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RIGIDNESS AND EXTENDED ARMENDARIZ PROPERTY

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Abstract. For a ring endomorphism α of a ring R, Krempa called α a *rigid* endomorphism if $a\alpha(a) = 0$ implies $a = 0$ for $a \in R$, and Hong et al. called R an α-*rigid* ring if there exists a rigid endomorphism α. Due to Rege and Chhawchharia, a ring R is called *Armendariz* if whenever the product of any two polynomials in $R[x]$ over R is zero, then so is the product of any pair of coefficients from the two polynomials. The Armendariz property of polynomials was extended to one of skew polynomials (i.e., α-Armendariz rings and α-skew Armendariz rings) by Hong et al. In this paper, we study the relationship between α-rigid rings and extended Armendariz rings, and so we get various conditions on the rings which are equivalent to the condition of being an α-rigid ring. Several known results relating to extended Armendariz rings can be obtained as corollaries of our results.

Throughout this paper, all rings are associative with identity. Given a ring R, the polynomial ring over R is denoted by $R[x]$. Recall that a ring R is called *reduced* if it has no nonzero nilpotent elements. Armendariz [1, Lemma 1] showed that for a reduced ring R, if any polynomial $f(x) = a_0 + a_1x + \cdots + a_mx^m$, $g(x) = b_0 + b_1x + \cdots + b_nx^n \in R[x]$ satisfies $f(x)g(x) = 0$, then $a_ib_j = 0$ for each i, j. Since then, Rege and Chhawchharia [12] called R an *Armendariz* ring if it satisfies this condition. Many properties of Armendariz rings have been studied by many authors [2, 3, 5, 6, 8, 10, 11, 12].

The reducedness and Armendariz property of a ring were extended as follows. For a ring R with a ring endomorphism $\alpha : R \to R$, a *skew polynomial* ring (also called an *Ore extension of endomorphism type*) $R[x; \alpha]$ of R is the ring obtained by giving the polynomial ring over R with the new multiplication $xr = \alpha(r)x$ for all $r \in R$. Recall that an endomorphism α of a ring R is called *rigid* [9] if $a\alpha(a) = 0$ implies $a = 0$ for $a \in R$, and a ring R is called α-*rigid* [4] if there exists a rigid endomorphism α of R. Note that any rigid endomorphism of a ring is a monomorphism, and α-rigid rings are reduced rings [4, Proposition 5]. On the other hand, the Armendariz property with respect to polynomials was extended to one of skew polynomials. A ring R is called α-*Armendariz* (resp.,

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α-skew Armendariz) [6, Definition 1.1] (resp., [5, Definition]) if for $p(x) = a_0 + a_1x + \cdots + a_mx^m$ and $q(x) = b_0 + b_1x + \cdots + b_nx^n$ in $R[x;\alpha]$, $p(x)q(x) = 0$ implies $a_ib_j = 0$ (resp., $a_i\alpha^i(b_j) = 0$) for all $0 \le i \le m$ and $0 \le j \le n$. It can be easily checked that every subring S with $\alpha(S) \subseteq S$ of an α-Armendariz ring (resp., an α-skew Armendariz ring) is also α-Armendariz (resp., α-skew Armendariz). Note that every α-rigid ring is α-Armendariz [6, Proposition 1.7], and every α-Armendariz ring is α-skew Armendariz [6, Theorem 1.8], but the converses do not hold by [6, Example 1.6] and [6, Example 1.9], respectively. Moreover R is an α-rigid ring if and only if $R[x;\alpha]$ is reduced [5, Proposition 3]. In [3], Chen and Tong showed the relationship between α-rigid rings and α-skew Armendariz rings. Motivated by their results, in this paper, we continue the study of α-Armendariz rings, improving several results in [3] and [6], and moreover we obtain various rings which are equivalent to α-rigid rings. Several known results relating to Armendariz rings can be obtained as corollaries of our results.

In [12, Remark 3.1], Rege and Chhawchharia showed that every $n \times n$ full matrix ring over any ring R is not I_R-Armendariz for $n \ge 2$ where I_R is an identity endomorphism of R. We also know that there exists a 2×2 full (and also upper triangular) matrix ring R with an endomorphism α such that R is not α-Armendariz by [6, Theorem 1.8] and [5, Example 13] in general. Hence, we consider the following.

