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SUBCLASSES OF k-UNIFORMLY CONVEX AND k-STARLIKE FUNCTIONS DEFINED BY SĂLĂGEAN OPERATOR

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Abstract. The main object of this paper is to introduce and investigate new subclasses of normalized analytic functions in the open unit disc $\mathbb{U}$, which generalize the familiar class of k-starlike functions. The various properties and characteristics for functions belonging to these classes derived here include (for example) coefficient inequalities, distortion theorems involving fractional calculus, extreme points, integral operators and integral means inequalities.

1. Introduction

Let $\mathcal{A}$ denote the class of functions f normalized by

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j, \tag{1.1}$$

which are *analytic* in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of functions which are *univalent* in $\mathbb{U}$. Suppose also that, for $0 \leq \alpha < 1$, $\mathcal{ST}(\alpha)$ and $\mathcal{CV}(\alpha)$ denote the classes of functions in $\mathcal{A}$ which are, respectively, starlike of order α and convex of order α in $\mathbb{U}$.

Sălăgean [10] introduced the following operator which is popularly known as the *Sălăgean derivative operator*:

$$D^0 f(z) = f(z), \quad D^1 f(z) = Df(z) = zf'(z)$$

and, in general,

$$D^n f(z) = D(D^{n-1} f(z)) \quad (n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\};\ \mathbb{N} = \{1, 2, 3, \ldots\}).$$

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We easily find from (1.1) that

$$D^n f(z) = z + \sum_{j=2}^{\infty} j^n a_j z^j \qquad (f \in \mathcal{A}\ ;\ n \in \mathbb{N}_0).$$

By $k - \mathcal{UCV}$, $0 \leq k < \infty$, we denote the class of all k-uniformly convex functions introduced in [4]. Recall that a function $f \in \mathcal{S}$ is said to be k-uniformly convex in $\mathbb{U}$, if the image of every circular arc contained in $\mathbb{U}$ with center at ζ, where $|\zeta| \leq k$, is convex. Note that the class $1 - \mathcal{UCV}$ coincides with the class $\mathcal{UCV}$ of uniformly convex functions, introduced in [2]. Moreover, for $k = 0$ we get the class of all convex univalent functions. For more details on uniformly type function see also [3]. It is known that $f \in k - \mathcal{UCV}$ if and only if it satisfies the following condition:

$$\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > k\left|\frac{zf''(z)}{f'(z)}\right| \quad z \in \mathbb{U},\ 0 \leq k < \infty.$$

For $k = 1$ we get one-variable characterization of $\mathcal{UCV}$ obtained in [7], and independently in [9].

We consider the class $k - \mathcal{ST}$, $0 \leq k < \infty$, of k-starlike functions (see [5]) which are associated with k-uniformly convex functions by the relation

$$f \in k - \mathcal{UCV} \Leftrightarrow zf' \in k - \mathcal{ST}.$$

Thus, the class $k - \mathcal{ST}, 0 \leq k < \infty$ is the subfamily of $\mathcal{S}$, consisting of functions that satisfy the analytic condition

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > k\left|\frac{zf'(z)}{f(z)} - 1\right| \quad z \in \mathbb{U}.$$

For our present investigation, we need the following two-variable analytic characterization of the classes of k-uniformly convex and k-starlike functions.

Theorem 1.1 ([4]). *Let $f \in \mathcal{S}$ and $0 \leq k < \infty$. Then $f \in k - \mathcal{UCV}$ if and only if*

$$\mathrm{Re}\left\{1 + \frac{(z-\zeta)f''(z)}{f'(z)}\right\} \geq 0 \quad z \in \mathbb{U},\ |\zeta| \leq k.$$

Now, from the Alexander type relation, we immediately get:

Theorem 1.2. *Let $f \in \mathcal{S}$ and $0 \leq k < \infty$. Then $f \in k - \mathcal{ST}$ if and only if*

$$\mathrm{Re}\left\{\frac{\zeta}{z} + \frac{(z-\zeta)f'(z)}{f(z)}\right\} \geq 0 \quad z \in \mathbb{U}, |\zeta| \leq k.$$

For $|\zeta| \leq k$ $(0 \leq k < \infty)$, $\alpha (0 \leq \alpha < 1)$, $m \in \mathbb{N}$, $n \in \mathbb{N}_0$, $m > n$ and $z \in \mathbb{U}$, let $\mathcal{ST}(k, m, n, \alpha)$ denote the family of analytic functions f of the form (1.1) such that

$$\mathrm{Re}\left\{\frac{\zeta}{z} + \frac{(z-\zeta)D^m f(z)}{zD^n f(z)}\right\} \geq \alpha, \tag{1.2}$$

where D^m is denoted the Sălăgean derivative operator.

