$$

for all $x_1, \cdots, x_d \in X$. So L is a generalized additive mapping. Putting $m = 0$ and letting $n \to \infty$ in (3.2), we get (3.iii).

Now, let $L' : X \to Y$ be another generalized additive mapping satisfying (3.iii). By Lemma 2.1, L and L' are additive. So we have

$$\begin{aligned} &\|L(x) - L'(x)\| = \frac{1}{2^n} \|L(2^n x) - L'(2^n x)\| \\ &\leq \frac{1}{2^n} (\|L(2^n x) - f(2^n x)\| + \|L'(2^n x) - f(2^n x)\|) \\ &\leq \frac{2}{2^{n+2}({}_{d-4}C_l - {}_{d-4}C_{l-4} + 1)} \widetilde{\varphi}(2^n x, 2^n x, 2^n x, 2^n x, \underbrace{0, \cdots, 0}_{d-4 \text{ times}}), \end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in X$. So we can conclude that $L(x) = L'(x)$ for all $x \in X$. This proves the uniqueness of L.

By the assumption, for each $u \in \mathcal{U}(A)$, we get

$$\begin{aligned} \|D_u L(x, \underbrace{0, \cdots, 0}_{d-1 \text{ times}})\| &= \lim_{n\to\infty} \frac{1}{2^n} \|D_u f(2^n x, \underbrace{0, \cdots, 0}_{d-1 \text{ times}})\| \\ &\leq \lim_{n\to\infty} \frac{1}{2^n} \varphi(2^n x, \underbrace{0, \cdots, 0}_{d-1 \text{ times}}) = 0 \end{aligned}$$

for all $x \in X$. So

$$2({}_{d-1}C_l -{}_{d-1} C_{l-1} + 1)L(\frac{ux}{2}) = ({}_{d-1}C_l -{}_{d-1} C_{l-1} + 1)uL(x)$$

for all $u \in \mathcal{U}(A)$ and all $x \in X$. Since L is additive,

$$L(ux) = 2L(\frac{ux}{2}) = uL(x) \tag{3.3}$$

for all $u \in \mathcal{U}(A)$ and all $x \in X$.

Now let $a \in A$ ($a \neq 0$) and M an integer greater than $4|a|$. Then $|\frac{a}{M}| < \frac{1}{4} < 1 - \frac{2}{3} = \frac{1}{3}$. By [2, Theorem 1], there exist three elements $u_1, u_2, u_3 \in \mathcal{U}(A)$ such that $3\frac{a}{M} = u_1 + u_2 + u_3$. So by (3.3)

$$\begin{aligned} L(ax) &= L(\frac{M}{3} \cdot 3\frac{a}{M}x) = M \cdot L(\frac{1}{3} \cdot 3\frac{a}{M}x) = \frac{M}{3}L(3\frac{a}{M}x) \\ &= \frac{M}{3}L(u_1x + u_2x + u_3x) = \frac{M}{3}(L(u_1x) + L(u_2x) + L(u_3x)) \\ &= \frac{M}{3}(u_1 + u_2 + u_3)L(x) = \frac{M}{3} \cdot 3\frac{a}{M}L(x) \\ &= aL(x) \end{aligned}$$

for all $a \in A$ and all $x \in X$. Hence

$$L(ax + by) = L(ax) + L(by) = aL(x) + bL(y)$$

for all $a, b \in A (a, b \neq 0)$ and all $x, y \in X$. And $L(0x) = 0 = 0L(x)$ for all $x \in X$. So the unique generalized additive mapping $L : X \to Y$ is an A-linear mapping. □

Corollary 3.2. *Let $r = 2$. Let θ and $p < 1$ be positive real numbers. Let $f : X \to Y$ be an odd mapping such that*

$$\|D_u f(x_1, \cdots, x_d)\| \le \theta \sum_{j=1}^{d} ||x_j||^p$$

for all $u \in \mathcal{U}(A)$ and all $x_1, \cdots, x_d \in X$. Then there exists a unique A-linear generalized additive mapping $L : X \to Y$ such that

$$\|f(x) - L(x)\| \le \frac{2\theta}{(2 - 2^p)({}_{d-4}C_l -{}_{d-4} C_{l-4} + 1)}||x||^p$$

for all $x \in X$.