A ring R can be extended to a ring

$$S_3(R) = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \mid a, b, c, d \in R \right\}$$

and an endomorphism α of R can also be extended to the endomorphism $\bar{\alpha} : S_3(R) \to S_3(R)$ defined by $\bar{\alpha}((a_{ij})) = (\alpha(a_{ij}))$. Recall that the trivial extension $T(R, M) = R \oplus M$ of R by M is isomorphic to the ring of all matrices $\left(\begin{smallmatrix} r & m \\ 0 & r \end{smallmatrix}\right)$, where $r \in R$ and $m \in M$ and the usual matrix operations are used. Hong et al. [6, Proposition 2.1] proved that, if R is an α-rigid ring, then $S_3(R)$ is $\bar{\alpha}$-Armendariz and so the trivial extension $T(R, R)$ of R is $\bar{\alpha}$-Armendariz. Now, we show that these are equivalent. First we state the following lemma.

Lemma 1. *Let α be an endomorphism of a ring R.*

(1) [6, Proposition 1.3(ii)] *If R is an α-Armendariz ring, then α is a monomorphism.*

(2) [6, Proposition 2.4] *R is an α-rigid ring if and only if for each $a \in R$, $\alpha^a(a)a = 0$ implies $a = 0$.*

Theorem 2. *Let α be an endomorphism of a ring R. Then the following are equivalent*:

(1) *R is an α-rigid ring.*

(2) $S_3(R) = \left\{ \left(\begin{smallmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{smallmatrix}\right) \mid a, b, c, d \in R \right\}$ *is an $\bar{\alpha}$-Armendariz ring.*

(3) *The trivial extension $T(R, R)$ of R is an $\bar{\alpha}$-Armendariz ring.*

Proof. Note that $T(R,R)$ is isomorphic to the subring

$$\left\{ \begin{pmatrix} a & b & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} \mid a, b \in R \right\}$$

of a ring $S_3(R)$ and each subring of an α-Armendariz ring is also α-Armendariz. Hence, it is enough to show that (3)$\Rightarrow$(1). Let $T(R,R)$ be $\bar{\alpha}$-Armendariz. Assume on the contrary that R is not α-rigid. By Lemma 1, there exists $0 \neq a \in R$ such that $\alpha(a)a = 0$ and $\alpha(a) \neq 0$. For $p(x) = \left(\begin{smallmatrix} \alpha(a) & 0 \\ 0 & \alpha(a) \end{smallmatrix}\right) + \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) x$, $q(x) = \left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right) + \left(\begin{smallmatrix} 0 & -1 \\ 0 & 0 \end{smallmatrix}\right) x \in T(R,R)[x;\bar{\alpha}]$, we have $p(x)q(x) = 0$, but $\left(\begin{smallmatrix} \alpha(a) & 0 \\ 0 & \alpha(a) \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & -1 \\ 0 & 0 \end{smallmatrix}\right) \neq 0$; which is a contradiction. Thus R is α-rigid. $\square$

If we take α as the identity endomorphism I_R of a ring R, then we have the following corollary which generalizes the results in [8, Proposition 2] and [10, Theorem 2.3].

Corollary 3. *For a ring R, the following are equivalent*:
(1) *R is a reduced ring.*
(2) *$S_3(R) = \left\{ \left(\begin{smallmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{smallmatrix}\right) \mid a, b, c, d \in R \right\}$ is an Armendariz ring.*
(3) *The trivial extension $T(R,R)$ of R is an Armendariz ring.*

Hong et al. [5, p. 261] showed that the ring

$$S_n(R) = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{pmatrix} \mid a, a_{ij} \in R \right\}$$

cannot be $\bar{\alpha}$-Armendariz for $n \geq 4$, even if R is an α-rigid ring. However, we obtain subrings of $S_n(R)$ for $n \geq 4$ which are $\bar{\alpha}$-Armendariz as follows.

From [11], $RA = \{rA \mid r \in R\}$ for any $A \in \mathrm{Mat}_n(R)$ where $\mathrm{Mat}_n(R)$ is the $n \times n$ full matrix ring and for $n \geq 2$, let $V = \sum_{i=1}^{n-1} E_{i(i+1)}$ where E_{ij}'s are the matrix units. For an even number $n = 2k(\geq 2)$, let

$$A_n^e(R) = \sum_{i=1}^{k} \sum_{j=k+i}^{n} RE_{ij}, \quad \text{and} \quad B_n^e(R) = \sum_{i=1}^{k+1} \sum_{j=k+i-1}^{n} RE_{ij};$$

and for an odd number $n = 2k+1(\geq 3)$, let

$$A_n^o(R) = \sum_{i=1}^{k+1} \sum_{j=k+i}^{n} RE_{ij}, \quad \text{and} \quad B_n^o(R) = \sum_{i=1}^{k+2} \sum_{j=k+i-1}^{n} RE_{ij}.$$

In addition, for $n \geq 2$ put
$A_n(R) = RI_n + RV + \cdots + RV^{k-1} + A_n^e(R)$ and
$B_n(R) = RI_n + RV + \cdots + RV^{k-2} + B_n^e(R)$ for $n = 2k$;

$A_n(R) = RI_n + RV + \cdots + RV^{k-1} + A_n^o(R)$ and
$B_n(R) = RI_n + RV + \cdots + RV^{k-2} + B_n^o(R)$ for $n = 2k+1$, where I_n is the unit matrix of $\mathrm{Mat}_n(R)$.