If we set $m = 1$, $n = 0$ and $\alpha = 0$ in (1.2), then $\mathcal{ST}(k, m, n, \alpha)$ reduces to the function class of k-starlike functions. On the other hand, if we choose $m = 2$, $n = 1$ and $\alpha = 0$, then we obtain the function class of k-uniformly convex functions.

Let $\mathcal{T}$ denote the subclass of $\mathcal{A}$ whose Taylor-Maclaurin expansion about $z = 0$ can be expressed in the following form:

$$f(z) = z - \sum_{j=2}^{\infty} a_j z^j, \;\; a_j \geq 0. \tag{1.3}$$

We shall denote by $\overline{\mathcal{ST}}(k, m, n, \alpha)$, the subclass of the functions in $\mathcal{ST}(k,m, n,\alpha)$ that has their non-zero Taylor-Maclaurin coefficients, from second onwards, all negative. Thus, we can write

$$\overline{\mathcal{ST}}(k, m, n, \alpha) = \mathcal{ST}(k, m, n, \alpha) \cap \mathcal{T}.$$

In this paper, the coefficient conditions for the class $k - \mathcal{ST}$ are extended to the class $\mathcal{ST}(k, m, n, \alpha)$ of the forms (1.2) above. Furthermore, we determine distortion theorems, extreme points, integral means inequalities and integral operators for the functions belong to the class $\overline{\mathcal{ST}}(k, m, n, \alpha)$.

2. Coefficient inequalities

In our first theorem, we introduce a sufficient coefficient bound for analytic functions in $\mathcal{ST}(k, m, n, \alpha)$.

Theorem 2.1. *If $f(z) \in \mathcal{A}$ satisfies the following inequality:*

$$\sum_{j=2}^{\infty} [(1 + k)j^m - (k + \alpha)j^n] |a_j| \leq 1 - \alpha \tag{2.1}$$

$$(0 \leq \alpha < 1, \; j \in \mathbb{N} \setminus \{1\}, \; m \in \mathbb{N}, \; n \in \mathbb{N} \cup \{0\}, \; m > n)$$

then $f(z) \in \mathcal{ST}(k, m, n, \alpha)$.

Proof. Suppose that (2.1) is true for $0 \leq \alpha < 1$, $j \in \mathbb{N} \setminus \{1\}$, $m \in \mathbb{N}$, $n \in \mathbb{N} \cup \{0\}$, $m > n$. For $f(z) \in \mathcal{A}$, let us define the function $F(z)$ by

$$F(z) = \frac{\zeta}{z} + \frac{(z - \zeta)D^m f(z)}{zD^n f(z)} - \alpha.$$

It suffices to show that

$$\left| \frac{F(z) - 1}{F(z) + 1} \right| < 1 \qquad (z \in \mathbb{U}).$$

We note that

$$\left| \frac{F(z) - 1}{F(z) + 1} \right|$$

$$= \left| \frac{\frac{\zeta}{z} + \frac{(z-\zeta)D^m f(z)}{zD^n f(z)} - \alpha - 1}{\frac{\zeta}{z} + \frac{(z-\zeta)D^m f(z)}{zD^n f(z)} - \alpha + 1} \right|$$

$$= \left| \frac{\zeta D^n f(z) + (z-\zeta)D^m f(z) - (1+\alpha)zD^n f(z)}{\zeta D^n f(z) + (z-\zeta)D^m f(z) + (1-\alpha)zD^n f(z)} \right|$$

$$= \left| \frac{\alpha + \zeta \sum_{j=2}^{\infty}(j^m - j^n)a_j z^{j-2} - \sum_{j=2}^{\infty}[j^m - (1+\alpha)j^n]a_j z^{j-1}}{(2-\alpha) - \zeta \sum_{j=2}^{\infty}(j^m - j^n)a_j z^{j-2} + \sum_{j=2}^{\infty}[j^m + (1-\alpha)j^n]a_j z^{j-1}} \right|$$

