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SOME TOEPLITZ OPERATORS ON WEIGHTED BERGMAN SPACES

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Abstract. We consider the problem to determine when a Toeplitz operator is bounded on weighted Bergman spaces. We show that Toeplitz operators induced by elements of some set are bounded and each element of the set is related with a Carleson measure on the weighted Bergman space.

1. Introduction

Let dA denote normalized Lebesgue area measure on the unit disk $\mathbb{D}$. For $\alpha > -1$, the weighted Bergman space A^2_α consists of the analytic functions in $L^2(\mathbb{D}, dA_\alpha)$, where $dA_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha dA(z)$. Then A^2_α is closed in $L^2(\mathbb{D}, dA_\alpha)$ and for each $z \in \mathbb{D}$, there is a reproducing kernel K^α_z in A^2_α such that $f(z) = \langle f, K^\alpha_z\rangle$ for all $f \in A^2_\alpha$, in fact, $K^\alpha_z(w) = \frac{1}{(1-\overline{z}w)^{2+\alpha}}$ and the normalized reproducing kernel k^α_z is the function $\frac{K^\alpha_z}{\|K^\alpha_z\|_{2,\alpha}}$, that is, $k^\alpha_z(w) = \frac{(1-|z|^2)^{1+\frac{\alpha}{2}}}{(1-\overline{z}w)^{2+\alpha}}$, where the norm $\| \ \|_{2,\alpha}$ and the inner product are taken in the space $L^2(\mathbb{D}, dA_\alpha)$.

A linear operator S on A^2_α induces a function $\widetilde{S}$ on $\mathbb{D}$ given by $\widetilde{S}(z) = \langle Sk^\alpha_z, k^\alpha_z\rangle$, $z \in \mathbb{D}$. The function $\widetilde{S}$ is called the Berezin transform of S.

For $u \in L^1(\mathbb{D}, dA)$, the Toeplitz operator T^α_u with symbol u is the operator on A^2_α defined by $T^\alpha_u(f) = P_\alpha(uf)$, $f \in A^2_\alpha$, where P_α is the orthogonal projection from $L^2(\mathbb{D}, dA_\alpha)$ onto A^2_α and let $\widetilde{u}$ denote $\widetilde{T^\alpha_u}$. Then the Toeplitz operator T^α_u is bounded whenever $u \in L^\infty(\mathbb{D}, dA)$ but every element of $L^1(\mathbb{D}, dA)$ dose not imply the boundness of the Toeplitz operator T^α_u. Since $L^\infty(\mathbb{D}, dA)$ is dense in $L^1(\mathbb{D}, dA)$, T^α_u is densely defined on A^2_α. We note that Berezin transforms and Carleson measures are useful tools in the study of Toeplitz operators ([2], [4], [5]). Using those tools, many mathematicians working in operator theory are characterized the boundedness and compactness of Toeplitz operators.

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In this paper, we prove that Toeplitz operators with special symbols are bounded and $\|uk_z^\alpha\|_{p,\alpha}$ having vanishing property implies the compactness of the Toeplitz operator T_u^α.

Section 3 contains some properties of special symbols, that is, each element of some set implies a Carleson measure and we deal with appropriate products of Toeplitz operators and Hankel operators.

2. Unitary operator and example

Let $\mathrm{Aut}(\mathbb{D})$ denote the set of all bianalytic maps of $\mathbb{D}$ onto $\mathbb{D}$. By Schwarz's lemma, each element of $\mathrm{Aut}(\mathbb{D})$ is a linear fractional transformation of the form $\lambda\varphi_z$, $|\lambda| = 1$, where $\varphi_z(w) = \frac{z-w}{1-\overline{z}w}$, $w \in \mathbb{D}$. Then $\varphi_z \circ \varphi_z$ is the identity function on $\mathbb{D}$ and $\mathrm{Aut}(\mathbb{D})$ is called the Möbius group under composition. For $\alpha > -1$ and $z \in \mathbb{D}$, we define $U_z^\alpha f(w) = f \circ \varphi_z(w)\frac{(1-|z|^2)^{\frac{\alpha}{2}+1}}{(1-\overline{z}w)^{\alpha+2}}$, $f \in L^2(\mathbb{D}, dA_\alpha)$, $w \in \mathbb{D}$. A simple compactation shows that U_z^α is an isometry. Since $(1-\overline{z}\varphi_z(w))^{\alpha+2} = \left(\frac{1-|z|^2}{1-\overline{z}w}\right)^{\alpha+2}$, $U_z^\alpha U_z^\alpha$ is the identity operator and hence $(U_z^\alpha)^* = (U_z^\alpha)^{-1} = U_z^\alpha$, that is, U_z^α is a self-adjoint unitary operator on A_α^2. Moreover, $U_z^\alpha(A_\alpha^z) = A_\alpha^2$ and $U_z^\alpha|_{A_\alpha^2}$ is also denoted by U_z^α and $U_z^\alpha 1 = k_z^\alpha(w)$.