Proposition 4. *Let α be an endomorphism of a ring R. The following are equivalent*:
(1) *R is an α-rigid ring.*
(2) *$A_n(R)$ is an $\bar{\alpha}$-Armendariz ring for $n = 2k+1 \geq 3$.*
(3) *$A_n(R) + RE_{1k}$ is an $\bar{\alpha}$-Armendariz ring for $n = 2k \geq 4$.*
(4) *$V_n(R) = RI_n + RV + RV^2 + \cdots + RV^{n-1}$ is an $\bar{\alpha}$-Armendariz ring for $n \geq 2$.*

Proof. Assume that R is an α-rigid ring. First, let $S = A_n(R)$ if $n = 2k+1 \geq 3$ and $S = A_n(R) + RE_{1k}$ if $n = 2k \geq 4$, and let $P(x) = C_0 + C_1x + \cdots + C_ux^u$ and $Q(x) = D_0 + D_1x + \cdots + D_vx^v$ in $S[x;\bar{\alpha}]$ with $P(x)Q(x) = 0$. We show that $C_iD_j = 0$ for $0 \leq i \leq u, 0 \leq j \leq v$. Let $p_{st}(x) = c_{st}^{(0)} + c_{st}^{(1)}x + \cdots + c_{st}^{(u)}x^u$ and $q_{st}(x) = d_{st}^{(0)} + d_{st}^{(1)}x + \cdots + d_{st}^{(v)}x^v$, where $c_{st}^{(i)}$ and $d_{st}^{(j)}$ are the (s,t)-entries of C_i and D_j, respectively for $0 \leq i \leq u$ and $0 \leq j \leq v$. Then, we can write that $P(x) = (p_{st}(x))$ and $Q(x) = (q_{st}(x))$ for $1 \leq s,t \leq n$ and then $(p_{st}(x))(q_{st}(x)) = 0$ in S. Since R is α-rigid, $R[x;\alpha]$ is reduced by [5, Proposition 3]. From [3, Lemma 2.4], we have $((p_{st}(x))(q_{st}(x)))_{st} = 0$ for $1 \leq s,t \leq n$. So $p_{sl}(x)q_{lt}(x) = 0$ in $R[x;\alpha]$ for $1 \leq l \leq n$. That is,

$$(c_{sl}^{(0)} + c_{sl}^{(1)}x + \cdots + c_{sl}^{(u)}x^u)(d_{lt}^{(0)} + d_{lt}^{(1)}x + \cdots + d_{lt}^{(v)}x^v) = 0.$$

Since R is an α-rigid, R is α-Armendariz by [6, Proposition 1.7]. Hence $c_{sl}^{(i)}d_{lt}^{(j)} = 0$ for $1 \leq s,t,l \leq n$, $0 \leq i \leq u$ and $0 \leq j \leq v$, and so $C_iD_j = (c_{st}^{(i)})(d_{st}^{(j)}) = 0$. Therefore S is $\bar{\alpha}$-Armendariz. This proves that (1) $\Rightarrow$(2) and (1) $\Rightarrow$(3).

Next, assume that R is an α-rigid ring. For $n = 2,3$, $V_n(R)$ is $\bar{\alpha}$-Armendariz by Theorem 2 and for $n \geq 4$, $V_n(R)$ is also $\bar{\alpha}$-Armendariz since $V_n(R)$ is a subring of $A_n(R)$ or $A_n(R) + RE_{1k}$ and $\alpha(V_n(R)) \subseteq V_n(R)$.

The converses follow the proof of Theorem 2, respectively. □

Corollary 5. *The following are equivalent for a ring R.*
(1) *R is a reduced ring.*
(2) *$A_n(R)$ is an Armendariz ring for $n = 2k+1 \geq 3$.*
(3) *$A_n(R) + RE_{1k}$ is an Armendariz ring for $n = 2k \geq 4$.*
(4) *$V_n(R) = RI_n + RV + RV^2 + \cdots + RV^{n-1}$ is an Armendariz ring for $n \geq 2$.*

If we define $\rho : V_n(R) \to R[x]/\langle x^n \rangle$ by $\rho(a_0I_n + a_1V + \cdots + a_{n-1}V^{n-1}) = a_0 + a_1x + \cdots + a_{n-1}x^{n-1} + \langle x^n \rangle$, then ρ is a ring isomorphism, where $\langle x^n \rangle$ is an ideal of $R[x]$ generated by x^n and $n \geq 2$. So we have the following corollary.

