Bull. Korean Math. Soc. **48** (2011), No. 1, pp. 197–211
DOI 10.4134/BKMS.2011.48.1.197

ON THE 2-BRIDGE KNOTS OF DUNWOODY $(1,1)$-KNOTS

Soo Hwan Kim and Yangkok Kim*

Abstract. Every $(1,1)$-knot is represented by a 4-tuple of integers (a,b,c,r), where $a>0$, $b\geq 0$, $c\geq 0$, $d=2a+b+c$, $r\in\mathbb{Z}_d$, and it is well known that all 2-bridge knots and torus knots are $(1,1)$-knots. In this paper, we describe some conditions for 4-tuples which determine 2 -bridge knots and determine all 4-tuples representing any given 2-bridge knot.

1. Introduction

In this note all manifolds will be assumed to be closed, connected and orientable and all $(1,1)$-knots are non-oriented if there is no special reference. In [6] Dunwoody introduced a family of 3-manifolds depending on six integer parameters which induce a class of knots. It was shown that all knots induced by Dunwoody manifolds are $(1,1)$-knots in [21]. Moreover all $(1,1)$-knots are induced by Dunwoody manifolds in [4]. In [12] and [21] a type of 4-tuples representing all 2-bridge knots was described. We here determine a type of 4-tuples representing all 2-bridge knots and their dual and mirror images from a different point of view. We also recall that a type of 4-tuples representing the torus knot $T(p,q)$ was determined in [1] and [15] when either $q\equiv\pm1 \mod p$ or $q\equiv\pm2 \mod p$.

Let (V_1,V_2) be a Heegaard splitting of a 3-manifold M with genus n. A properly embedded disc D in the handlebody V_2 is called a *meridian disc* of V_2 if cutting V_2 along D yields a handlebody of genus $n-1$. A collection of n mutually disjoint meridian discs $\{D_i\}$ in V_2 is called a *complete system* of meridian discs of V_2 if cutting V_2 along $\cup_i D_i$ gives a 3-ball. Let α_i denote the 1-sphere ∂D_i which lies in the closed orientable surface $\partial V_1=\partial V_2$ of genus n. The system is said to be a *Heegaard diagram* of the 3-manifold M and denoted by $(V_1;\alpha_1,\alpha_2,\ldots,\alpha_n)$. Moreover, the system $(V_2;\beta_1,\beta_2,\ldots,\beta_n)$ is called a *dual Heegaard diagram* of the 3-manifold M if $\{D_i\}$ is a complete system of n mutually disjoint meridian discs in V_1 and β_i is the 1-sphere ∂D_i which

Received June 22, 2009; Revised October 9, 2009.
2010 *Mathematics Subject Classification.* Primary 57M25; Secondary 57M27.
Key words and phrases. (1,1)-knot, (1,1)-decomposition, cyclic branched covering, crystallization, Dunwoody manifold, Heegaard splitting, Heegaard diagram, 2-bridge knot, torus knot.
* This work was supported by Dong-eui University Grant(2009AA091).

lies in the closed orientable surface $\partial V_1 = \partial V_2$ of genus n. In other words, $(V_2; \beta_1, \beta_2, \ldots, \beta_n)$ is the dual Heegaard diagram of $(V_1; \alpha_1, \alpha_2, \ldots, \alpha_n)$.

Let M be a 3-manifold and K a knot in M. Then the pair (M, K) has a $(1,1)$-*decomposition* if there exists a Heegaard splitting of genus one $(V_1, K_1) \cup_\phi (V_2, K_2)$ of (M, K) such that $(V_1; \alpha_1)$ is a Heegaard diagram of M and $K_1 \subset V_1$ and $K_2 \subset V_2$ are properly embedded trivial arcs, where ϕ is an attaching homeomorphism. We call the knot K an $(1,1)$-knot. Note that M turns out to be a lens space $L(p,q)$, and we assume to include $\mathbb{S}^3 = L(1,0)$ but not $\mathbb{S}^1 \times \mathbb{S}^2 = L(0,1)$ in this note. By the dual $(1,1)$-decomposition of (M, K) we mean that $(V_2; \beta_1)$ is the dual Heegaard diagram of M. Thus the $(1,1)$-knot K does not change under such a dual process. We refer to [1], [2], [4]-[6], [8]-[10], [12]-[24] for definitions and fundamental results on $(1,1)$-knots and $(1,1)$-decompositions.

In Section 2, we introduce a set $\mathcal{D}$ of 4-tuples of integers (a, b, c, r) such that $a > 0$, $b \geq 0$, $c \geq 0$, $r \in \mathbb{Z}_d$, where $d = 2a + b + c$, inducing the $(1,1)$-decomposition of (M, K) determined by two permutations. Furthermore we determine conditions for a 4-tuple $(a, 0, 1, r)$ to be contained in $\mathcal{D}$ and give the formula for its dual decomposition. In Section 3, we determine the forms of 4-tuples in $\mathcal{D}$ representing all 2-bridge knots and their mirror images by using the method of crystallization in [11]. As an application, for any 2-bridge knot K, we can show that there exist two $(1,1)$-decompositions representing K. Generally we find other forms of 4-tuples representing K by means of the dual process and homeomorphic property, which are different from the forms for K obtained from Theorem 3.1. As a consequence, we show that there exist at most four $(1,1)$-decompositions representing K, which has also been shown in [16] by using Heegaard splittings of the exteriors of 2-bridge knots.

2. The $(1,1)$-decompositions and its dual decompositions

We introduce the $(1,1)$-decompositions of (M, K) determined by two permutations and a 4-tuple of integers (a, b, c, r) such that $a > 0$, $b \geq 0$, $c \geq 0$, $r \in \mathbb{Z}_d$, where $d = 2a + b + c$, as follows.