$$\leq \frac{\alpha + |\zeta| \sum_{j=2}^{\infty}(j^m - j^n)|a_j||z|^{j-2} + \sum_{j=2}^{\infty}[j^m - (1+\alpha)j^n]|a_j||z|^{j-1}}{(2-\alpha) - |\zeta| \sum_{j=2}^{\infty}(j^m - j^n)|a_j||z|^{j-2} - \sum_{j=2}^{\infty}[j^m + (1-\alpha)j^n]|a_j||z|^{j-1}}$$

$$< \frac{\alpha + k \sum_{j=2}^{\infty}(j^m - j^n)|a_j| + \sum_{j=2}^{\infty}[j^m - (1+\alpha)j^n]|a_j|}{(2-\alpha) - k \sum_{j=2}^{\infty}(j^m - j^n)|a_j| - \sum_{j=2}^{\infty}[j^m + (1-\alpha)j^n]|a_j|}.$$

The last expression is bounded above by 1, if

$$\alpha + k \sum_{j=2}^{\infty}(j^m - j^n)|a_j| + \sum_{j=2}^{\infty}(j^m - (1+\alpha)j^n)|a_j|$$
$$\leq (2-\alpha) - k \sum_{j=2}^{\infty}(j^m - j^n)|a_j| - \sum_{j=2}^{\infty}(j^m + (1-\alpha)j^n)|a_j|$$

which is equivalent to our condition (2.1). This completes the proof of our theorem. □

Remark 1. If we set $m = 1$, $n = 0$ and $\alpha = 0$, in Theorem 2.1 above, we obtain the coefficient inequality for the class $k - \mathcal{ST}$, which is given in [5].

In the following theorem it is shown that the condition (2.1) is also necessary for functions $f(z)$ of the form (1.3) to be in the class $f \in \overline{\mathcal{ST}}(k, m, n, \alpha)$.

Theorem 2.2. *Let $f(z) \in \mathcal{T}$. Then $f(z) \in \overline{\mathcal{ST}}(k, m, n, \alpha)$ if and only if*

$$\sum_{j=2}^{\infty} [(1+k)j^m - (k+\alpha)j^n]\, a_j \leq 1 - \alpha \tag{2.2}$$

for some $0 \leq \alpha < 1$, $j \in \mathbb{N} \setminus \{1\}$, $m \in \mathbb{N}$, $n \in \mathbb{N}_0$ $(m > n)$.

Proof. Since $\overline{\mathcal{ST}}(k,m,n,\alpha) \subset \mathcal{ST}(k,m,n,\alpha)$, we only need to prove the "only if" part of the theorem. For functions $f(z) \in \mathcal{T}$, we note that the condition

$$\operatorname{Re}\left\{\frac{\zeta}{z} + \frac{(z-\zeta)D^m f(z)}{zD^n f(z)}\right\} \geq \alpha$$

is equivalent to

$$\operatorname{Re}\left\{\frac{1-\zeta\sum_{j=2}^{\infty} j^n a_j z^{j-2} - (z-\zeta)\sum_{j=2}^{\infty} j^m a_j z^{j-2}}{1-\sum_{j=2}^{\infty} j^n a_j z^{j-1}}\right\} \geq \alpha.$$

If we choose z and ζ real and letting $z \to 1^-$ and $\zeta \to -k^+$, we have,

$$\frac{1+k\sum_{j=2}^{\infty} j^n a_j - (1+k)\sum_{j=2}^{\infty} j^m a_j}{1-\sum_{j=2}^{\infty} j^n a_j} \geq \alpha$$

or

$$\sum_{j=2}^{\infty} \left[(1+k)j^m - (k+\alpha)j^n\right] a_j \leq 1-\alpha$$

which is equivalent to (2.2). And so the proof is complete. □

3. Distortion theorems involving fractional calculus

In this section, we shall prove several distortion theorems for functions to general class $\overline{\mathcal{ST}}(k,m,n,\alpha)$. Each of these theorems would involve certain operators of fractional calculus (that is, fractional integrals and fractional derivatives) which are defined as follows (see, for details [8, 12]).