For a linear operator S on A_α^2, we define the conjugate operator S_z by $U_z^\alpha S U_z^\alpha$.

Now we are ready to state useful properties.

Lemma 2.1. *For $u \in L^1(\mathbb{D}, dA)$ and $z \in \mathbb{D}$, $(T_u^\alpha)_z = T_{u\circ\varphi_z}^\alpha$.*

Proof. Since $(T_u^\alpha)_z = U_z^\alpha T_u^\alpha U_z^\alpha$ and $(U_z^\alpha)^{-1} = U_z^\alpha$, it is enough to show that $U_z^\alpha T_u^\alpha = T_{u\circ\varphi_z}^\alpha U_z^\alpha$. Take any f in A_α^2 and any w in $\mathbb{D}$. Then

$$
\begin{aligned}
& U_z^\alpha T_u^\alpha(f)(w) \\
&= U_z^\alpha P_\alpha(uf)(w) \\
&= P_\alpha(uf)(\varphi_z(w))\frac{(1-|z|^2)^{\frac{\alpha}{2}+1}}{(1-\overline{z}w)^{\alpha+2}} \\
&= (\alpha+1)\int_{\mathbb{D}} \frac{u(t)f(t)(1-|t|^2)^\alpha}{(1-\varphi_z(w)\overline{t})^{2+\alpha}} dA(t)\frac{(1-|z|^2)^{\frac{\alpha}{2}+1}}{(1-\overline{z}w)^{\alpha+2}} \\
&= (\alpha+1)\int_{\mathbb{D}} u(t)f(t)(1-|t|^2)^\alpha \frac{(1-\overline{z}w)^{2+\alpha}}{(1-\overline{z}w-z\overline{t}+w\overline{t})^{2+\alpha}} dA(t)\frac{(1-|z|^2)^{\frac{\alpha}{2}+1}}{(1-\overline{z}w)^{\alpha+2}} \\
&= (\alpha+1)\int_{\mathbb{D}} u(t)f(t)(1-|t|^2)^\alpha \frac{(1-|z|^2)^{\frac{\alpha}{2}+1}}{(1-\overline{z}w-z\overline{t}+w\overline{t})^{2+\alpha}} dA(t) \\
&= (\alpha+1)\int_{\mathbb{D}} u\circ\varphi_z(s) f\circ\varphi_z(s)(1-|\varphi_z(s)|^2)^\alpha \frac{(1-|z|^2)^{\frac{\alpha}{2}+1}|\varphi_z'(s)|^2 dA(s)}{(1-\overline{z}w-z\overline{\varphi_z(s)}+w\overline{\varphi_z(s)})^{2+\alpha}}
\end{aligned}
$$

$$= (\alpha+1)\int_{\mathbb{D}} u\circ\varphi_z(s) f\circ\varphi_z(s)\frac{\left(1-|z|^2\right)^{\alpha}\left(1-|s|^2\right)^{\alpha}}{|1-\overline{z}s|^{2\alpha}}(1-|z|^2)^{\frac{\alpha}{2}+1}\frac{\left|(1-|z|^2)\right|^2}{|1-\overline{z}s|^4}$$
$$\times\ \frac{(1-z\overline{s})^{2+\alpha}dA(s)}{\left(1-|z|^2\right)^{2+\alpha}(1-w\overline{s})^{2+\alpha}}$$
$$= (\alpha+1)\int_{\mathbb{D}} u\circ\varphi_z(s) f\circ\varphi_z(s)\frac{\left(1-|s|^2\right)^{\alpha}(1-|z|^2)^{\frac{\alpha}{2}+1}(1-z\overline{s})^{2+\alpha}dA(s)}{(1-\overline{z}s)^{\alpha}(1-z\overline{s})^{\alpha}(1-\overline{z}s)^{2}(1-z\overline{s})^{2}(1-w\overline{s})^{2+\alpha}}$$
$$= (\alpha+1)\int_{\mathbb{D}} u\circ\varphi_z(s) f\circ\varphi_z(s)\frac{(1-|z|^2)^{\frac{\alpha}{2}+1}\left(1-|s|^2\right)^{\alpha}}{(1-\overline{z}s)^{\alpha+2}(1-w\overline{s})^{2+\alpha}}dA(s)$$
$$= \int_{\mathbb{D}} u\circ\varphi_z(s)U_z^{\alpha}f(s)\frac{dA_\alpha(s)}{(1-w\overline{s})^{2+\alpha}}$$
$$= P_\alpha(u\circ\varphi_z U_z^{\alpha} f)(w).$$