Let $\{m^+, m^-\}$ be a set of circles with each other different orientations, and $X^+ = \{1, 2, \ldots, d\}$ and $X^- = \{\overline{1}, \overline{2}, \ldots, \overline{d}\}$ sets of d vertices in m^+ and m^-, respectively. We define each of 2-cycles in the permutation α to be the ends of curves connecting m^+ and m^- or themselves as the rule of Figure 1:

$$\begin{aligned} \alpha \;=\; & (1, d)(2, d-1)(3, d-2) \cdots (a, d-a+1) \\ & (a+1, \overline{a+c+1}) \cdots (a+b, \overline{a+c+b}) \\ & (a+b+1, \overline{a+1}) \cdots (a+b+c, \overline{a+c}) \\ & (\overline{1}, \overline{d})(\overline{2}, \overline{d-1}) \cdots (\overline{a}, \overline{d-a+1})) \end{aligned}$$

and

$$\beta = (1, \overline{1-r})(2, \overline{2-r}) \cdots (j, \overline{j-r}) \cdots (d, \overline{d-r}),$$

where all numbers are under mod d. We note that a disk m, called a meridian disk, is obtained by the corresponding points in m^+ and m^- via β. For example $(r, \overline{r-r})$ means that the number r of m^+ is identified with the number $\overline{r-r} = \overline{0} = \overline{d}$ in m^-. Thus $\alpha\beta$ determines the disjoint simple closed curves on the genus one solid torus, denoted by H, with the meridian disk m.

We consider a trivial arc K_1 in H such that $K_1 \cap \partial H = \partial K_1$, and ∂K_1 is situated inside the bigons determined by 2-cycles $(1, d)$ and $(\overline{1}, \overline{d})$ as shown Figure 1.

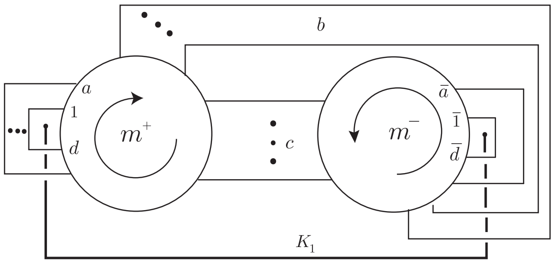

FIGURE 1. A trivial arc K_1 and the solid torus H determined by α and β

Assume that $|\alpha\beta|$ is the number of disjoint cycles in $\alpha\beta$ and that T is the number of disjoint simple closed curves on ∂H. Since consecutive two cycles in $\alpha\beta$ determine a simple closed curve on ∂H, we have $|\alpha\beta| = 2T$. Thus a set of 4-tuples of integers inducing the $(1,1)$-decompositions of (M, K) is

$$\mathcal{D} = \{(a, b, c, r) | a > 0, b \geq 0, c \geq 0, d = 2a + b + c, r \in \mathbb{Z}_d, |\alpha\beta| = 2\}.$$

For each (a, b, c, r) in $\mathcal{D}$, we denote the corresponding $(1,1)$-decomposition and $(1,1)$-knot of (M, K) by the *Dunwoody* $(1,1)$*-decomposition* $D(a, b, c, r)$ and the *Dunwoody* $(1,1)$*-knot* $K(a, b, c, r)$, respectively. For each $(a, b, c, r) \in \mathcal{D}$, M is to be a lens space because of $T = 1$. By [4], every $(1,1)$-knot can be represented by the Dunwoody $(1,1)$-knot $K(a, b, c, r)$. However this representation need not be unique. For example, both $K(1, 3, 4, 7)$ and $K(2, 1, 4, 4)$ represent a pretzel knot $P(-2, 3, 7)$ which is a $(1,1)$-knot as was mentioned in [21]. In the following, we give a condition for $(a, 0, 1, r)$ to lie in $\mathcal{D}$ and the formula for the dual decomposition of $D(a, 0, 1, r)$.

Theorem 2.1. *A* 4*-tuple* $(a, 0, 1, r)$ *lies in* $\mathcal{D}$ *if and only if there is a positive integer* k *such that* $2kr \equiv a \mod(2a+1)$ *or* $(2k-1)r \equiv a \mod(2a+1)$.

Proof. Let m and l be a meridian disk determined by β and a simple closed curve determined by $\alpha\beta$, respectively, of $D(a, 0, 1, r)$. We note that two vertices i and $1-i$ are connected in m^+ and, $\overline{i}$ and $\overline{(2r+1)-i}$ are connected in m^- if $i \neq a+1$ and $\overline{i} \neq \overline{r+a+1}$. We denote these connections by writing edges $[i, 1-i]$ and $[\overline{i}, \overline{(2r+1)-i}]$. Then there are $2a+1$ edges on $D(a, 0, 1, r)$

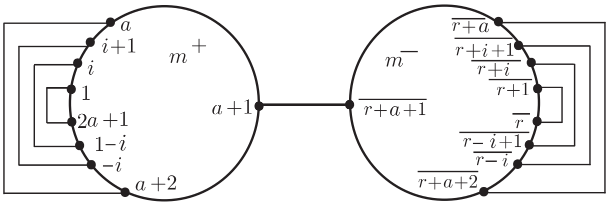

FIGURE 2. $D(a,0,1,r)$

inclusive of the edge $[a+1,\overline{r+a+1}]$ connecting m^+ and m^- (See Figure 2 where all numbers are under modulo $2a+1$). Moreover a simple closed curve on $D(a,0,1,r)$ can be expressed by a closed paths of edges. For example a closed path is of the form:

$$[i,1-i],[\overline{1-i},\overline{(2r+1)-(1-i)}],\ldots,[\overline{(2r+1)-i},\overline{i}].$$

For simplicity we denote the above closed path by

$$1-i\to\overline{(2r+1)-(1-i)}\to\cdots\to\overline{i}$$

obtained by writing the terminal vertex of each edge. The $2m$th number and the $(2m-1)$th number in the path starting from i are $\overline{m(2r+1)-(m-1)-i}$ and $(m-1)-(m-1)(2r+1)+i$ respectively if it does not contain the edge $[a+1,\overline{r+a+1}]$. Similarly the $2m$th number and the $(2m-1)$th number in the path starting from $\overline{i}$ are $m-(m-1)(2r+1)-i$ and $\overline{(m-1)(2r+1)-(m-1)+i}$ respectively if it does not contain the edge $[a+1,\overline{r+a+1}]$. We now claim that the number of curves in $D(a,0,1,r)$ is 1. Equivalently a closed path from 0 to $\overline{1}$ has $2a+1$ vertices. There are two cases as follows:

Case i) The closed path contains the edge $[a+1,\overline{r+a+1}]$. That is,

$$(*)\qquad \underbrace{0\to\overline{(2r+1)-1}\to\cdots\to\overline{a+1}}_{2k\text{ vertices}}\to\underbrace{\overline{r+a+1}\to\cdots\to\overline{1}}_{2\ell-1\text{ vertices}}$$

(i) For $0\to\overline{(2r+1)-1}\to\cdots\to\overline{a+1}$,

$$k(2r+1)-k+1+0\equiv a+1 \mod(2a+1)$$

or

$$2kr\equiv a \mod(2a+1).$$

(ii) For $\overline{r+a+1}\to\cdots\to\overline{1}$,

$$(\ell-1)(2r+1)-(\ell-1)+(r+a+1)\equiv 1 \mod(2a+1)$$

or

$$(2\ell-1)r\equiv -a \mod(2a+1).$$

The path $(*)$ has $2k+(2\ell-1)=2a+1$ vertices and so $k+\ell=a+1$. Hence the relation $(2\ell-1)r\equiv -a \mod(2a+1)$ in (ii) is equivalent to $2kr\equiv a \mod(2a+1)$ in (i).