Definition 3.1. The fractional integral of order δ is defined, for a function f, by

$$D_z^{-\delta} f(z) = \frac{1}{\Gamma(\delta)} \int_0^z \frac{f(\xi)}{(z-\xi)^{1-\delta}} d\xi\,, \ \ \delta > 0 \tag{3.1}$$

where f is an analytic function in a simply-connected region of z-plane containing the origin, and the multiplicity of $(z-\xi)^{\delta-1}$ is removed by requiring, $\log(z-\xi)$ to be real when $z-\xi > 0$.

Definition 3.2. The fractional derivative of order δ is defined, for a function f, by

$$D_z^{\delta} f(z) = \frac{1}{\Gamma(1-\delta)} \frac{d}{dz} \int_0^z \frac{f(\xi)}{(z-\xi)^{\delta}} d\xi\,, \ \ \ 0 \leq \delta < 1 \tag{3.2}$$

where f is constrained, and the multiplicity of $(z-\xi)^{-\delta}$ is removed, as in Definition 3.1.

Definition 3.3. Under the hypotheses of Definition 3.2, the fractional derivative of order $(n+\delta)$ is defined by

$$D_z^{n+\delta} f(z) = \frac{d^n}{dz^n} D_z^{\delta} f(z),$$

where $0 \leq \delta < 1$ and $n \in \mathbb{N}_0$.

By virtue of Definitions 3.1, 3.2 and 3.3, we have

$$D_z^{-\delta} z^j = \frac{\Gamma(j+1)}{\Gamma(j+\delta+1)} z^{j+\delta} \qquad (j \in \mathbb{N}, \delta > 0)$$

and

$$D_z^{\delta} z^j = \frac{\Gamma(j+1)}{\Gamma(j-\delta+1)} z^{j-\delta} \qquad (j \in \mathbb{N}, 0 \leq \delta < 1).$$

Theorem 3.1. *Let the function $f(z)$ given by* (1.3) *be in the class $\overline{\mathcal{ST}}(k,m,n,\alpha)$. Then*

$$\left|D_z^{-\delta} f(z)\right| \geq \frac{|z|^{1+\delta}}{\Gamma(2+\delta)} \left\{1 - \frac{1-\alpha}{(2+\delta)\left[(1+k)2^{m-1} - (k+\alpha)2^{n-1}\right]} |z|\right\}$$

and

$$\left|D_z^{-\delta} f(z)\right| \leq \frac{|z|^{1+\delta}}{\Gamma(2+\delta)} \left\{1 + \frac{1-\alpha}{(2+\delta)\left[(1+k)2^{m-1} - (k+\alpha)2^{n-1}\right]} |z|\right\}$$

for $\delta > 0$ and $z \in \mathbb{U}$. Each of these results are sharp.

Proof. Let

$$\begin{aligned} F(z) &= \Gamma(2+\delta) z^{-\delta} D_z^{-\delta} f(z) \\ &= z - \sum_{j=2}^{\infty} \frac{\Gamma(j+1)\Gamma(2+\delta)}{\Gamma(j+1+\delta)} a_j z^j \\ &= z - \sum_{j=2}^{\infty} \Psi(j) a_j z^j, \end{aligned}$$

where

$$\Psi(j) = \frac{\Gamma(j+1)\Gamma(2+\delta)}{\Gamma(j+1+\delta)} \qquad (j \in \mathbb{N} \setminus \{1\}).$$

Since $\Psi(j)$ is a decreasing function of j, we can write

$$0 < \Psi(j) \leq \Psi(2) = \frac{2}{2+\delta}. \tag{3.3}$$

Furthermore, in view of Theorem 2.2, we have

$$\left[(1+k)2^m - (k+\alpha)2^n\right] \sum_{j=2}^{\infty} a_j \leq \sum_{j=2}^{\infty} \left[(1+k)j^m - (k+\alpha)j^n\right] a_j \leq 1-\alpha,$$

which evidently yields

$$\sum_{j=2}^{\infty} a_j \leq \frac{1-\alpha}{(1+k)2^m-(k+\alpha)2^n}. \tag{3.4}$$

Therefore by using (3.3) and (3.4), we can see that

$$|F(z)| \geq |z| - \Psi(2)|z|^2 \sum_{j=2}^{\infty} a_j \geq |z| - \frac{1-\alpha}{(2+\delta)[(1+k)2^{m-1}-(k+\alpha)2^{n-1}]}|z|^2$$

and

$$|F(z)| \leq |z| + \Psi(2)|z|^2 \sum_{j=2}^{\infty} a_j \leq |z| + \frac{1-\alpha}{(2+\delta)[(1+k)2^{m-1}-(k+\alpha)2^{n-1}]}|z|^2$$

which proves Theorem 3.1.