Thus $(T_u^{\alpha})_z = T_{u\circ\varphi_z}^{\alpha}$. □

Corollary 2.2. *If $u_1,\ldots,u_n \in L^1(\mathbb{D}, dA)$, then*

$$U_z^{\alpha}T_{u_1}\cdots T_{u_n}U_z^{\alpha} = T_{u_1\circ\varphi_z}\cdots T_{u_n\circ\varphi_z}.$$

Proof. It follows from the fact that $U_z^{\alpha}U_z^{\alpha}$ is the identity. □

Proposition 2.3. *For $u \in L^1(\mathbb{D}, dA)$ and $z \in \mathbb{D}$, $\widetilde{T_u^{\alpha}}\circ\varphi_z = (\widetilde{T_u^{\alpha}})_z$.*

Proof. Since $\widetilde{(T_u^{\alpha})_z} = \widetilde{T_{u\circ\varphi_z}^{\alpha}}$, it is enough to show that $\widetilde{T_{u\circ\varphi_z}^{\alpha}} = \widetilde{T_u^{\alpha}}\circ\varphi_z$. Take any t in $\mathbb{D}$. Then

$$\widetilde{T_u^{\alpha}}\circ\varphi_z(t)$$
$$= \langle T_u^{\alpha}k_{\varphi_z(t)}^{\alpha}, k_{\varphi_z(t)}^{\alpha}\rangle$$
$$= \langle P_\alpha(uk_{\varphi_z(t)}^{\alpha}), k_{\varphi_z(t)}^{\alpha}\rangle$$
$$= \langle uk_{\varphi_z(t)}^{\alpha}, k_{\varphi_z(t)}^{\alpha}\rangle$$
$$= \int_{\mathbb{D}} u(x)k_{\varphi_z(t)}^{\alpha}(x)\overline{k_{\varphi_z(t)}^{\alpha}(x)}dA_\alpha(x)$$
$$= \int_{\mathbb{D}} u(x)\Big(\frac{(1-|z|^2)(1-|t|^2)}{|1-\overline{z}t|^2}\Big)^{1+\frac{\alpha}{2}}\Big(\frac{1-z\overline{t}}{1-z\overline{t}-\overline{z}x+\overline{t}z}\Big)^{2+\alpha}\Big(\frac{(1-|z|^2)(1-|t|^2)}{|1-\overline{z}t|^2}\Big)^{1+\frac{\alpha}{2}}$$
$$\times\Big(\frac{1-\overline{z}t}{1-\overline{z}t-z\overline{x}+t\overline{x}}\Big)^{2+\alpha}(\alpha+1)(1-|x|^2)^{\alpha}dA(x)$$
$$= \int_{\mathbb{D}} u(x)(1-|t|^2)^{1+\frac{\alpha}{2}}\Big(\frac{1}{(1-\overline{z}t)(1-z\overline{t})}\Big)^{1+\frac{\alpha}{2}}\Big(\frac{1-z\overline{t}}{1-z\overline{t}-\overline{z}x+\overline{t}x}\Big)^{2+\alpha}(1-|t|^2)^{1+\frac{\alpha}{2}}$$
$$\times (1-|z|^2)^{2+\alpha}\frac{1}{\Big(1-\overline{z}t)(1-z\overline{t}\Big)}^{1+\frac{\alpha}{2}}\Big(\frac{1-\overline{z}t}{1-\overline{z}t-z\overline{x}+t\overline{x}}\Big)^{2+\alpha}(\alpha+1)(1-|x|^2)^{\alpha}dA(x)$$