Case ii) The path contains the edge $[\overline{r+a+1}, a+1]$. That is,

$$(**) \qquad \underbrace{0 \to \cdots \to r+a+1}_{2k-1 \text{ vertices}} \to \underbrace{a+1 \to \cdots \to \overline{1}}_{2\ell \text{ vertices}}$$

(i) For $0 \to \cdots \to r+a+1$,

$$(k-1) - (k-1)(2r+1) + 0 \equiv r+a+1 \mod(2a+1)$$

or

$$(2k-1)r \equiv a \mod(2a+1).$$

(ii) For $a+1 \to \cdots \to \overline{1}$,

$$\ell(2r+1) - (\ell-1) - (a+1) \equiv 1 \mod(2a+1)$$

or

$$2\ell r \equiv -a \mod(2a+1).$$

The path $(**)$ has $(2k-1)+2\ell = 2a+1$ vertices and so $k+\ell = a+1$. Hence the relation $2\ell r \equiv -a \mod(2a+1)$ in (ii) is equivalent to $(2k-1)r \equiv a \mod(2a+1)$ in (i). $\square$

For any $(1,1)$-knot K in $\mathbb{S}^3$, let $(\mathbb{S}^3, K) = (V_1, K_1) \cup_\phi (V_2, K_2)$ be a $(1,1)$-decomposition of K. Then there exists the Dunwoody $(1,1)$-decomposition $D(a,b,c,r)$ of K such that $D(a,b,c,r)$ has the Heegaard diagram (V_1, α_1) of (V_1, K_1), where α_1 is the oriented simple closed curve l on ∂V_1 determined by $\alpha\beta$. This means that $D(a,b,c,r)$ can be regarded as one in Figure 1. By the definition of the dual process, we can obtain the dual $(1,1)$-decomposition of $D(a,b,c,r)$, denoted by $Du(a,b,c,r)$ or $D(a',b',c',r')$, which has the Heegaard diagram (V_2, β_1) of (V_2, K_2). Then β_1 is the oriented simple closed curve on ∂V_2 as the image of meridian curve m on V_1 by ϕ and denoted by l'. In fact, this is understood easily from the attaching homeomorphism ϕ defined the images of simple closed curve l and meridian curve m on V_1 by the meridian curve m' and simple closed curve l' on V_2, respectively. We now find the dual $(1,1)$-decomposition of $D(a,0,1,r)$ in $\mathcal{D}$. Let $D(a,0,1,r)$ be a $(1,1)$-decomposition of $(\mathbb{S}^3, K)$. Then there exist three types of areas as follows (see Figure 2 where all indices are taken under modulo $2a+1$):

(1) two bigons $[1, 2a+1]$ and $[\overline{r}, \overline{r+1}]$ at m^+ and m^- respectively,

(2) $2(a-1)$ quadrilaterals $[i, i+1, 2a+1-i, 2a+2-i]$ at m^+ and $[\overline{i+r}, \overline{i+r+1}, \overline{r-i+1}, \overline{r-i}]$ at m^- where $1 \le i \le a-1$,

(3) an octagon $[a, a+2, a+1, \overline{r+a+1}, \overline{r+a+2}, \overline{r+a}, \overline{r+a+1}, a+1]$.

For simplicity we denote the quadrilateral $[i, i+1, 2a+2-i, 2a+2-i+1]$ at m^+ by the line with endpoints $\widehat{i}$ and $2\widehat{a+1}-i$, and the quadrilateral $[\overline{i+r}, \overline{i+r+1}, \overline{r-i+1}, \overline{r-i}]$ at m^- by the line with endpoints $\widehat{\overline{r+i}}$ and $\overline{r+\widehat{2a}-i+1}$ to get a graph as shown in Figure 3.

Through the process that changes $\{m, l\}$ into $\{l', m'\}$, the roles of m and l in each area will be interchanged in $Du(a,0,1,r)$, so if 1 in the bigon $[1, 2a+1]$ is a

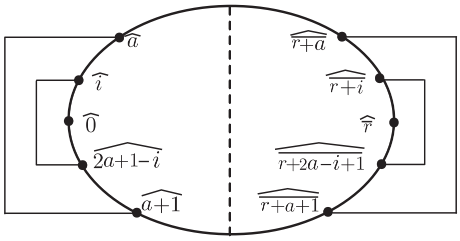

FIGURE 3. Three areas of $D(a,0,1,r)$

starting point in $D(a,0,1,r)$, then $2a+1$ will be a starting point in $Du(a,0,1,r)$. Similarly, the point $r+1$ in the bigon $[\overline{r},\overline{r+1}]$ is going to situate on r' in $Du(a,0,1,r)$. On $D(a,0,1,r)$, we note that a is the number of areas (1) and (2) that is connected from the bigon $[1,2a+1]$ to a quadrilateral with $(a,a+2)$ which is to be a side of an octagon. Therefore, on $Du(a,0,1,r)$, a' determined by l' is equal to the number of areas (1) and (2) that is connected along parts of m from the bigon $[2a+1,1]$ to a quadrilateral which is connected with a side of an octagon. Since each area is preserved in $D(a,0,1,r)$ and $Du(a,0,1,r)$, we have $c' = 2a+1-a'$. We also note that the r'th term of the following cycle is the number $r+1$ or $\overline{r+1}$:

$$0 = 2a+1 \to \cdots \to r+1 \text{ or } \overset{r'\text{th term}}{\overset{\downarrow}{}}\overline{r+1} \to \cdots$$

which is a cycle along l starting from $0 = 2a+1$ and determine m' on $Du(a,0,1,r)$. Vice versa we can obtain $Du(a,0,c,r)$ from $D(a,0,c,r)$ by the dual process of above. Summarizing, we formulate as follows.