Finally, since the equalities are attained for the function $f(z)$ defined by

$$D_z^{-\delta} f(z) = \frac{z^{1+\delta}}{\Gamma(2+\delta)} \left\{1 - \frac{1-\alpha}{(2+\delta)[(1+k)2^{m-1}-(k+\alpha)2^{n-1}]} z\right\}$$

or, equivalently, by

$$f(z) = z - \frac{1-\alpha}{(1+k)2^m-(k+\alpha)2^n} z^2, \tag{3.5}$$

our proof of Theorem 3.1 is completed. □

Corollary 3.1. *Under the hypothesis of Theorem 3.1, $D_z^{-\delta} f(z)$ is included in a disc with its center at the origin and radius r_1 given by*

$$r_1 = \frac{1}{\Gamma(2+\delta)} \left\{1 - \frac{1-\alpha}{(2+\delta)\,[(1+k)2^{m-1}-(k+\alpha)2^{n-1}]}\right\}.$$

Theorem 3.2. *Let the function $f(z)$ given by (1.3) be in the class $\overline{\mathcal{ST}}(k,m,n,\alpha)$. Then*

$$\left|D_z^{\delta} f(z)\right| \geq \frac{|z|^{1-\delta}}{\Gamma(2-\delta)} \left\{1 - \frac{1-\alpha}{(2-\delta)\,[(1+k)2^{m-1}-(k+\alpha)2^{n-1}]}|z|\right\}$$

and

$$\left|D_z^{\delta} f(z)\right| \leq \frac{|z|^{1-\delta}}{\Gamma(2-\delta)} \left\{1 + \frac{1-\alpha}{(2-\delta)\,[(1+k)2^{m-1}-(k+\alpha)2^{n-1}]}|z|\right\}$$

for $0 \leq \delta < 1$ and $z \in \mathbb{U}$. Equalities are attained for the function $f(z)$ defined by

$$D_z^{\delta} f(z) = \frac{z^{1-\delta}}{\Gamma(2-\delta)} \left\{1 - \frac{1-\alpha}{(2-\delta)\,[(1+k)2^{m-1}-(k+\alpha)2^{n-1}]} z\right\}.$$

Proof. Let

$$\begin{aligned} G(z) &= \Gamma(2-\delta) z^{\delta} D_z^{\delta} f(z) \\ &= z - \sum_{j=2}^{\infty} \frac{\Gamma(j)\Gamma(2-\delta)}{\Gamma(j+1-\delta)} j a_j z^j \\ &= z - \sum_{j=2}^{\infty} \Lambda(j) j a_j z^j, \end{aligned}$$

where

$$\Lambda(j) = \frac{\Gamma(j)\Gamma(2-\delta)}{\Gamma(j+1-\delta)} \qquad (j \in \mathbb{N} \setminus \{1\}).$$

Since $\Lambda(j)$ is a decreasing function of j, we can write

$$0 < \Lambda(j) \leq \Lambda(2) = \frac{1}{2-\delta}. \tag{3.6}$$

Furthermore, in view of Theorem 2.2, we have

$$\left[((1+k)2^{m-1} - (k+\alpha)2^{n-1}\right] \sum_{j=2}^{\infty} j a_j \leq \sum_{j=2}^{\infty} \left[(1+k)j^m - (k+\alpha)j^n\right] a_j \leq 1-\alpha,$$

which evidently yields

$$\sum_{j=2}^{\infty} j a_j \leq \frac{1-\alpha}{(1+k)2^{m-1} - (k+\alpha)2^{n-1}}. \tag{3.7}$$