$$= \int_{\mathbb{D}} u(x)(1-|t|^2)^{1+\frac{\alpha}{2}} \Big(\frac{1-\overline{z}x}{1-\overline{z}x-\overline{t}z+\overline{t}x}\Big)^{2+\alpha} (1-|t|^2)^{1+\frac{\alpha}{2}} \Big(\frac{1-z\overline{x}}{1-z\overline{x}-t\overline{z}+t\overline{x}}\Big)^{2+\alpha}$$
$$\times \frac{(1-|z|^2)^{\alpha+2}}{|1-\overline{z}x|^{2\alpha+4}} dA_\alpha(x)$$
$$= \int_{\mathbb{D}} u\circ\varphi_z(s) k_t^\alpha(s) \overline{k_t^\alpha(s)} dA_\alpha(s)$$
$$= \langle u\circ\varphi_z k_t^\alpha, k_t^\alpha\rangle$$
$$= \langle P_\alpha(u\circ\varphi_z k_t^\alpha), k_t^\alpha\rangle$$
$$= \langle T^\alpha_{u\circ\varphi_z} k_t^\alpha, k_t^\alpha\rangle$$
$$= \widetilde{T^\alpha_{u\circ\varphi_z}}(t).$$

This completes the proof. □

We will show that the Toeplitz operators with special symbols are bounded. To do so, we need the following proposition, in fact, the following proposition holds for every linear operator on A^2_α.

Proposition 2.4. *If S is a linear operator on A^2_α and $z, w \in \mathbb{D}$, then $(S_z)^* = (S^*)_z$ and $SK_z^\alpha(w) = \overline{S^*K_w^\alpha(z)}$.*

Proof. Take any f, g in A^2_α. Since $\langle S_z f, g\rangle = \langle U_z^\alpha S U_z^\alpha f, g\rangle = \langle f, U_z^\alpha S^* U_z^\alpha g\rangle = \langle f, (S^*)_z g\rangle$, $(S_z)^* = (S^*)_z$. For the 2nd equality, $SK_z^\alpha(w) = \langle SK_z^\alpha, K_w^\alpha\rangle = \langle K_z^\alpha, S^*K_w^\alpha\rangle = \overline{S^*K_w^\alpha(z)}$. □

Since $K_z^\alpha(w) = \frac{1}{(1-\overline{z}w)^{2+\alpha}} = \sum_{n=0}^\infty \frac{\Gamma(n+2+\alpha)}{n!\Gamma(2+\alpha)}\overline{z}^n w^n$, $k_z^\alpha(w) = (1-|z|^2)^{1+\frac{\alpha}{2}} \times \sum_{n=0}^\infty \frac{\Gamma(n+2+\alpha)}{n!\Gamma(2+\alpha)}\overline{z}^n w^n$. We define $S(\sum a_n w^n) = \sum a_n(-w)^n$. Then $S(K_z^\alpha(w)) = \sum_{n=0}^\infty \frac{\Gamma(n+2+\alpha)}{n!\Gamma(2+\alpha)}\overline{z}^n(-1)^n w^n = K_z^\alpha(-w)$ and S has an infinite-dimensional range and S is an isometry and invertible, that is, $S^* = S^{-1} = S$. Thus S is not compact. Since $\widetilde{S}(z) = \langle Sk_z^\alpha, k_z^\alpha\rangle = (1-|z|^2)^{2+\alpha}\langle SK_z^\alpha, K_z^\alpha\rangle = (1-|z|^2)^{2+\alpha}K_z^\alpha(-z) = \left(\frac{1-|z|^2}{1+|z|^2}\right)^{2+\alpha}$, $\lim_{z\to\partial\mathbb{D}}\widetilde{S}(z) = 0$ and hence the vanishing property does not imply the compactness of operators.

3. Toeplitz operators with special symbols

This section deals with Toeplitz operators with special symbols. We begin by constructing some set and show that each element of the set implies a bounded linear operator. Recall that P_α is the orthogonal projection from $L^2(\mathbb{D}, dA_\alpha)$ onto A^2_α and for $z \in \mathbb{D}$ and $f \in L^2(\mathbb{D}, dA_\alpha)$, $P_\alpha(f)(z) = \langle P_\alpha(f), K_z^\alpha\rangle = \int_{\mathbb{D}} f(w)\overline{K_z^\alpha(w)}dA_\alpha(w)$. Moreover, we extend the domain of P_α to $L^1(\mathbb{D}, dA_\alpha)$ and for $f \in A^1_\alpha$, $f(z) = \int_{\mathbb{D}} \frac{f(w)dA_\alpha(w)}{(1-z\overline{w})^{2+\alpha}}$, $z \in \mathbb{D}$ (see [5]).