Theorem 2.2. *Let $D(a',0,c',r')$ be the dual of $D(a,0,1,r)$. Then a' is the positive integer $m(< a)$ satisfying one of the following conditions*:

(1) $2r(m-1) \equiv a \mod(2a+1)$,
(2) $2r(m-1) \equiv a+1 \mod(2a+1)$,
(3) $-2(m-2)r \equiv r+a \mod(2a+1)$,
(4) $-2(m-2)r \equiv r+a+1 \mod(2a+1)$.

Furthermore $c' = 2a-2a'+1$ and r' is as follows:

$$r' = \begin{cases} 2m \text{ if } (2m-1)r \equiv 0 \mod(2a+1) \\ 2m-1 \text{ if } (2m-1)r+1 \equiv 0 \mod(2a+1) \\ s+t \text{ if } sr \equiv a \mod(2a+1) \text{ and } tr \equiv -a \mod(2a+1), \text{ where } s,t < a. \end{cases}$$

Proof. We note that in the three areas form of $D(a,0,1,r)$, two points $\widehat{i}$ and $\widehat{2a+1-i}$ are connected in the left part, and two points $\widehat{\overline{r+i}}$ and $\overline{r+\widehat{2a-i}+1}$ are connected in the right part (See Figure 3). As in the proof of Theorem 2.1, a cycle starting from $\widehat{0}$ is of the form:

$$\widehat{0} \to \overline{\widehat{2r-0}} \to (2a\widehat{+1)}-2r \to \overline{2r-((2\widehat{a+1)}-2r)} \to \cdots$$

where all numbers are under modulo $2a+1$. We note that the $2m$th point and the $(2m+1)$th point in a cyclic starting from $\overline{\widehat{0}}$ is $-(\widehat{m-2})\cdot 2r$ and $\overline{(m\widehat{-1)}\cdot 2r}$ respectively. As we see in the argument above a' is the number of edges in a sequence starting the point $\widehat{0}$ and ending $\widehat{a}$, $\widehat{a+1}$, $\overline{\widehat{r+a}}$ or $\overline{r\widehat{+a+}1}$. That is, a' is the positive integer m such that (i) $(m-1)\cdot 2r \equiv a$ or $a+1$ or (ii) $-(m-2)\cdot 2r \equiv r+a$ or $r+a+1$ under modulo $2a+1$.

By the argument above r' is the number of vertices in a path starting the point $0 = 2a+1$ and ending $r+1$ or $\overline{r+1}$ (See Figure 2). We first consider the case that the path has the vertex $r+1$ or $\overline{r+1}$ before crossing the edge connecting $a+1$ and $\overline{r+a+1}$. In this case $(m-1)-(m-1)(2r+1)+0 \equiv r+1$ or $m(2r+1)-(m-1)-0 \equiv r+1$. If $(m-1)-(m-1)(2r+1)+0 \equiv r+1$ or equivalently $(2m-1)r+1 \equiv 0 \mod(2a+1)$, then the number of vertices are $2m-1$. If $m(2r+1)-(m-1)-0 \equiv r+1$ or equivalently $(2m-1)r \equiv 0 \mod(2a+1)$, then the number of vertices are $2m$. We now consider the other one case by case as follows:

Case 1) The closed path contains the edge $[a+1, \overline{r+a+1}]$. That is,

$$\underbrace{0 \to \overline{(2r+1)-1} \to \cdots \to \overline{a+1}}_{2k \text{ vertices}} \to \underbrace{\overline{r+a+1} \to \cdots \to r+1}_{2\ell \text{ vertices}}$$

(i) For $0 \to \overline{(2r+1)-1} \to \cdots \to \overline{a+1}$,

$$k(2r+1)-k+1+0 \equiv a+1 \mod(2a+1)$$

or

$$2kr \equiv a \mod(2a+1).$$

(ii) For $\overline{r+a+1} \to \cdots \to r+1$,

$$\ell-(\ell-1)(2r+1)-(r+a+1) \equiv r+1 \mod(2a+1)$$

$$\ell-(\ell-1)(2r+1)-(r+a+1) \equiv r+1 \mod(2a+1)$$

or

$$2\ell r \equiv a \mod(2a+1).$$

Thus the number of all vertices in the path $(*)$ is $2k+2\ell$.

Case 2) The closed path contains the edge $[a+1, \overline{r+a+1}]$. That is,

$$\underbrace{0 \to \overline{(2r+1)-1} \to \cdots \to \overline{a+1}}_{2k \text{ vertices}} \to \underbrace{\overline{r+a+1} \to \cdots \to \overline{r+1}}_{2\ell-1 \text{ vertices}}$$

(i) For $0 \to \overline{(2r+1)-1} \to \cdots \to \overline{a+1}$, $2kr \equiv a \mod(2a+1)$.

(ii) For $\overline{r+a+1} \to \cdots \to \overline{r+1}$,

$$(\ell-1)(2r+1)-(\ell-1)+(r+a+1) \equiv r+1 \mod(2a+1)$$

or

$$2(\ell-1)r \equiv -a \mod(2a+1).$$

We note that (i) and (ii) are not compatible.

Case 3) The path contains the edge $[\overline{r+a+1}, a+1]$. That is,

$$\underbrace{0 \to \cdots \to r+a+1}_{2k-1 \text{ vertices}} \to \underbrace{a+1 \to \cdots \to r+1}_{2\ell-1 \text{ vertices}}$$

(i) For $0 \to \cdots \to r+a+1$,

$$(k-1)-(k-1)(2r+1)+0 \equiv r+a+1 \mod(2a+1)$$

or

$$(2k-1)r \equiv a \mod(2a+1).$$

(ii) For $a+1 \to \cdots \to r+1$,

$$(\ell-1)-(\ell-1)(2r+1)+(a+1) \equiv r+1 \mod(2a+1)$$

or

$$(2\ell-1)r \equiv a \mod(2a+1).$$

Thus the number of all vertices in the path $(*)$ is $(2k-1)+(2\ell-1)$.