Proof. Define $\varphi(x_1, \cdots, x_d) = \theta \sum_{j=1}^{d} ||x_j||^p$, and apply Theorem 3.1. □

THEOREM 3.3. *Let $r = 2$. Let $f : X \to Y$ be an odd mapping for which there is a function $\varphi : X^d \to [0, \infty)$ satisfying* (3.ii) *such that*

$$\widetilde{\varphi}(x_1, \cdots, x_d) := \sum_{j=1}^{\infty} 2^j \varphi(\frac{x_1}{2^j}, \cdots, \frac{x_d}{2^j}) < \infty$$

for all $x_1, \cdots, x_d \in X$. Then there exists a unique A-linear generalized additive mapping $L : X \to Y$ such that

$$\|f(x) - L(x)\| \le \frac{1}{4({}_{d-4}C_l - {}_{d-4}C_{l-4} + 1)} \widetilde{\varphi}(x, x, x, x, \underbrace{0, \cdots, 0}_{d-4 \text{ times}})$$

for all $x \in X$.

Proof. Note that $f(0) = 0$ and $f(-x) = -f(x)$ for all $x \in X$ since f is an odd mapping. Let $u = 1 \in \mathcal{U}(A)$. Putting $x_1 = x_2 = x_3 = x_4 = x$ and $x_5 = \cdots = x_d = 0$ in (3.ii), we have

$$\|f(2x) - 2f(x)\| \le \frac{1}{2({}_{d-4}C_l - {}_{d-4}C_{l-4} + 1)} \varphi(x, x, x, x, \underbrace{0, \cdots, 0}_{d-4 \text{ times}})$$

for all $x \in X$. So we get

$$\|f(x) - 2f(\frac{x}{2})\| \le \frac{1}{2({}_{d-4}C_l - {}_{d-4}C_{l-4} + 1)} \varphi(\frac{x}{2}, \frac{x}{2}, \frac{x}{2}, \frac{x}{2}, \underbrace{0, \cdots, 0}_{d-4 \text{ times}})$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 3.1. $\square$

COROLLARY 3.4. *Let $r = 2$. Let θ and $p > 1$ be positive real numbers. Let $f : X \to Y$ be an odd mapping such that*

$$\|D_u f(x_1, \cdots, x_d)\| \le \sum_{j=1}^{d} \theta ||x_j||^p$$

for all $u \in \mathcal{U}(A)$ and all $x_1, \cdots, x_d \in X$. Then there exists a unique A-linear generalized additive mapping $L : X \to Y$ such that

$$\|f(x) - L(x)\| \le \frac{2\theta}{(2^p - 2)({}_{d-4}C_l - {}_{d-4}C_{l-4} + 1)} ||x||^p$$

for all $x \in X$.

Proof. Define $\varphi(x_1, \cdots, x_d) = \theta \sum_{j=1}^{d} ||x_j||^p$, and apply Theorem 3.3. $\square$

THEOREM 3.5. *Let $r = 4$. Let $f : X \to Y$ be an odd mapping for which there is a function $\varphi : X^d \to [0, \infty)$ satisfying* (3.ii) *such that*

$$\widetilde{\varphi}(x_1, \cdots, x_d) := \sum_{j=1}^{\infty} 2^j \varphi(\frac{x_1}{2^j}, \cdots, \frac{x_d}{2^j}) < \infty$$

for all $x_1, \cdots, x_d \in X$. Then there exists a unique A-linear generalized additive mapping $L : X \to Y$ such that

$$\|f(x) - L(x)\| \leq \frac{1}{8({}_{d-4}C_{l-1} -{}_{d-4} C_{l-3})} \widetilde{\varphi}(x, x, x, x, \underbrace{0, \cdots, 0}_{d-4 \text{ times}})$$

for all $x \in X$.