Corollary 6 ([6, Proposition 2.4]). *Let α be an endomorphism of a ring R. Then the factor ring $R[x]/\langle x^n\rangle$ is $\bar{\alpha}$-Armendariz if and only if R is an α-rigid ring.*

Remark 7. Observe that $B_n^e(R)$ for $n = 2k \geq 2$, $B_n^o(R)$ for $n = 2k+1 \geq 3$ and $B_n(R)$ for $n \geq 2$ are not $\bar{\alpha}$-Armendariz rings, even though R is an α-Armendariz ring by the same method as in [11, Example 1.1], since the endomorphism α of an α-Armendariz ring R preserves identity by [6, Corollary 1.4(i)].

The following example shows that there exists an Armendariz ring R with an endomorphism α such that R is not α-skew Armendariz.

Example 8. Let $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ where $\mathbb{Z}_2$ is the ring of integers modulo 2. Then R is a commutative reduced ring, and so it is Armendariz. Let $\alpha : R \to R$ be an endomorphism defined by $\alpha((a,b)) = (b,a)$. Then R is not α-skew Armendariz by [5, Example 2], and so R is not α-Armendariz, either by [6, Theorem 1.8].

However, we have the following.

Proposition 9. *Let α be an endomorphism of a ring R. If the skew polynomial ring $R[x;\alpha]$ of R is an Armendariz ring, then R is α-skew Armendariz.*

Proof. Assume that $R[x;\alpha]$ is Armendariz. Let $p(x)q(x) = 0$, where $p(x) = a_0 + a_1x + \cdots + a_mx^m$ and $q(x) = b_0 + b_1x + \cdots + b_nx^n$ in $R[x;\alpha]$. We show that $a_i\alpha^i(b_j) = 0$ for all i,j. Set $f(y) = a_0 + (a_1x)y + \cdots + (a_mx^m)y^m$ and $g(y) = b_0 + (b_1x)y + \cdots + (b_nx^n)y^n$ in $(R[x;\alpha])[y]$. Then $f(y)g(y) = 0$, since $p(x)q(x) = 0$ and y commutes with x. Since $R[x;\alpha]$ is Armendariz, we have $a_ix^ib_jx^j = 0$, and so $a_i\alpha^i(b_j) = 0$ for all i,j. Therefore R is an α-skew Armendariz ring. □

Corollary 10 ([5, Corollary 4]). *Let α be an endomorphism of a ring R. If R is α-rigid, then R is an α-skew Armendariz ring.*

Proof. Let R be an α-rigid ring. Then $R[x;\alpha]$ is reduced by [5, Proposition 3] and so $R[x;\alpha]$ is Armendariz. Therefore R is an α-skew Armendariz ring by Proposition 9. □

Observe that the conclusion of Proposition 9 cannot be replaced by the condition "R is α-Armendariz" by the next example.

Example 11. Let R be the polynomial ring $\mathbb{Z}_2[x]$ over $\mathbb{Z}_2$, the ring of integers modulo 2, and let the endomorphism α of R be defined by $\alpha(f(x)) = f(0)$ for $f(x) \in \mathbb{Z}_2[x]$. Then R is a reduced α-skew Armendariz ring by [5, Example 5], but R is not α-Armendariz by [6, Example 1.9]. Now, we show that $S = R[y;\alpha]$ is an Armendariz ring. Let $f(T) = f_0 + f_1T + \cdots + f_mT^m$ and $g(T) = g_0 + g_1T + \cdots + g_nT^n \in S[T]$ with $f(T)g(T) = 0$. We also let $f_i = \sum_{s=0}^{u_i} f_{i_s}(x)y^s$ and $g_j = \sum_{t=0}^{v_j} g_{j_t}(x)y^t$ where $f_{i_s}(x), g_{j_t}(x) \in \mathbb{Z}_2[x]$ for each $0 \leq i \leq m$ and $0 \leq j \leq n$. Without loss of generality, assume that $f_{i_s}(x) \neq 0$ and $g_{j_t}(x) \neq 0$

in $\mathbb{Z}_2[x]$ for all $1 \leq s \leq u_i$, $0 \leq t \leq v_j$, $0 \leq i \leq m$ and $0 \leq j \leq n$. Then we have the following system of equations:

$$
\begin{aligned}
&(0) \quad f_0 g_0 = 0 \ ; \\
&(1) \quad f_0 g_1 + f_1 g_0 = 0 \ ; \\
&\quad \vdots \\
&(k) \quad f_0 g_k + f_1 g_{k-1} + \cdots + f_{k-1} g_1 + f_k g_0 = 0 \ ; \\
&(k+1) \ f_0 g_{k+1} + f_1 g_k + \cdots + f_k g_1 + f_{k+1} g_0 = 0 \ ; \\
&\quad \vdots \\
&(m+n) \ f_m g_n = 0 \ .
\end{aligned}
$$