Case 4) The path contains the edge $[\overline{r+a+1}, a+1]$. That is,

$$\underbrace{0 \to \cdots \to r+a+1}_{2k-1 \text{ vertices}} \to \underbrace{a+1 \to \cdots \to \overline{r+1}}_{2\ell \text{ vertices}}$$

(i) For $0 \to \cdots \to r+a+1$,

$$(k-1)-(k-1)(2r+1)+0 \equiv r+a+1 \mod(2a+1)$$

or

$$(2k-1)r \equiv a \mod(2a+1).$$

(ii) For $a+1 \to \cdots \to \overline{r+1}$,

$$\ell(2r+1)-(\ell-1)-(a+1) \equiv r+1 \mod(2a+1)$$

or

$$(2\ell-1)r \equiv -a \mod(2a+1).$$

We note that (i) and (ii) are not compatible. □

Example 1. Let $D(a', 0, c', r')$ be the dual of $D(5, 0, 1, 2)$. Then $-2(3-2)\cdot 2 \equiv 2+5 \mod(2a+1)$. Thus $a' = 3$ and so $c' = 2\cdot 5 - 2\cdot 3 + 1 = 5$. Moreover $(2\cdot 3-1)\cdot 2+1 \equiv 0 \mod(2\cdot 5+1)$. Thus $r' = 2\cdot 3 - 1 = 5$. Therefore $D(a', 0, c', r') = D(3, 0, 5, 5)$ (See Figure 4).

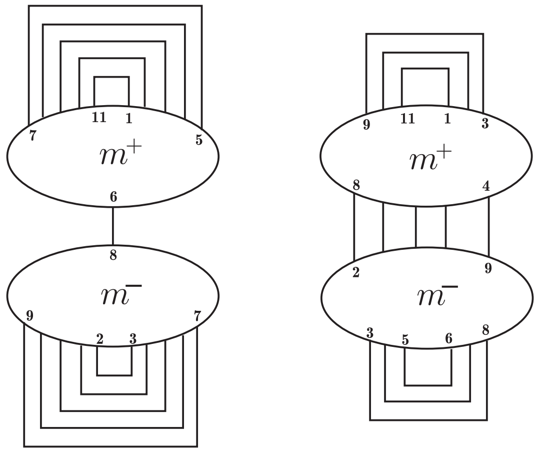

FIGURE 4. $D(5, 0, 1, 2)$ and its dual $D(3, 0, 5, 5)$

3. The (1, 1)-decompositions of 2-bridge knots

Since every 2-bridge knot is invertible, the 2-bridge knot $\mathfrak{b}(p, q)$ is equivalent to $\mathfrak{b}(p, q - p)$, where p is odd. By $\mathfrak{b}^*(p, q)$ we denote a mirror image $\mathfrak{b}(p, -q)$ of $\mathfrak{b}(p, q)$. However the classification of the 2-bridge knot follows from that of the lens space up to orientation-preserving homeomorphism as follows:

$$-L(p, q) = L(-p, q) = L(p, -q),\ L(p, q) = L(-p, -q) = L(p, q + kp)$$

for any integer k. Here $-L(p, q)$ denotes the same manifold as $L(p, q)$ but with orientation reversed. Thus we can assume that all 2-bridge knots are of the form $\mathfrak{b}(d, h)$ such that d is odd and h is even. We note that $\mathfrak{b}(d, -h) = \mathfrak{b}(d, 2d - h)$ for each $h < d$. In this section, we show that the 4-tuples $(a, 0, 1, r)$ in $\mathcal{D}$ are representing all 2-bridge knots $\mathfrak{b}(2a + 1, 2r)$ in $\mathbb{S}^3$ and that there are at most four (1, 1)-decompositions for each 2-bridge knot by using the dual decomposition of $D(a, 0, 1, r)$. It was proved in [5] Theorem 4.2(iv) that the (1, 1)-knot $K(2k - 2, 0, 1, 1)$ is equivalent to the 2-bridge knot $\mathfrak{b}(4k - 3, 2)$. From now on we write $D_2(a, b, c, r)$ for the Heegaard diagram $D_2(a, b, c, r, 0)$ inducing Dunwoody manifold.

Theorem 3.1. *A $D(a, 0, 1, r)$ is a (1, 1)-decomposition of $(\mathbb{S}^3, K)$, where K is the 2-bridge knot $\mathfrak{b}(2a + 1, 2r)$ in $\mathbb{S}^3$.*

Proof. Let $\mathfrak{b}(2a + 1, 2r)$ be a 2-bridge knot with $(2a + 1, 2r) = 1$. Then $(2a + 1)sa + 2rta = a$ for some integers s, t. That is, $2tar \equiv a \mod(2a + 1)$ and so $(a, 0, 1, r) \in \mathcal{D}$ by Theorem 2.1. Note that the lens space $L(p, q)$ is the 2-fold cyclic branched covering of $\mathbb{S}^3$ branched over a unique 2-bridge knot or link of type (p, q). Therefore it is sufficient to show that the Heegaard diagram

$D_2(a, 0, 1, r)$ represents the 2-fold cyclic branched covering of $\mathbb{S}^3$ over the 2-bridge knot $\mathfrak{b}(d, 2r)$, where $d = 2a + 1$. Since $D_2(a, 0, 1, r)$ is the genus two Heegaard diagram, we obtain the crystallization associated with $D_2(a, 0, 1, r)$, denoted $\Gamma(d, r)$, by using the method of Lemmas 3 and 4 in [11] as (a) and (b) in Figure 5. On the other hand, the crystallization $\Gamma(d, r)$ can be obtained