Let $MK = \{u \in L^1(\mathbb{D}, dA) : \sup_{\lambda\in\mathbb{D}} \|uk_\lambda^\alpha\|_{p,\alpha} < \infty$ for every $p \in (1,\infty)\}$, where $\|\cdot\|_{p,\alpha}$ is the norm on $L^p(\mathbb{D}, dA_\alpha)$. Since $\|uk_\lambda\|_{p,\alpha} = \|\overline{u}k_\lambda\|_{p,\alpha}$, MK is closed under the formation of conjugations.

Lemma 3.1. *For any $u \in MK$, $(T_u^\alpha)^* = T_{\overline{u}}^\alpha$.*

Proof. Take any f, g in A_α^2. Since $\langle T_u^\alpha f, g\rangle = \langle P_\alpha(uf), g\rangle = \langle uf, g\rangle = \langle f, \overline{u}g\rangle = \langle f, P_\alpha(\overline{u}g)\rangle = \langle f, T_{\overline{u}}^\alpha g\rangle$, $(T_u)^* = T_{\overline{u}}^\alpha$. □

Lemma 3.2. *Suppose $u \in MK$, $z \in \mathbb{D}$, and $p \in (1,\infty)$. Then there is a constant $c > 0$ such that $\|(T_u^\alpha)_z 1\|_{p,\alpha} \le c\|uk_z^\alpha\|_{p,\alpha}$.*

Proof. Since $(T_u^\alpha)_z = U_z^\alpha T_u^\alpha U_z^\alpha$ and $U_z^\alpha 1 = k_z^\alpha$, $\|(T_u^\alpha)_z 1\|_{p,\alpha} = \|U_z^\alpha T_u^\alpha U_z^\alpha 1\|_{p,\alpha}$ $= \|U_z^\alpha P_\alpha(uk_z^\alpha)\|_{p,\alpha} \le \|P_\alpha\|\|uk_z^\alpha\|_{p,\alpha}$ and hence $\|(T_u^\alpha)_z 1\|_{p,\alpha} \le c\|uk_z^\alpha\|_{p,\alpha}$ for some c. □

Suppose $f \in A_\alpha^2$ and $w \in \mathbb{D}$. Then $(T_u^\alpha f)(w) = \langle T_u^\alpha f, K_w^\alpha\rangle = \langle f, (T_u^\alpha)^* K_w^\alpha\rangle$ $= \int_{\mathbb{D}} f(z)\overline{(T_u^\alpha)^* K_w^\alpha(z)}dA_\alpha(z) = \int_{\mathbb{D}} f(z)(T_u^\alpha K_z^\alpha)(w)dA_\alpha(z)$, here the last equality follows from Proposition 2.4. Thus T_u^α is the integral operator with kernel $(T_u^\alpha K_z^\alpha)(w)$.

Lemma 3.3. *Suppose $0 < t$, $s \in (1,\infty)$ and $u \in MK$. Then there is a constant $c > 0$ such that $\int_{\mathbb{D}} \frac{|(T_u^\alpha K_z^\alpha)(w)|}{(1-|w|^2)^t} \le \frac{c\|uk_z^\alpha\|_{s,\alpha}}{(1-|z|^2)^t}$ for all $z \in \mathbb{D}$ and $\int_{\mathbb{D}} \frac{|(T_u^\alpha K_z^\alpha)(w)|}{(1-|z|^2)^t} dA_\alpha(z) \le \frac{c\|\overline{u}k_w^\alpha\|_{s,\alpha}}{(1-|w|^2)^t}$ for all $w \in \mathbb{D}$.*