Therefore by using (3.6) and (3.7), we can see that

$$|G(z)| \geq |z| - \Lambda(2)|z|^2 \sum_{j=2}^{\infty} j a_j \geq |z| - \frac{1-\alpha}{(2-\delta)[(1+k)2^{m-1} - (k+\alpha)2^{n-1}]} |z|^2$$

and

$$|G(z)| \leq |z| + \Lambda(2)|z|^2 \sum_{j=2}^{\infty} j a_j \leq |z| + \frac{1-\alpha}{(2-\delta)[(1+k)2^{m-1} - (k+\alpha)2^{n-1}]} |z|^2$$

which give the inequalities of Theorem 3.2. Since equalities are attained for the function $f(z)$ defined by

$$D_z^{\delta} f(z) = \frac{z^{1-\delta}}{\Gamma(2-\delta)} \left\{ 1 - \frac{1-\alpha}{(2-\delta)\left[(1+k)2^{m-1} - (k+\alpha)2^{n-1}\right]} z \right\}$$

that is, by (3.5). We complete the assertion of Theorem 3.2. □

Corollary 3.2. *Under the hypothesis of Theorem 3.2, $D_z^{\delta} f(z)$ is included in a disc with its center at the origin and radius r_2 given by*

$$r_2 = \frac{1}{\Gamma(2-\delta)} \left\{ 1 - \frac{1-\alpha}{(2-\delta)\left[(1+k)2^{m-1} - (k+\alpha)2^{n-1}\right]} \right\}.$$

4. Extreme points for $\mathcal{ST}(k, m, n, \alpha)$

Theorem 4.1. *Let* $f_1(z) = z$ *and*

$$f_j(z) = z - \frac{1-\alpha}{(1+k)j^m - (k+\alpha)j^n} z^j \qquad (j \in \mathbb{N} \setminus \{1\}). \tag{4.1}$$

Then $f(z) \in \overline{\mathcal{ST}}(k, m, n, \alpha)$ *if and only if it can be expressed in the following form*:

$$f(z) = \sum_{j=1}^{\infty} \lambda_j f_j(z),$$

where $\lambda_j \geq 0$ *and* $\sum_{j=1}^{\infty} \lambda_j = 1$.

Proof. Suppose that

$$f(z) = \sum_{j=1}^{\infty} \lambda_j f_j(z) = z - \sum_{j=2}^{\infty} \lambda_j \frac{1-\alpha}{(1+k)j^m - (k+\alpha)j^n} z^j.$$

Then from Theorem 2.2, we have

$$\begin{aligned} &\sum_{j=2}^{\infty} [(1+k)j^m - (k+\alpha)j^n] \frac{1-\alpha}{(1+k)j^m - (k+\alpha)j^n} \lambda_j \\ &= (1-\alpha) \sum_{j=2}^{\infty} \lambda_j \\ &= (1-\alpha)(1-\lambda_1) \leq 1-\alpha. \end{aligned}$$

Thus, in view of Theorem 2.2, we find that $f(z) \in \overline{\mathcal{ST}}(k, m, n, \alpha)$.

Conversely, suppose that $f(z) \in \overline{\mathcal{ST}}(k, m, n, \alpha)$. Then, since

$$a_j \leq \frac{1-\alpha}{(1+k)j^m - (k+\alpha)j^n} \qquad (j \in \mathbb{N} \setminus \{1\}),$$

we may set

$$\lambda_j = \frac{(1+k)j^m - (k+\alpha)j^n}{1-\alpha} a_j \quad (j \in \mathbb{N} \setminus \{1\})$$

and

$$\lambda_1 = 1 - \sum_{j=2}^{\infty} \lambda_j.$$

Thus, clearly, we have

$$f(z) = \sum_{j=1}^{\infty} \lambda_j f_j(z).$$

This completes the proof of the theorem. □

Corollary 4.1. *The extreme points of the class* $\overline{\mathcal{ST}}(k,m,n,\alpha)$ *are given by*

$$f_1(z) = z$$

and

$$f_j(z) = z - \frac{1-\alpha}{(1+k)j^m - (k+\alpha)j^n} z^j \qquad (j \in \mathbb{N} \setminus \{1\}) .$$

Theorem 4.2. *The class* $\overline{\mathcal{ST}}(k,m,n,\alpha)$ *is a convex set.*