We claim that $f_{0_0}(x) = f_{1_0}(x) = \cdots = f_{m_0}(x) = 0$ and each $g_{j_t}(x)$ has no constant term for $0 \leq t \leq v_j$ and $0 \leq j \leq n$. We proceed by induction on $i + j$. Since R is α-skew Armendariz and $f_0 g_0 = 0$ from Eq.(0), we obtain $f_{0_s}(x)\alpha^s(g_{0_t}(x)) = 0$ for all $0 \leq s \leq u_0$ and $0 \leq t \leq v_0$, and so $f_{0_0}(x) = 0$ and $g_{0_t}(0) = 0$ for $0 \leq t \leq v_0$. So each $g_{0_t}(x)$ has no constant term for $0 \leq t \leq v_0$. This proves for $i + j = 0$. Now suppose that our claim is true for $i+j \leq k-1$. By the induction hypothesis and Eq.(k), we get $0 = f_0 g_k + f_k g_0 = (f_{0_1}(x)y + f_{0_2}(x)y^2 + \cdots + f_{0_{u_0}}(x)y^{u_0})(g_{k_0}(x) + g_{k_1}(x)y + \cdots + g_{k_{v_k}}(x)y^{v_k}) + f_{k_0}(x)(g_{0_0}(x) + g_{0_1}(x)y + \cdots + g_{0_{v_0}}(x)y^{v_0}) = f_{k_0}(x)g_{0_0}(x) + [f_{k_0}(x)g_{0_1}(x) + f_{0_1}(x)\alpha(g_{k_0}(x))]y + [f_{k_0}(x)g_{0_2}(x) + f_{0_1}(x)\alpha(g_{k_1}(x)) + f_{0_2}(x)\alpha^2(g_{k_0}(x))]y^2 + \cdots + f_{0_{u_0}}(x)\alpha^{u_0}(g_{k_{v_k}}(x))y^{u_0+v_k}$. Then $f_{k_0}(x) = 0$, and so we have the following:

$$
\begin{aligned}
&\text{(i)} \quad f_{0_1}(x)\alpha(g_{k_0}(x)) = 0 \ ; \\
&\text{(ii)} \quad f_{0_1}(x)\alpha(g_{k_1}(x)) + f_{0_2}(x)\alpha^2(g_{k_0}(x)) = 0 \ ; \\
&\quad \vdots \\
&\text{(iii)} \quad f_{0_1}(x)\alpha(g_{k_{i-1}}(x)) + f_{0_2}(x)\alpha^2(g_{k_{i-2}}(x)) + \cdots + f_{0_i}(x)\alpha^i(g_{k_0}(x)) = 0 \ ; \\
&\quad \vdots \\
&\text{(iv)} \quad f_{0_{u_0}}(x)\alpha^{u_0}(g_{k_{v_k}}(x)) = 0 \ .
\end{aligned}
$$

Hence, $g_{k_0}(0) = g_{k_1}(0) = \cdots = g_{k_{v_k}}(0) = 0$, and so $g_{k_t}(x)$ has no constant term for all $0 \leq t \leq v_k$. Thus $f_i = \sum_{s=1}^{u_i} f_{i_s}(x)y^s$, $g_j = \sum_{t=0}^{v_j} g_{j_t}(x)y^t$ and each g_{j_t} has no constant term for $0 \leq i \leq m$, $0 \leq t \leq v_j$ and $0 \leq j \leq n$, and so $f_i g_j = 0$ for all i, j. Therefore $S = R[y;\alpha] = (\mathbb{Z}_2[x])[y;\alpha]$ is Armendariz.

The following extends the result in [3, Lemma 3.8].

Proposition 12. *Let α be an endomorphism of a ring R. If S is a ring and $\sigma : R \to S$ is a ring isomorphism, then we have the following.*

(1) *R is an α-rigid ring if and only if S is a $\sigma\alpha\sigma^{-1}$-rigid ring.*

(2) *R is an α-Armendariz ring if and only if S is a $\sigma\alpha\sigma^{-1}$-Armendariz ring.*

(3) *R is an α-skew Armendariz ring if and only if S is a $\sigma\alpha\sigma^{-1}$-skew Armendariz ring.*

Proof. (1) For $a \in R$, there exists $a' \in S$ such that $\sigma(a) = a'$ since σ is bijective, and so $a\alpha(a) = 0$ if and only if $\sigma(a)(\sigma\alpha\sigma^{-1})\sigma(a) = 0$ if and only if $a'(\sigma\alpha\sigma^{-1})(a') = 0$. This yields that R is α-rigid if and only if S is $\sigma\alpha\sigma^{-1}$-rigid.