Proof. Since $k_z^\alpha = (1-|z|^2)^{1+\frac{\alpha}{2}} K_z^\alpha$, $T_u^\alpha K_z^\alpha = \frac{U_z^\alpha (T_u^\alpha)_z 1}{(1-|z|^2)^{1+\frac{\alpha}{2}}} = \frac{(T_u^\alpha)1\circ\varphi_z \frac{(1-|z|^2)^{\frac{\alpha}{2}+1}}{(1-\overline{z}w)^{\alpha+2}}{}_z}{(1-|z|^2)^{1+\frac{\alpha}{2}}}$ and hence

$$\begin{aligned}
&\int_{\mathbb{D}} \frac{|(T_u^\alpha K_z^\alpha)(w)|}{(1-|w|^2)^t} dA_\alpha(w) \\
&= \int_{\mathbb{D}} \frac{|((T_u^\alpha)_z 1)\circ\varphi_z(w)| \frac{(1-|z|^2)^{\frac{2}{\alpha}+1}}{|1-\overline{z}w|^{\alpha+2}} (1-|w|^2)^\alpha dA(w)}{(1-|z|^2)^{\frac{\alpha}{2}+1}(1-|w|^2)^t} \\
&= \int_{\mathbb{D}} \frac{|((T_u^\alpha)_z 1)(\lambda)| \frac{1}{|1-\overline{z}\varphi_z(\lambda)|^{\alpha+2}}}{(1-|\varphi_z(\lambda)|^2)^t} (1-|\varphi_z(\lambda)|^2)^\alpha |\varphi_z'(\lambda)|^2 dA(\lambda) \\
&= \int_{\mathbb{D}} \frac{|((T_u^\alpha)_z 1)(\lambda)| \frac{|1-\overline{z}\lambda|^{\alpha+2}}{(1-|z|^2)^{\alpha+2}}}{\frac{(1-|z|^2)^t(1-|\lambda|^2)^t}{|1-\overline{z}\lambda|^{2t}}} \frac{(1-|z|^2)^\alpha(1-|\lambda|^2)^\alpha}{|1-\overline{z}\lambda|^{2\alpha}} \frac{(1-|z|^2)^2}{|1-\overline{z}\lambda|^4} dA(\lambda) \\
&= \frac{1}{(1-|z|^2)^t} \int_{\mathbb{D}} \frac{|((T_u^\alpha)_z 1)(\lambda)|}{|1-\overline{z}\lambda|^{2-2t+\alpha}} \frac{1}{(1-|\lambda|^2)^{t-\alpha}} dA(\lambda) \\
&\le \frac{1}{(1-|z|^2)^t} \|(T_u^\alpha)_z 1\|_{s,\alpha} \Big(\int_{\mathbb{D}} \frac{1}{|1-\overline{z}\lambda|^{(2-2t+\alpha)s'}(1-|\lambda|^2)^{ts'-\alpha}} dA(\lambda) \Big)^{\frac{1}{s'}},
\end{aligned}$$

where s and s' are conjugate exponents.

By Lemma 3.2, there is a constant c_1, such that $\|(T_u^\alpha)_z 1\|_{s,\alpha} \le c_1 \|uk_z^\alpha\|_{s,\alpha}$.

Let $c = c_1\Big(\int_{\mathbb{D}} \frac{1}{|1-\overline{z}\lambda|^{(2-2t+\alpha)s'}(1-|\lambda|^2)^{ts'-\alpha}} dA(\lambda)\Big)^{\frac{1}{s'}}$. If c is infinity, then trivially the inequality holds and hence $\int_{\mathbb{D}} \frac{|(T_u^\alpha K_z^\alpha)(w)|}{(1-|w|^2)^t} dA_\alpha(w) \le \frac{c\|uk_z^\alpha\|_{s,\alpha}}{(1-|z|^2)^t}$. Proposition 2.4 and Lemma 3.1 and $\overline{MK} \subset MK$ imply that there is a constant c such that $\int_{\mathbb{D}} \frac{|(T_u^\alpha K_z^\alpha)(w)|}{(1-|z|^2)^t} dA_\alpha(z) = \int_{\mathbb{D}} \frac{|(T_{\overline{u}}^\alpha K_w^\alpha)(z)|}{(1-|z|^2)^t} dA_\alpha(z) \le \frac{c\|\overline{u}k_w^\alpha\|_{s,\alpha}}{(1-|w|^2)^t}$. This completes the proof. $\square$

Suppose $\alpha \ne 0$. Then there is $t > 0$ such that $s' = \frac{2\alpha}{4t-2-\alpha} > 1$ and

$$\int_{\mathbb{D}} \frac{1}{|1-\overline{z}\lambda|^{(2-2t+\alpha)s'}(1-|\lambda|^2)^{ts'-\alpha}} dA(\lambda) = \int_{\mathbb{D}} \frac{1}{|1-\overline{z}\lambda|^{\frac{6}{5}}(1-|\lambda|^2)^{\frac{3}{5}}} dA(\lambda).$$

Axler's paper ([1], Lemma 4) asserts that the last integral is finite. In Lemma 3.3, c is finite.