Proof. Suppose that each of the functions $f_i(z)$, $(i = 1, 2)$ given by

$$f_i(z) = z - \sum_{j=2}^{\infty} a_{j,i} z^j \quad (a_{j,i} \geq 0, i = 1, 2) \tag{4.2}$$

is in the class $\overline{\mathcal{ST}}(k,m,n,\alpha)$. It is sufficient to show that the function $g(z)$ defined by

$$g(z) = \eta f_1(z) + (1-\eta) f_2(z) \quad (0 \leq \eta < 1)$$

is also in the class $\overline{\mathcal{ST}}(k,m,n,\alpha)$. Since

$$g(z) = z - \sum_{j=2}^{\infty} [\eta a_{j,1} + (1-\eta) a_{j,2}] z^j,$$

with the aid of Theorem 2.2, we have

$$\begin{aligned} &\sum_{j=2}^{\infty} [(1+k)j^m - (k+\alpha)j^n][\eta a_{j,1} + (1-\eta) a_{j,2}] \\ &= \eta \sum_{j=2}^{\infty} [(1+k)j^m - (k+\alpha)j^n] a_{j,1} + (1-\eta) \sum_{j=2}^{\infty} [(1+k)j^m - (k+\alpha)j^n] a_{j,2} \\ &\leq \eta(1-\alpha) + (1-\eta)(1-\alpha) = 1-\alpha \end{aligned}$$

which implies that $g(z) \in \overline{\mathcal{ST}}(k,m,n,\alpha)$. Hence $\overline{\mathcal{ST}}(k,m,n,\alpha)$ is a convex set. □

5. Integral means inequalities and integral operators

Definition 5.1. For two functions f and g, analytic in $\mathbb{U}$, we say that the function $f(z)$ is subordinate to $g(z)$ in $\mathbb{U}$, and write

$$f(z) \prec g(z) \qquad (z \in \mathbb{U}),$$

if there exists a Schwarz function $w(z)$, analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ such that

$$f(z) = g(w(z)) \qquad (z \in \mathbb{U}).$$

In particular, if the function g is univalent in $\mathbb{U}$, the above subordination is equivalent to

$$f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

See also Duren [1].

In 1925, Littlewood [6] proved the following subordination theorem.

Theorem 5.1 (Littlewood [6])**.** *If f and g are analytic in $\mathbb{U}$ with $f \prec g$, then for $\mu > 0$ and $z = re^{i\theta}(0 < r < 1)$*

$$\int_0^{2\pi} |f(z)|^\mu \, d\theta \leqq \int_0^{2\pi} |g\,(z)|^\mu \, d\theta.$$

We will make use of Theorem 5.1 to prove:

Theorem 5.2. *Let $f(z) \in \overline{\mathcal{ST}}(k, m, n, \alpha)$ and $f_2(z)$ is defined by*

$$f_2(z) = z - \frac{1-\alpha}{(1+k)2^m - (k+\alpha)2^n} z^2.$$

If there exists an analytic function $w(z)$ given by

$$w(z) = \frac{(1+k)2^m - (k+\alpha)2^n}{1-\alpha} \sum_{j=2}^{\infty} a_j z^{j-1},$$

then for $z = re^{i\theta}$ and $0 < r < 1$,

$$\int_0^{2\pi} \left|f(re^{i\theta})\right|^\mu d\theta \leq \int_0^{2\pi} \left|f_2(re^{i\theta})\right|^\mu d\theta \qquad (\mu > 0).$$

Proof. We must show that

$$\int_0^{2\pi} \left|1 - \sum_{j=2}^{\infty} a_j z^{j-1}\right|^\mu d\theta \leq \int_0^{2\pi} \left|1 - \frac{1-\alpha}{(1+k)2^m - (k+\alpha)2^n} z\right|^\mu d\theta.$$

By applying Littlewood's subordination theorem, it would suffice to show that

$$1 - \sum_{j=2}^{\infty} a_j z^{j-1} \prec 1 - \frac{1-\alpha}{(1+k)2^m - (k+\alpha)2^n} z.$$

By setting

$$1 - \sum_{j=2}^{\infty} a_j z^{j-1} = 1 - \frac{1-\alpha}{(1+k)2^m - (k+\alpha)2^n} w(z)$$

we find that

$$w(z) = \frac{(1+k)2^m - (k+\alpha)2^n}{1-\alpha} \sum_{j=2}^{\infty} a_j z^{j-1}$$

which readily yields $w(0) = 0$.