Proof. Note that $f(0) = 0$ and $f(-x) = -f(x)$ for all $x \in X$ since f is an odd mapping. Let $u = 1 \in \mathcal{U}(A)$. Putting $x_1 = x_2 = x_3 = x_4 = x$ and $x_5 = \cdots = x_d = 0$ in (3.ii), we have

$$\|f(\frac{x}{2}) - \frac{1}{2} f(x)\| \leq \frac{1}{16({}_{d-4}C_{l-1} -{}_{d-4} C_{l-3})} \varphi(x, x, x, x, \underbrace{0, \cdots, 0}_{d-4 \text{ times}})$$

for all $x \in X$. So we get

$$\|f(x) - 2f(\frac{x}{2})\| \leq \frac{1}{8({}_{d-4}C_{l-1} -{}_{d-4} C_{l-3})} \varphi(x, x, x, x, \underbrace{0, \cdots, 0}_{d-4 \text{ times}})$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 3.1. □

COROLLARY 3.6. *Let $r = 4$. Let θ and $p > 1$ be positive real numbers. Let $f : X \to Y$ be an odd mapping such that*

$$\|D_u f(x_1, \cdots, x_d)\| \leq \sum_{j=1}^{d} \theta ||x_j||^p$$

for all $u \in \mathcal{U}(A)$ and all $x_1, \cdots, x_d \in X$. Then there exists a unique A-linear generalized additive mapping $L : X \to Y$ such that

$$\|f(x) - L(x)\| \leq \frac{\theta}{(2^p - 2)({}_{d-4}C_{l-1} -{}_{d-4} C_{l-3})} ||x||^p$$

for all $x \in X$.

Proof. Define $\varphi(x_1, \cdots, x_d) = \theta \sum_{j=1}^{d} ||x_j||^p$, and apply Theorem 3.5. □

THEOREM 3.7. *Let $r = 4$. Let $f : X \to Y$ be an odd mapping for which there is a function $\varphi : X^d \to [0, \infty)$ satisfying* (3.ii) *such that*

$$\widetilde{\varphi}(x_1, \cdots, x_d) := \sum_{j=0}^{\infty} \frac{1}{2^j} \varphi(2^j x_1, \cdots, 2^j x_d) < \infty$$

for all $x_1, \cdots, x_d \in X$. Then there exists a unique A-linear generalized additive mapping $L : X \to Y$ such that

$$\|f(x) - L(x)\| \le \frac{1}{8({}_{d-4}C_{l-1} - {}_{d-4}C_{l-3})} \widetilde{\varphi}(x, x, x, x, \underbrace{0, \cdots, 0}_{d-4 \text{ times}})$$

for all $x \in X$.

Proof. Note that $f(0) = 0$ and $f(-x) = -f(x)$ for all $x \in X$ since f is an odd mapping. Let $u = 1 \in \mathcal{U}(A)$. Putting $x_1 = x_2 = x_3 = x_4 = x$ and $x_5 = \cdots = x_d = 0$ in (3.ii), we have

$$\|f(\frac{x}{2}) - \frac{1}{2} f(x)\| \le \frac{1}{16({}_{d-4}C_{l-1} - {}_{d-4}C_{l-3})} \varphi(x, x, x, x, \underbrace{0, \cdots, 0}_{d-4 \text{ times}})$$

for all $x \in X$. So we get

$$\|f(x) - \frac{1}{2} f(2x)\| \le \frac{1}{16({}_{d-4}C_{l-1} - {}_{d-4}C_{l-3})} \varphi(2x, 2x, 2x, 2x, \underbrace{0, \cdots, 0}_{d-4 \text{ times}})$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 3.1. □

COROLLARY 3.8. *Let $r = 4$. Let θ and $p < 1$ be positive real numbers. Let $f : X \to Y$ be an odd mapping such that*

$$\|D_u f(x_1, \cdots, x_d)\| \le \sum_{j=1}^{d} \theta ||x_j||^p$$

for all $u \in \mathcal{U}(A)$ and all $x_1, \cdots, x_d \in X$. Then there exists a unique A-linear generalized additive mapping $L : X \to Y$ such that

$$\|f(x) - L(x)\| \le \frac{\theta}{(2 - 2^p)({}_{d-4}C_{l-1} - {}_{d-4}C_{l-3})} ||x||^p$$

for all $x \in X$.