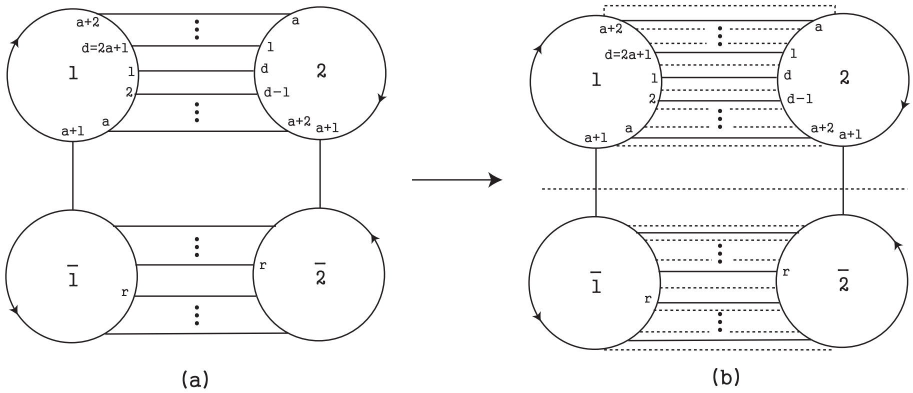

FIGURE 5. The crystallization associated with $D_2(a, 0, 1, r)$

directly from Figure 5(b) by the following process. Let C_1 and C_2 be two circles with $2d$ vertices on the plane, corresponding to the circles 1 and 2 of Figure 5(b), and A a circle with 2 vertices on the same plane, corresponding to a middle dotted line of Figure 5(b). Then we draw $2d - 1$ parallel lines connecting C_1 and C_2, and a line connecting A and C_i for each $i = 1, 2$ as in Figure 6. Finally, each point of, say C_1, C_i for each $i = 1, 2$ is identified by a point following after going $2r$ edges along counterclockwise orientation around C_1, connecting to C_2 or A, say C_2, and finally going $2r$ edges along clockwise orientation around C_2. Note that points that arrive inside A in the process are fixed. If we define a pair of identified two points by the process to be the same number or alphabet, we obtain the crystallization $\Gamma(d, r)$ from Figure 6. From the facts of [7] and [3] we know that the crystallization $\Gamma(d, r)$ constructed above represents a Dunwoody manifold with a genus two Heegaard splitting. Since every genus two Heegaard splitting is 2-symmetric, there is an axis in the interior of C_i for each $i = 1, 2$ as depicted in Figure 6. We denote the axes in the interiors of three circles C_1, C_2, and A by X_1, X_2, and X_A respectively. In fact, let A_1 (resp. A_2) be a point in C_1 (resp. C_2) which is connected to X_A. Then the fixed points of X_1 (resp. X_2) in the interior of C_1 (resp. C_2) are points obtained by a counterclockwise rotation of $d - r$ edges from A_1 (resp. A_2) and a clockwise rotation of r edges from A_1 (resp. A_2). In the sense of [7], let E be the set of edges in the interiors of C_1 and C_2 joining vertices by the reflections with respect to the axes X_1 and X_2 and D the set of edges in the exteriors of C_1, C_2 and A. Then we obtain the 2-bridge knot K so that X_1

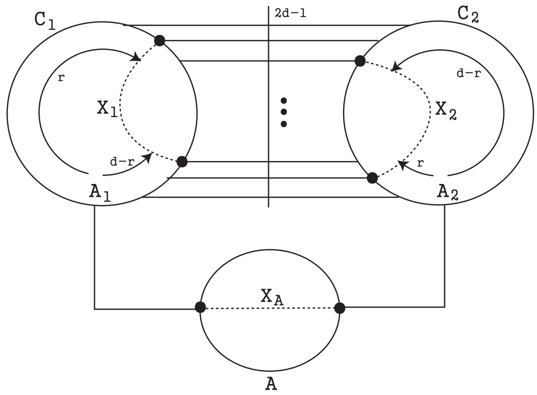

FIGURE 6. The crystallization $\Gamma(d, r)$ and its three axes

and X_2 are bridges and the edges of $E \cup D \cup X_A$ give only undercrossings. We can easily see that the knot K is the 2-bridge knot $\mathfrak{b}(d, 2r)$ from the Schubert diagram. □

Corollary 3.2. *$D(a, 0, 1, (2a+1) - r)$ is the mirror image of $D(a, 0, 1, r)$ for two 4-tuples $(a, 0, 1, r)$ and $(a, 0, 1, (2a+1) - r)$ in $\mathcal{D}$.*

Proof. By Theorem 3.1, $D(a, 0, 1, r)$ is the 2-bridge knot $\mathfrak{b}(2a + 1, 2r)$ and $D(a, 0, 1, (2a+1) - r)$ is the 2-bridge knot $\mathfrak{b}((2a+1), 2(2a+1) - 2r)$. Now we just note that $\mathfrak{b}((2a+1), 2(2a+1) - 2r) = \mathfrak{b}((2a+1), -2r)$ is the mirror image of $\mathfrak{b}(2a + 1, 2r)$. □

By Theorem 3.1 and the fact that the 2-fold cyclic branched covering of $\mathbb{S}^3$ branched over a unique knot $\mathfrak{b}(2a + 1, 2r)$ is the lens space $L(2a + 1, 2r)$, we have the following.

Corollary 3.3. *$D_2(a, 0, 1, r)$ is homeomorphic to $D_2(a', 0, 1, r')$ if and only if $a = a'$ and $4rr' \equiv \pm 1 \mod(2a + 1)$.*

For a given 2-bridge knot K in $\mathbb{S}^3$, the $(0, 2)$-decomposition of $(\mathbb{S}^3, K)$ is the Heegaard splitting of genus zero $B_1 \cup_P B_2$ such that $B_1 \cap K = b_1 \cup b_2$ and $B_2 \cap K = b_3 \cup b_4$, where $b_1 \cup b_2$ and $b_3 \cup b_4$ are properly embedded trivial arcs in B_1 and B_2, respectively and $P : (\partial B_2, \partial(b_3 \cup b_4)) \to (\partial B_1, \partial(b_1 \cup b_2))$ is an attaching homeomorphism. Then b_1, b_2, b_3, b_4 are the closures of the components of $K - \partial B_1$ or $K - \partial B_2$. Let $V_1 = B_1 \cup N(b_3, B_2)$, $K_1 = b_1 \cup b_3 \cup b_2$, $V_2 = Cl(B_2 \setminus N(b_3, B_2))$, $K_2 = b_4$. Then each V_i is a solid torus and K_i is a trivial arc in V_i for $i = 1, 2$. Hence $(\mathbb{S}^3, K)$ admits the $(1,1)$-decomposition $(V_1, K_1) \cup_\phi (V_2, K_2)$, where $\phi : (\partial V_2, \partial K_2) \to (\partial V_1, \partial K_1)$ is an attaching homeomorphism extended by P. Moreover, by using b_1, b_2, b_4 for b_3, we can obtain other three $(1,1)$-decompositions of $(\mathbb{S}^3, K)$. Thus there are four $(1,1)$-decompositions of $(\mathbb{S}^3, K)$ for any 2-bridge knot K in $\mathbb{S}^3$.