Furthermore, using (3.4), we obtain

$$\begin{aligned}|w(z)| &= \left| \frac{(1+k)2^m - (k+\alpha)2^n}{1-\alpha} \sum_{j=2}^{\infty} a_j z^{j-1} \right| \\ &\le |z| \frac{(1+k)2^m - (k+\alpha)2^n}{1-\alpha} \sum_{j=2}^{\infty} a_j \\ &\le |z| < 1.\end{aligned}$$

This completes the proof of the theorem. □

Definition 5.2 ([11])**.** Let $I_c : \mathcal{A} \to \mathcal{A}$ be the integral operator defined by $f = I_c(F)$, where $c \in (-1, \infty)$, $F \in \mathcal{A}$ and

$$f(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} F(t) dt. \tag{5.1}$$

We note that, if $F \in \mathcal{A}$ is a function of the form (1.3), then

$$f(z) = I_c F(z) = z - \sum_{j=2}^{\infty} \frac{c+1}{c+j} a_j z^j. \tag{5.2}$$

Theorem 5.3. *If* $F(z) = z - \sum_{j=2}^{\infty} a_j z^j \in \overline{\mathcal{ST}}(k, m, n, \alpha)$*, then* $f(z) = I_c F(z) \in \overline{\mathcal{ST}}(k, m, n, \alpha)$*, where* I_c *is the integral operator defined by* (5.1).

Proof. We have $f(z) = z - \sum_{j=2}^{\infty} b_j z^j$, where $b_j = \frac{c+1}{c+j} a_j$, $c \in (-1, \infty), j = 2, 3, \ldots$. Thus $b_j < a_j$, for $j = 2, 3 \ldots$. Using the condition (2.2) for $F(z)$ we obtain

$$\sum_{j=2}^{\infty} [(1+k)j^m - (k+\alpha)j^n] b_j \le \sum_{j=2}^{\infty} [(1+k)j^m - (k+\alpha)j^n] a_j \le 1 - \alpha.$$

This completes our proof. □

Definition 5.3. We consider the integral operator $I_{c+\gamma} : \mathcal{A} \to \mathcal{A}$, $0 < u \le 1$, $1 \le \gamma < \infty$, $0 < c < \infty$, defined by

$$f(z) = I_{c+\gamma}(F(z)) = (c+\gamma) \int_0^1 u^{c+\gamma-2} F(uz) du. \tag{5.3}$$

Remark 2. If $F(z) = z + \sum_{j=2}^{\infty} a_j z^j$, then from (5.3) we obtain

$$f(z) = z + \sum_{j=2}^{\infty} \frac{c+\gamma}{c+j+\gamma-1} a_j z^j.$$

Also, we notice that $0 < \frac{c+\gamma}{c+j+\gamma-1} < 1$, where $0 < c < \infty$, $j \ge 2$, $1 \le \gamma < \infty$.

Remark 3. It is easy to prove that for $F(z) \in \mathcal{T}$ and $f(z) = I_{c+\gamma}(F(z))$, we have $f(z) \in \mathcal{T}$, where $I_{c+\gamma}$ is the integral operator defined by (5.3).

Theorem 5.4. *Let $F(z)$ be in the class $\overline{\mathcal{ST}}(k,m,n,\alpha)$, $F(z) = z - \sum_{j=2}^{\infty} a_j z^j$, $a_j \geq 0$, $j \geq 2$. Then $f(z) = I_{c+\gamma}(F(z)) \in \mathcal{ST}(k,m,n,\alpha)$, where $I_{c+\gamma}$ is the integral operator defined by* (5.3).

Proof. Since $F(z) \in \overline{\mathcal{ST}}(k,m,n,\alpha)$, we have

$$\sum_{j=2}^{\infty}[(1+k)j^m - (k+\alpha)j^n]a_j < 1-\alpha,$$

where $0 \leq k < \infty$, $0 \leq \alpha < 1$, $m \in \mathbb{N}$, $n \in \mathbb{N}_0$ and $m > n$. Let $f(z) = z - \sum_{j=2}^{\infty} b_j z^j$, where

$$b_j = \frac{c+\gamma}{c+\gamma+j-1}a_j \geq 0 \quad \text{and} \quad 0 < \frac{c+\gamma}{c+\gamma+j-1} < 1.$$

Thus we can write

$$[(1+k)j^m - (k+\alpha)j^n]b_j \leq [(1+k)j^m - (k+\alpha)j^n]a_j.$$

Therefore,

$$\sum_{j=2}^{\infty}[(1+k)j^m - (k+\alpha)j^n]b_j \leq \sum_{j=2}^{\infty}[(1+k)j^m - (k+\alpha)j^n]a_j \leq 1-\alpha.$$

This completes our proof. □

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