Proof. Define $\varphi(x_1, \cdots, x_d) = \theta \sum_{j=1}^{d} ||x_j||^p$, and apply Theorem 3.7. □

4. Isomorphisms in unital C^*-algebras

Throughout this section, assume that $\mathcal{A}$ is a unital C^*-algebra with norm $||\cdot||$ and unit e, and that $\mathcal{B}$ is a unital C^*-algebra with norm $\|\cdot\|$. Let $\mathcal{U}(\mathcal{A})$ be the set of unitary elements in $\mathcal{A}$.

We are going to investigate C^*-algebra isomorphisms between unital C^*-algebras.

Theorem 4.1. *Let $r = 2$. Let $h : \mathcal{A} \to \mathcal{B}$ be an odd bijective mapping satisfying $h(2^n uy) = h(2^n u)h(y)$ for all $u \in \mathcal{U}(\mathcal{A})$, all $y \in \mathcal{A}$, and $n = 0, 1, 2, \cdots$, for which there exists a function $\varphi : \mathcal{A}^d \to [0, \infty)$ such that*

$$\text{(4.i)} \qquad \widetilde{\varphi}(x_1, \cdots, x_d) := \sum_{j=0}^{\infty} \frac{1}{2^j} \varphi(2^j x_1, \cdots, 2^j x_d) < \infty,$$

$$\|D_\mu h(x_1, \cdots, x_d)\| \le \varphi(x_1, \cdots, x_d),$$

$$\text{(4.ii)} \qquad \|h(2^n u^*) - h(2^n u)^*\| \le \varphi(\underbrace{2^n u, \cdots, 2^n u}_{d \text{ times}})$$

for all $\mu \in \mathbb{T}^1 := \{\lambda \in \mathbb{C} \mid \ |\lambda| = 1\}$, all $u \in \mathcal{U}(\mathcal{A})$, $n = 0, 1, 2, \cdots$, and all $x_1, \cdots, x_d \in \mathcal{A}$. Assume that (4.iii) $\lim_{n\to\infty} \frac{1}{2^n} h(2^n e)$ is invertible. Then the odd bijective mapping $h : \mathcal{A} \to \mathcal{B}$ is a C^*-algebra isomorphism.

Proof. Consider the C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ as left Banach modules over the unital C^*-algebra $\mathbb{C}$. By Theorem 3.1, there exists a unique $\mathbb{C}$-linear generalized additive mapping $H : \mathcal{A} \to \mathcal{B}$ such that

$$\text{(4.iv)} \qquad \|h(x) - H(x)\|$$

$$\le \frac{1}{4({}_{d-4}C_l - {}_{d-4}C_{l-4} + 1)} \widetilde{\varphi}(x, x, x, x, \underbrace{0, \cdots, 0}_{d-4 \text{ times}})$$

for all $x \in \mathcal{A}$. The generalized additive mapping $H : \mathcal{A} \to \mathcal{B}$ is given by

$$H(x) = \lim_{n\to\infty} \frac{1}{2^n} h(2^n x)$$

for all $x \in \mathcal{A}$.

By (4.i) and (4.ii), we get

$$H(u^*) = \lim_{n\to\infty} \frac{1}{2^n} h(2^n u^*) = \lim_{n\to\infty} \frac{1}{2^n} h(2^n u)^*$$
$$= (\lim_{n\to\infty} \frac{1}{2^n} h(2^n u))^* = H(u)^*$$

for all $u \in \mathcal{U}(\mathcal{A})$. Since H is $\mathbb{C}$-linear and each $x \in \mathcal{A}$ is a finite linear combination of unitary elements (see [3, Theorem 4.1.7]), i.e., $x = \sum_{j=1}^m \lambda_j u_j$ $(\lambda_j \in \mathbb{C}, u_j \in \mathcal{U}(\mathcal{A}))$,

$$H(x^*) = H(\sum_{j=1}^m \overline{\lambda_j} u_j^*) = \sum_{j=1}^m \overline{\lambda_j} H(u_j^*) = \sum_{j=1}^m \overline{\lambda_j} H(u_j)^*$$
$$= (\sum_{j=1}^m \lambda_j H(u_j))^* = H(\sum_{j=1}^m \lambda_j u_j)^* = H(x)^*$$

for all $x \in \mathcal{A}$.