Theorem 3.4. *For each $u \in MK$, T_u^α is bounded.*

Proof. Let $h(\lambda) = \frac{1}{(1-|\lambda|^2)^t}$. By the above observation and Lemma 3.3,

$$\int_{\mathbb{D}} \frac{|(T_u^\alpha K_z^\alpha)(w)|}{(1-|w|^2)^t} dA_\alpha(w) \le c_1 h(z) \text{ and } \int_{\mathbb{D}} \frac{|(T_u^\alpha K_z^\alpha)(w)|}{(1-|z|^2)^t} dA_\alpha(z) \le c_2 h(w),$$

where $c_1 = c \sup \|uk_z^\alpha\|_{s,\alpha}$ and $c_2 = c \sup \|\overline{u}k_w^\alpha\|_{s,\alpha}$.

The Schur's test (see page 126 of [3]) implies that T_u^α is bounded and $\|T_u^\alpha\| \le \sqrt{c_1 c_2}$. $\square$

Recall that T_u^α is the integral operator with kernel $(T_u^\alpha K_z^\alpha)(w)$, that is, $(T_u^\alpha f)(w) = \int_{\mathbb{D}} f(z)(T_u^\alpha K_z^\alpha)(w) dA(z)$. For $0 < r < 1$, we define an operator T_r^α on A_α^2 by $(T_r^\alpha f)(w) = \int_{r\mathbb{D}} f(z) T_u^\alpha K_z^\alpha(w) dA(z)$, $f \in A_\alpha^2$. Since $\int_{\mathbb{D}}\int_{\mathbb{D}} |T_u^\alpha K_z^\alpha(w)\chi_{r\mathbb{D}}(z)|^2 dA_\alpha(w) dA_\alpha(z) = \int_{r\mathbb{D}}\int_{\mathbb{D}} |T_u^\alpha K_z^\alpha(w)|^2 dA_\alpha(w) dA_\alpha(z) = \int_{r\mathbb{D}} \|T_u^\alpha K_z^\alpha\|_2^2 dA_\alpha(z) \le \|T_u^\alpha\|^2 \int_{r\mathbb{D}} \|K_z^\alpha\|_2^2 dA(z) < \infty$, $T_u^\alpha K_z^\alpha(w)\chi_{r\mathbb{D}}(z) \in L^2(\mathbb{D}\times\mathbb{D}, dA_\alpha \times dA_\alpha)$ and hence T_r^α is a Hilbert-Schmidt operator. Thus each T_r^α is compact. Let $h(\lambda) = \Big(\frac{1}{1-|\lambda|^2}\Big)^t$. By Lemma 3.3 and Theorem 3.4,

$$\int_{\mathbb{D}} |T_u^\alpha K_z^\alpha(w)\chi_{\mathbb{D}\setminus r\mathbb{D}}(z)| h(w) dA_\alpha(w) \le c_1 h(z) \quad \text{and}$$
$$\int_{\mathbb{D}} |T_u^\alpha K_z^\alpha(w)\chi_{\mathbb{D}\setminus r\mathbb{D}}(z)| h(z) dA_\alpha(z) \le c_2 h(w),$$

where $c_1 = c \sup_{r \le |z| < 1} \|uk_z^\alpha\|_{s,\alpha}$ and $c_2 = c \sup \|\overline{u}k_w^\alpha\|_{s,\alpha}$. The Schur's test implies that $\|T_u^\alpha - T_r^\alpha\| \le c_1^{\frac{1}{2}} c_2^{\frac{1}{2}}$. If $\lim_{z\to\partial\mathbb{D}} \|uk_z^\alpha\|_{s,\alpha} = 0$, then $\lim_{r\to 1^-} c_1 = 0$ and hence $\lim_{r\to 1^-} \|T_u^\alpha - T_r^\alpha\| = 0$. Since each T_r^α is compact, T_u^α is also a compact operator. Thus we have the following theorem.

Theorem 3.5. *Let $u \in MK$. If $\lim_{z\to\partial\mathbb{D}} \|uk_z^\alpha\|_{p,\alpha} = 0$ for every $p \in (1,\infty)$, then T_u^α is a compact operator.*

We note that A_α^p consists of the analytic functions in $L^p(\mathbb{D}, dA_\alpha)$. Suppose μ is a finite positive Borel measure on $\mathbb{D}$ and $p > 1$. Recall that if $i_p : A_\alpha^p \to L^p(\mathbb{D}, d\mu)$ is bounded, then μ is called a Carleson measure on the Bergman space A_α^p and Carleson measures are very useful tools in operator theory.