(2) and (3) Similarly, $p(x) = \sum_{i=0}^{m} a_i x^i$, $q(x) = \sum_{j=0}^{n} b_j x^j \in R[x;\alpha]$ if and only if $p^{'}(x) = \sum_{i=0}^{m} a_i^{'} x^i$, $q^{'}(x) = \sum_{j=0}^{n} b_j^{'} x^j \in S[x;\sigma\alpha\sigma^{-1}]$, letting $\sigma(a_i) = a_i^{'}, \sigma(b_j) = b_j^{'}$ for all i, j since σ is bijective. Then $p(x)q(x) = 0$ in $R[x;\alpha]$ if and only if $\sum_{i+j=k} a_i\alpha^i(b_j) = 0$ for each $0 \le k \le m+n$ if and only if $\sum_{i+j=k} \sigma(a_i\alpha^i(b_j)) = 0$ for each $0 \le k \le m+n$ if and only if $\sum_{i+j=k} \sigma(a_i)(\sigma\alpha\sigma^{-1})^i\sigma(b_j) = 0$ for each $0 \le k \le m+n$, since $(\sigma\alpha\sigma^{-1})^t = \sigma\alpha^t\sigma^{-1}$ for any positive integer t if and only if $\sum_{i+j=k} a_i^{'}(\sigma\alpha\sigma^{-1})^i(b_j^{'}) = 0$ for each $0 \le k \le m+n$ if and only if $p^{'}(x)q^{'}(x) = 0$ in $S[x;\sigma\alpha\sigma^{-1}]$. Hence, for all i, j, $a_ib_j = 0$ if and only if $a_i^{'}b_j^{'} = 0$; and $a_i\alpha^i(b_j) = 0$ if and only if $\sigma(a_i)(\sigma\alpha\sigma^{-1})^i\sigma(b_j) = 0$ if and only if $a_i^{'}(\sigma\alpha\sigma^{-1})^i(b_j^{'}) = 0$. The proof is completed. □

Recall that if α is an endomorphism of a ring R, then the map $\bar{\alpha} : R[x] \to R[x]$ defined by $\bar{\alpha}(\sum_{i=0}^{m} a_i x^i) = \sum_{i=0}^{m} \alpha(a_i)x^i$ is an endomorphism of the polynomial ring $R[x]$ and clearly this map extends α. The *Laurent polynomial ring* $R[x, x^{-1}]$ with an indeterminate x, consists of all formal sums $\sum_{i=k}^{n} a_i x^i$, where $a_i \in R$ and k, n are (possibly negative) integers. The map $\bar{\alpha} : R[x, x^{-1}] \to R[x, x^{-1}]$ defined by $\bar{\alpha}(\sum_{i=k}^{n} a_i x^i) = \sum_{i=k}^{n} \alpha(a_i)x^i$ extends α and also is an endomorphism of $R[x, x^{-1}]$.

Theorem 13. *Let α be an endomorphism of a ring R. The following are equivalent*:

(1) *R is an α-rigid ring.*

(2) *$R[x]$ is an $\bar{\alpha}$-rigid ring.*

(3) *$R[x, x^{-1}]$ is an $\bar{\alpha}$-rigid ring.*

Proof. (1)⇒(2) Assume that R is α-rigid, but $R[x]$ is not $\bar{\alpha}$-rigid. Then there exists a nonzero $f(x) = \sum_{i=0}^{n} a_i x^i \in R[x]$ such that $f(x)\bar{\alpha}(f(x)) = 0$. Suppose that $a_k \neq 0$ and $a_0 = \cdots = a_{k-1} = 0$ where $0 \le k \le n$. Then $0 = f(x)\bar{\alpha}(f(x)) = (a_kx^k + \cdots + a_nx^n)(\alpha(a_k)x^k + \cdots + \alpha(a_n)x^n)$ yields $a_k\alpha(a_k) = 0$, and so $a_k = 0$; which is a contradiction. Thus $R[x]$ is $\bar{\alpha}$-rigid.

(2)⇒(3) Let $f(x) \in R[x, x^{-1}]$ with $f(x)\bar{\alpha}(f(x)) = 0$. Then there exists a positive integer n such that $f_1(x) = f(x)x^n \in R[x]$, and so $f_1(x)\bar{\alpha}(f_1(x)) = 0$. Since $R[x]$ is $\bar{\alpha}$-rigid, we obtain $f_1(x) = 0$, and hence $f(x) = 0$. Thus $R[x, x^{-1}]$ is $\bar{\alpha}$-rigid.