Since $h(2^n uy) = h(2^n u)h(y)$ for all $u \in \mathcal{U}(\mathcal{A})$, all $y \in \mathcal{A}$, and all $n = 0, 1, 2, \cdots$,

$$H(uy) = \lim_{n\to\infty} \frac{1}{2^n} h(2^n uy) = \lim_{n\to\infty} \frac{1}{2^n} h(2^n u)h(y) = H(u)h(y) \tag{4.1}$$

for all $u \in \mathcal{U}(\mathcal{A})$ and all $y \in \mathcal{A}$. By the additivity of H and (4.1),

$$2^n H(uy) = H(2^n uy) = H(u(2^n y)) = H(u)h(2^n y)$$

for all $u \in \mathcal{U}(\mathcal{A})$ and all $y \in \mathcal{A}$. Hence

$$H(uy) = \frac{1}{2^n} H(u)h(2^n y) = H(u)\frac{1}{2^n} h(2^n y) \tag{4.2}$$

for all $u \in \mathcal{U}(\mathcal{A})$ and all $y \in \mathcal{A}$. Taking the limit in (4.2) as $n \to \infty$, we obtain

$$H(uy) = H(u)H(y) \tag{4.3}$$

for all $u \in \mathcal{U}(\mathcal{A})$ and all $y \in \mathcal{A}$. Since H is $\mathbb{C}$-linear and each $x \in \mathcal{A}$ is a finite linear combination of unitary elements, i.e., $x = \sum_{j=1}^m \lambda_j u_j$ $(\lambda_j \in \mathbb{C}, u_j \in \mathcal{U}(\mathcal{A}))$, it follows from (4.3) that

$$H(xy) = H(\sum_{j=1}^m \lambda_j u_j y) = \sum_{j=1}^m \lambda_j H(u_j y) = \sum_{j=1}^m \lambda_j H(u_j)H(y)$$
$$= H(\sum_{j=1}^m \lambda_j u_j)H(y) = H(x)H(y)$$

for all $x, y \in \mathcal{A}$.

By (4.1) and (4.3),

$$H(e)H(y) = H(ey) = H(e)h(y)$$

for all $y \in \mathcal{A}$. Since $\lim_{n\to\infty} \frac{1}{2^n} h(2^n e) = H(e)$ is invertible,

$$H(y) = h(y)$$

for all $y \in \mathcal{A}$.

Therefore, the odd bijective mapping $h : \mathcal{A} \to \mathcal{B}$ is a C^*-algebra isomorphism, as desired. □

Corollary 4.2. *Let $r = 2$. Let θ and $p < 1$ be positive real numbers. Let $h : \mathcal{A} \to \mathcal{B}$ be an odd bijective mapping satisfying $h(2^n uy) = h(2^n u)h(y)$ for all $u \in \mathcal{U}(\mathcal{A})$, all $y \in \mathcal{A}$, and all $n = 0, 1, 2, \cdots$, such that*

$$\|D_\mu h(x_1, \cdots, x_d)\| \le \theta \sum_{j=1}^{d} ||x_j||^p,$$
$$\|h(2^n u^*) - h(2^n u)^*\| \le d\ 2^{pn}\theta$$

for all $\mu \in \mathbb{T}^1$, all $u \in \mathcal{U}(\mathcal{A})$, $n = 0, 1, 2, \cdots$, and all $x_1, \cdots, x_d \in \mathcal{A}$. Assume that $\lim_{n\to\infty} \frac{1}{2^n} h(2^n e)$ is invertible. Then the odd bijective mapping $h : \mathcal{A} \to \mathcal{B}$ is a C^*-algebra isomorphism.