Let K be a 2-bridge knot $\mathfrak{b}(2a+1, 2r)$ in $\mathbb{S}^3$. Then there is a $(1,1)$-decomposition $D(a, 0, 1, r)$ of $(\mathbb{S}^3, K) = (V_1, K_1) \cup_\phi (V_2, K_2)$ such that $K(a, 0, 1, r)$ represents the 2-bridge knot K by Theorem 3.1. Since $D(a, 0, 1, r)$ is a weakly K-reducible, there exist a meridian disk D_1 in V_1 meeting with K_1 at a single point and a K_2-cancelling disk D_2 properly embedded in V_2 such that $\partial D_1 \cap \partial D_2 = \emptyset$ and $|\partial D_1 \cap l| = 1$ as examples in Figure 7. In fact, Figure 7 depicts $D(2, 0, 3, 3)$ and its dual decomposition $D(3, 0, 1, 2)$ representing a 2-bridge knot $\mathfrak{b}(7, 4)$ and their K-compressing disks. Let m and l be the meridian and longitude curves

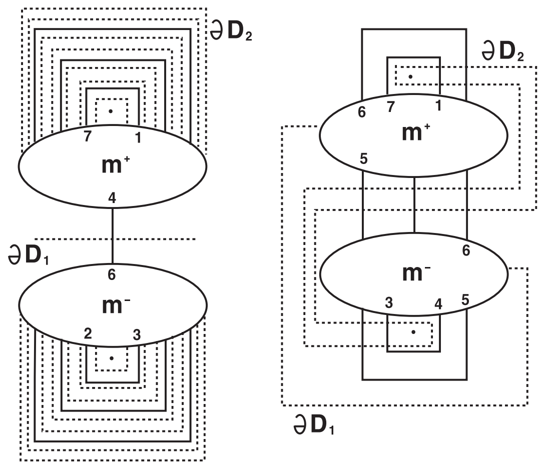

FIGURE 7. K-compressing disks of $D(3, 0, 1, 2)$ and its dual $D(2, 0, 3, 3)$

in V_1 determined by β and $\alpha\beta$, respectively, as in Figure 1. Then, in Figure 8, if $b = 1$, ∂D_1 meets with a part line $(a+1, r-a)$ of l at a single point and is disjoint with m. By duality, let m' and l' be meridian and longitude in V_2 corresponding to l and m in V_1, respectively, by attaching map ϕ. Then, on $Du(a, b, c, r)$, $\partial\phi(D_1)$ meets with m' at a single point and is disjoint with l'. Therefore we have $\partial\phi(D_1) \cap \partial\phi(D_2) = \emptyset$, and so $Du(a, 0, 1, r)$ is a weakly K-reducible. Note that $D(a, 0, c, r)$ is equal to $D(a, c, 0, r)$ for each $c \geq 1$, up to isotopy moves. In the following theorem, we only consider $D(a, b, c, r)$ with $c \neq 0$.

Theorem 3.4. *Let $D(a, b, c, r)$ be a $(1,1)$-decomposition represent a 2-bridge knot and $D(a', b', c', r')$ the its dual decomposition. Then* $\{b = 0 \text{ and } b' = 0\}$ *and* $\{c = 1 \text{ or } c' = 1\}$.

Theorem 3.5. *For any 2-bridge knot K in $\mathbb{S}^3$, there exist at most four $(1,1)$-decompositions of $(\mathbb{S}^3, K)$ determined by four parameters in $\mathcal{D}$.*

Proof. Let K be the 2-bridge knots $\mathfrak{b}(2a+1, 2r)$. We note that if $\mathfrak{b}(2a+1, 2r)$ is amphicheiral, then $\mathfrak{b}(2a+1, 2r) = \mathfrak{b}(2a+1, -2r)$ or $2r(-2r) \equiv -4r^2 \equiv$

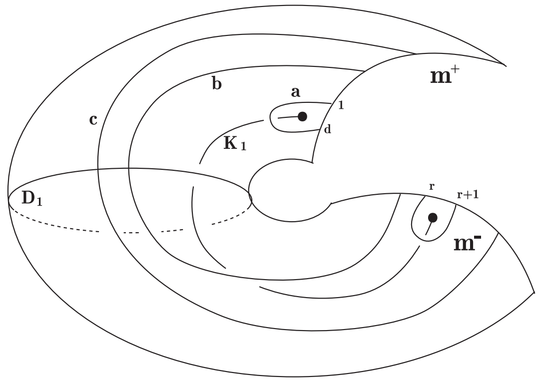

FIGURE 8. A meridian disk D_1 and (V_1, K_1)

$1 \mod(2a+1)$. If $\mathfrak{b}(2a+1, 2r)$ is not amphicheiral, then $2r$ has a unique inverse $2s \neq 2r$. That is, $2r\cdot 2s \equiv 1 \mod 2a+1$ in a unit group of $\mathbb{Z}_{2a+1}$. By Theorem 3.1 $D(a,0,1,r)$ and $D(a,0,1,s)$ represent $\mathfrak{b}(2a+1, 2r)$. Thus we have four $(1,1)$-decompositions for $\mathfrak{b}(2a+1, 2r)$, namely $D(a,0,1,r)$, $Du(a,0,1,r)$, $D(a,0,1,s)$, and $Du(a,0,1,s)$. If $\mathfrak{b}(2a+1, 2r)$ is amphicheiral, $\mathfrak{b}(2a+1, 2r) = \mathfrak{b}(2a+1, 2(2a+1)-2r)$. Then by Theorem 3.1 $D(a,0,1,r)$ and $D(a,0,1,(2a+1)-r)$ represents $\mathfrak{b}(2a+1, 2r)$. Thus we have four $(1,1)$-decompositions for $\mathfrak{b}(2a+1, 2r)$, namely $D(a,0,1,r)$, $Du(a,0,1,r)$, $D(a,0,1,(2a+1)-r)$, and $Du(a,0,1,(2a+1)-r)$. □

We note that the result of Theorem 3.5 has been shown in [16] by using Heegaard splittings of the exteriors of 2-bridge knots. As an application of Theorem 3.4, we now determine the $(1,1)$-decompositions for all 2-bridge knots of type $\mathfrak{b}(11, 6)$. We note that 6 has the inverse 2 in a unit group of $\mathbb{Z}_{11}$. That is, $6 \cdot 2 \equiv 1 \mod 11$. By Theorem 3.1 $D(5,0,1,3)$ and $D(5,0,1,1)$ represent $\mathfrak{b}(11,6)$. Thus we have four $(1,1)$-decompositions for $\mathfrak{b}(11,6)$, namely $D(5,0,1,3)$, $Du(5,0,1,3) = D(2,0,7,7)$, $D(5,0,1,1)$, and $Du(5,0,1,1) = D(5,0,1,10)$.