(3)⇒(1) R is α-rigid as a subring of $R[x, x^{-1}]$ when R is $\bar{\alpha}$-rigid. □

Corollary 14. (1) *Let R be a reduced ring with an endomorphism α. Then R is α-Armendariz if and only if $R[x]$ is $\bar{\alpha}$-Armendariz.*

(2) [2, Proposition 6] *Let R be a reduced ring and α be a monomorphism of R. Then R is α-skew Armendariz if and only if $R[x]$ is $\bar{\alpha}$-skew Armendariz.*

Proof. It follows from [2, Theorem 1], [6, Proposition 1.7] and Theorem 13. $\square$

Related to Corollary 14, notice that there exists a reduced α-skew Armendariz ring which is not α-Armendariz (Example 11).

Let α_γ be an endomorphism of a ring R_γ for each $\gamma \in \Gamma$. For the product $\prod_{\gamma\in\Gamma} R_\gamma$ of R_γ and the endomorphism $\bar{\alpha} : \prod_{\gamma\in\Gamma} R_\gamma \to \prod_{\gamma\in\Gamma} R_\gamma$ defined by $\bar{\alpha}((a_\gamma)) = (\alpha_\gamma(a_\gamma))$, it can be easily checked that $\prod_{\gamma\in\Gamma} R_\gamma$ is $\bar{\alpha}$-rigid if and only if each R_γ is α_γ-rigid.

Recall that for an endomorphism α and an ideal I of a ring R, I is called an *α-ideal* if $\alpha(I) \subseteq I$, and if I is an α-ideal of R, then $\bar{\alpha} : R/I \to R/I$ defined by $\bar{\alpha}(a+I) = \alpha(a) + I$ for $a \in R$ is an endomorphism of the factor ring R/I. The homomorphic image of an α-rigid ring is not $\bar{\alpha}$-rigid, in general. The following example shows that there exists a ring R with an automorphism α such that R/I is $\bar{\alpha}$-rigid for a non-zero α-ideal I of R, but R is not α-rigid.

Example 15. Let $R = \left(\begin{smallmatrix} F & F \\ 0 & F \end{smallmatrix}\right)$ where F is a field, and α be defined by $\alpha\left(\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right)\right) = \left(\begin{smallmatrix} a & -b \\ 0 & c \end{smallmatrix}\right)$. Note that R is not α-Armendariz by [6, Example 1.12], and so it is not α-rigid. However, for a nonzero proper ideal $I = \left(\begin{smallmatrix} 0 & F \\ 0 & 0 \end{smallmatrix}\right)$ of R, it can be easily checked that $\alpha(I) \subseteq I$ and R/I is $\bar{\alpha}$-rigid.

Let α be an automorphism of a ring R. Suppose that there exists the classical right quotient ring $Q(R)$ of R. Then for any $ab^{-1} \in Q(R)$ where $a, b \in R$ with b regular, the induced map $\bar{\alpha} : Q(R) \to Q(R)$ defined by $\bar{\alpha}(ab^{-1}) = \alpha(a)\alpha(b)^{-1}$ is also an endomorphism. Note that R is α-rigid if and only if $Q(R)$ is $\bar{\alpha}$-rigid.

Let R be an algebra over a commutative ring S. Recall that the *Dorroh extension* of R by S is the ring $D = R \times S$ with operations $(r_1, s_1) + (r_2, s_2) = (r_1+r_2, s_1+s_2)$ and $(r_1, s_1)(r_2, s_2) = (r_1r_2+s_1r_2+s_2r_1, s_1s_2)$, where $r_1, r_2 \in R$ and $s_1, s_2 \in S$. For an endomorphism α of R and the Dorroh extension D of R by S, $\bar{\alpha} : D \to D$ defined by $\bar{\alpha}(r, s) = (\alpha(r), s)$ is an S-algebra homomorphism.

In the following, we give some other example of α-rigid rings. Observe that for an α-rigid ring R, every subring S of R with $\alpha(S) \subseteq S$ is clearly α-rigid, and R is reduced with $\alpha(e) = e$ for $e^2 = e \in R$ by [4, Proposition 5].

Proposition 16. *Let α be an endomorphism of a ring R.*

(1) *R is an α-rigid ring if and only if eR and $(1-e)R$ are α-rigid for $e^2 = e \in R$.*

(2) *If R is an α-rigid ring and S is a reduced ring, then the Dorroh extension D of R by S is $\bar{\alpha}$-rigid.*

Proof. (1) It is enough to show that R is α-rigid. Suppose that eR and $(1-e)R$ are α-rigid. Let $a\alpha(a) = 0$ for $a \in R$. Then $0 = ea\alpha(ea)$ and $0 = (1-e)a\alpha((1-e)a)$. By hypothesis, we get $ea = 0$ and $(1-e)a = 0$, and so $a = 0$. Thus R is α-rigid.