Proof. Define $\varphi(x_1, \cdots, x_d) = \theta \sum_{j=1}^{d} ||x_j||^p$, and apply Theorem 4.1. □

Theorem 4.3. *Let $r = 2$. Let $h : \mathcal{A} \to \mathcal{B}$ be an odd bijective mapping satisfying $h(2^n uy) = h(2^n u)h(y)$ for all $u \in \mathcal{U}(\mathcal{A})$, all $y \in \mathcal{A}$, and $n = 0, 1, 2, \cdots$, for which there exists a function $\varphi : \mathcal{A}^d \to [0, \infty)$ satisfying* (4.i), (4.ii), *and* (4.iii) *such that*

$$\|D_\mu h(x_1, \cdots, x_d)\| \le \varphi(x_1, \cdots, x_d) \tag{4.v}$$

for $\mu = 1, i$, and all $x_1, \cdots, x_d \in \mathcal{A}$. If $h(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in \mathcal{A}$, then the odd bijective mapping $h : \mathcal{A} \to \mathcal{B}$ is a C^*-algebra isomorphism.

Proof. Put $\mu = 1$ in (4.v). By the same reasoning as in the proof of Theorem 4.1, there exists a unique generalized additive mapping $H : \mathcal{A} \to \mathcal{B}$

satisfying (4.iv). By the same reasoning as in the proof of [5, Theorem], the generalized additive mapping $H : \mathcal{A} \to \mathcal{B}$ is $\mathbb{R}$-linear.

Put $\mu = i$ in (4.v). By the same method as in the proof of Theorem 4.1, one can obtain that

$$H(ix) = \lim_{n\to\infty} \frac{1}{2^n} h(2^n ix) = \lim_{n\to\infty} \frac{i}{2^n} h(2^n x) = iH(x)$$

for all $x \in \mathcal{A}$.

For each element $\lambda \in \mathbb{C}$, $\lambda = s + it$, where $s, t \in \mathbb{R}$. So

$$\begin{aligned} H(\lambda x) &= H(sx + itx) = sH(x) + tH(ix) = sH(x) + itH(x) \\ &= (s + it)H(x) = \lambda H(x) \end{aligned}$$

for all $\lambda \in \mathbb{C}$ and all $x \in \mathcal{A}$. So

$$H(\zeta x + \eta y) = H(\zeta x) + H(\eta y) = \zeta H(x) + \eta H(y)$$

for all $\zeta, \eta \in \mathbb{C}$, and all $x, y \in \mathcal{A}$. Hence the generalized additive mapping $H : \mathcal{A} \to \mathcal{B}$ is $\mathbb{C}$-linear.

The rest of the proof is the same as in the proof of Theorem 4.1. □

References

[1] P. Găvruta, *A generalization of the Hyers–Ulam–Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. **184** (1994), 431–436.

[2] R. V. Kadison and G. Pedersen, *Means and convex combinations of unitary operators*, Math. Scand. **57** (1985), 249–266.

[3] R. V. Kadison and J. R. Ringrose, *Fundamentals of the Theory of Operator Algebras*, Elementary Theory, Academic Press, New York, 1983.

[4] C. Park, *On the stability of the linear mapping in Banach modules*, J. Math. Anal. Appl. **275** (2002), 711–720.

[5] Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297–300.

[6] ———, *On the stability of the quadratic functional equation and its applications*, Studia Univ. Babes-Bolyai **XLIII (3)** (1998), 89–124.

[7] ———, *The problem of S.M. Ulam for approximately multiplicative mappings*, J. Math. Anal. Appl. **246** (2000), 352–378.

[8] ———, *On the stability of functional equations in Banach spaces*, J. Math. Anal. Appl. **251** (2000), 264–284.

[9] Th.M. Rassias and P. Šemrl, *On the Hyers–Ulam stability of linear mappings*, J. Math. Anal. Appl. **173** (1993), 325–338.

[10] Th. M. Rassias and K. Shibata, *Variational problem of some quadratic functionals in complex analysis*, J. Math. Anal. Appl. **228** (1998), 234–253.
[11] S. M. Ulam, *Problems in Modern Mathematics*, Wiley, New York, 1960.

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