Corollary 3.6. *Let $\mathfrak{b}(2a+1, 2r)$ be a 2-bridge knot. Then all 4-tuples $(a,0,c,r)$ in $\mathcal{D}$ representing $\mathfrak{b}(2a+1, 2r)$ have the same value $2a+c = 2a+1$.*

Proof. By Theorem 3.1, there are at most four 4-tuples $(a,0,c,r) \in \mathcal{D}$ representing $\mathfrak{b}(2a+1, 2r)$. Here we just note that $D(a,0,c,r)$ and $Du(a,0,c,r)$ have the same value $2a+c = 2a+1$ by Theorem 2.2. □

Corollary 3.6 shows that all 4-tuples $(a,0,c,r)$ in $\mathcal{D}$ representing $\mathfrak{b}(2a+1, 2r)$ have the same value $2a+c = 2a+1$. We propose a conjecture that all 4-tuples (a,b,c,r) in $\mathcal{D}$ representing a given $(1,1)$-knot in the lens space has the same value $2a+b+c$.

Remark. In [14], we denote that numbers $b+c$, d and p have the same parities and so the number $d = 2a+b+c$ of the 6-tuple (d, a, b, c, r, s) or 4-tuple (a, b, c, r) in the paper should be assumed as odd type.

References

[1] Kh. Aydin, I. Gultekyn, and M. Mulazzani, *Torus knots and Dunwoody manifolds*, Siberian Math. J. **45** (2004), no. 1, 1–6.

[2] J. Berge, *The knots in $D^2 \times S^1$ which have nontrivial Dehn surgeries that yield $D^2 \times S^1$*, Topology Appl. **38** (1991), no. 1, 1–19.

[3] M. R. Casall and L. Grasselli, *2-symmetric crystallizations and 2-fold branched coverings of S^3*, Discrete Math. **87** (1991), no. 1, 9–22.

[4] A. Cattabriga and M. Mulazzani, *All strongly-cyclic branched coverings of $(1,1)$-knots are Dunwoody manifolds*, J. London Math. Soc. (2) **70** (2004), no. 2, 512–528.

[5] A. Cavicchioli, B. Ruini, and F. Spaggiari, *On a conjecture of M. J. Dunwoody*, Algebra Colloq. **8** (2001), no. 2, 169–218.

[6] M. J. Dunwoody, *Cyclic presentations and 3-manifolds*, Groups—Korea '94 (Pusan), 47–55, de Gruyter, Berlin, 1995.

[7] M. Ferry, *Crystallizations of 2-fold branched coverings of S^3*, Proc. Amer. Math. Soc., Vol.73 (1979), 271–276.

[8] H. Fujii, *Geometric indices and the Alexander polynomial of a knot*, Proc. Amer. Math. Soc. **124** (1996), no. 9, 2923–2933.

[9] D. Gabai, *Surgery on knots in solid tori*, Topology **28** (1989), no. 1, 1–6.

[10] ———, *1-bridge braids in solid tori*, Topology Appl. **37** (1990), no. 3, 221–235.

[11] C. Gagliardi, *Extending the concept of genus to dimension n*, Proc. Amer. Math. Soc. **81** (1981), no. 3, 473–481.

[12] L. Grasselli and M. Mulazzani, *Genus one 1-bridge knots and Dunwoody manifolds*, Forum Math. **13** (2001), no. 3, 379–397.

[13] C. Hayashi, *Genus one 1-bridge positions for the trivial knot and cabled knots*, Math. Proc. Cambridge Philos. Soc. **125** (1999), no. 1, 53–65.

[14] S. H. Kim, *On spatial theta-curves with the same $(Z_2 \oplus Z_2)$-fold and 2-fold branched covering*, Note Mat. **23** (2004/05), no. 1, 111–122.

[15] S. H. Kim and Y. Kim, *Torus knots and 3-manifolds*, J. Knot Theory Ramifications **13** (2004), no. 8, 1103–1119.

[16] T. Kobayashi, *Classification of unknotting tunnels for two bridge knots*, Proceedings of the Kirbyfest (Berkeley, CA, 1998), 259–290 (electronic), Geom. Topol. Monogr., 2, Geom. Topol. Publ., Coventry, 1999.

[17] K. Morimoto and M. Sakuma, *On unknotting tunnels for knots*, Math. Ann. **289** (1991), no. 1, 143–167.

[18] K. Morimoto, M. Sakuma, and Y. Yokota, *Examples of tunnel number one knots which have the property "$1+1=3$"*, Math. Proc. Cambridge Philos. Soc. **119** (1996), no. 1, 113–118.

[19] ———, *Identifying tunnel number one knots*, J. Math. Soc. Japan **48** (1996), no. 4, 667–688.

[20] M. Mulazzani, *Cyclic presentations of groups and cyclic branched coverings of $(1,1)$-knots*, Bull. Korean Math. Soc. **40** (2003), no. 1, 101–108.

[21] H. J. Song and S. H. Kim, *Dunwoody 3-manifolds and $(1,1)$-decomposiable knots*, Proc. Workshop in pure Math. (edited by Jongsu Kim and Sungbok Hong), Vol. 19 (2000), 193–211.

[22] Y. Q. Wu, *Incompressibility of surfaces in surgered 3-manifolds*, Topology **31** (1992), no. 2, 271–279.

[23] ______, *∂-reducing Dehn surgeries and* 1*-bridge knots*, Math. Ann. **295** (1993), no. 2, 319–331.
[24] ______, *Incompressible surfaces and Dehn surgery on* 1*-bridge knots in handlebodies*, Math. Proc. Cambridge Philos. Soc. **120** (1996), no. 4, 687–696.

Soo Hwan Kim
Department of Mathematics
Dong-eui University
Pusan 614-714, Korea
E-mail address: sootopology@hanmail.net

Yangkok Kim
Department of Mathematics
Dong-eui University
Pusan 614-714, Korea
E-mail address: ykkim@deu.ac.kr