(2) Let $(r, s) \in D$ with $(r, s)\bar{\alpha}(r, s) = 0$. Then $r\alpha(r) + s\alpha(r) + sr = 0$ and $s^2 = 0$. Since S is reduced, we get $s = 0$. Thus $r\alpha(r) = 0$, and so $r = 0$ since R is α-rigid. Hence, $(r, s) = 0$, and therefore the Dorroh extension D is $\bar{\alpha}$-reduced. $\square$

For any endomorphism α of a ring R, R is α-rigid if and only if $R[x;\alpha]$ is reduced by [5, Proposition 3], but there exists a semiprime ring R with an automorphism α such that the skew polynomial ring $R[x;\alpha]$ is not semiprime by the following example.

Example 17. Let F be a field and $F_i = F$ for $i \in \mathbb{Z}$. Let R be a F-subalgebra of $\prod_{i\in\mathbb{Z}} F_i$ generated by $\oplus_{i\in\mathbb{Z}} F_i$ and $1_{\prod_{i\in\mathbb{Z}} F_i}$. Let α be an automorphism of R defined by $\alpha((a_i)) = (a_{i+1})$. Then

$$R = \{(a_i) \in \prod_{i\in\mathbb{Z}} F_i \mid a_i \text{ is eventually constant}\}$$

is reduced and von Neumann regular, but $R[x;\alpha]$ is not semiprime by [7, Example 4.3].

Lemma 18. *For a ring R, the following are equivalent*:
(1) *R is a semiprime ring.*
(2) *For $a, b \in R$, $aRb = 0$ implies $aR \cap Rb = 0$.*

Proof. Suppose that R is semiprime and $aRb = 0$ for $a, b \in R$. Let $c \in aR \cap Rb$. Then $c = ar = sb$ for some $r, s \in R$. So $cRc = (ar)R(sb) \subseteq aRb = 0$, and thus $c = 0$. The converse is obvious. $\square$

A ring R is called *semicommutative* if $ab = 0$ implies $aRb = 0$ for $a, b \in R$; and so every reduced ring is semicommutative.

Theorem 19. *Let α be an endomorphism of a ring R. The following are equivalent*:
(1) *R is an α-rigid ring.*
(2) *For $p(x), q(x) \in R[x;\alpha]$, $p(x)q(x) = 0$ implies $p(x)R[x;\alpha] \cap R[x;\alpha]q(x) = 0$.*

Proof. Assume that R is α-rigid. By [5, Proposition 3], $R[x;\alpha]$ is reduced and so it is semiprime and semicommutative. If $p(x)q(x) = 0$ for $p(x), q(x) \in R[x;\alpha]$, then $p(x)R[x;\alpha]q(x) = 0$, and thus $p(x)R[x;\alpha] \cap R[x;\alpha]q(x) = 0$ by Lemma 18. Conversely, assume (2). Let $a\alpha(a) = 0$ for $a \in R$. For $p(x) = ax = q(x) \in R[x;\alpha]$, $p(x)q(x) = a\alpha(a)x^2 = 0$ and so $(ax)R[x;\alpha] \cap R[x;\alpha](ax) = 0$ by hypothesis. Then $ax = 0$, and hence $a = 0$. Therefore R is α-rigid. $\square$

Corollary 20. *For a ring R, the following are equivalent*:
(1) *R is a reduced ring.*
(2) *$R[x]$ is a reduced ring.*
(3) *For $a, b \in R$, $ab = 0$ implies $aR \cap Rb = 0$.*
(4) *For $f(x), g(x) \in R[x]$, $f(x)g(x) = 0$ implies $f(x)R[x] \cap R[x]g(x) = 0$.*

Proof. It follows from Theorem 13 and Theorem 19. $\square$

Paralleled to Theorem 2 and Proposition 4 in this paper, Chen and Tong [3] proved the relationship between α-rigid rings and $\bar{\alpha}$-skew Armendariz rings. However, we note that Theorem 19 shows that the results in [3, Theorems 3.11

and 3.12(15)] is meaningless: In [3, Theorem 3.11], Chen and Tong claimed that the trivial extension $T(R, R)$ of a ring R is $\bar{\alpha}$-skew Armendariz (equivalently, R is an α-rigid ring by [3, Theorem 3.4]) if and only if R is an α-skew Armendariz ring and for $p(x), q(x) \in R[x;\alpha]$, $p(x)q(x) = 0$ implies $p(x)R[x;\alpha] \cap R[x;\alpha]q(x) = 0$.

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