Proposition 3.6. *For $u \in MK$, $|u|dA_\alpha$ is a Carleson measure on A_α^p.*

Proof. It is enough to show that $\widetilde{u}$ is bounded. For $w \in \mathbb{D}$, $|\widetilde{u}(w)| = |\langle T_u^\alpha k_w^\alpha, k_w^\alpha\rangle| = |\langle P_\alpha(uk_w^\alpha), k_w^\alpha\rangle| = |\langle uk_w^\alpha, k_w^\alpha\rangle| \le \|uk_w^\alpha\|_{2,\alpha} < \infty$. Thus $|u|dA_\alpha$ is a Carleson measure on A_α^p. □

Corollary 3.7. *For $u \in MK$, T_u^α is a bounded linear operator.*

Proof. It follows from the fact that $|u|dA_\alpha$ is a Carleson measure. □

Using the concept of a Carleson measure, we can give another proof for Theorem 3.5.

Proposition 3.8. *If $\|uk_z^\alpha\|_{p,\alpha} \to 0$ as $z \to \partial\mathbb{D}$ for every $p \in (1,\infty)$ and $u \in MK$, then T_u^α is compact.*

Proof. Let's show that $|u|dA_\alpha$ is a vanishing Carleson measure on the Bergman space A_α^p. To do so, it is enough to show that $\lim_{|z|\to 1^-} |\widetilde{u}|(z) = 0$. For $z \in \mathbb{D}$, $|\widetilde{u}(z)| = |\langle T_u^\alpha k_z^\alpha, k_z^\alpha\rangle| = |\langle uk_z^\alpha, k_z^\alpha\rangle| \le \|uk_z^\alpha\|_{2,\alpha}\|k_z^\alpha\|_{2,\alpha} = \|uk_z^\alpha\|_{2,\alpha}$. The property $\lim_{|z|\to 1^-} \|uk_z^\alpha\|_{2,\alpha} = 0$ implies that $|u|dA_\alpha$ is a vanishing Carleson measure. Thus $T_{|u|}^\alpha$ is compact. Since $|\int_{\mathbb{D}} |f_u|^2 u dA_\alpha| \le \int_{\mathbb{D}} |f_u|^2 |u| dA_\alpha$, T_u^α is also compact. □

We define an operator $H_u^\alpha : A_\alpha^2 \to (A_\alpha^2)^\perp$ by

$$H_u^\alpha(g) = (I - P_\alpha)(ug), \ g \in A_\alpha^2.$$

Then H_u^α is called the Hankel operator on the weighted Bergman space with symbol u. Clearly H_u^α is densely defined for any $u \in L^1(\mathbb{D}, dA)$ and if $u \in L^\infty(\mathbb{D}, dA)$, then H_u^α is bounded with $\|H_u^\alpha\| \le \|u\|_\infty$.

Proposition 3.9. *If $u^2 \in MK$, then H_u^α is bounded.*

Proof. Take any f in A_α^2. Then $\|H_u^\alpha(f)\|_{2,\alpha}^2 = \|(I - P_\alpha)(uf)\|_{2,\alpha}^2 \le \|uf\|_{2,\alpha}^2 = \int_{\mathbb{D}} |f(z)|^2 |u(z)|^2 dA_\alpha(z)$. By Proposition 3.6, $|u|^2 dA_\alpha$ is a Carleson measure on A_α^2 and hence there is a constant $c < \infty$ such that $\int_{\mathbb{D}} |f(z)|^2 |u(z)|^2 dA_\alpha(z) \le c \int_{\mathbb{D}} |f(z)|^2 dA_\alpha(z)$. Thus H_u^α is bounded. □

Corollary 3.10. (1) *Suppose $u^2 \in MK$. Then $(H_u^\alpha)_z 1$ and $H_u^\alpha k_z^\alpha$ are in $L^2(\mathbb{D}, dA_\alpha)$ for every $z \in \mathbb{D}$.*

(2) *Suppose $u^2 \in MK$ and $z \in \mathbb{D}$. Then $H_u^\alpha \circ \varphi_z$ is bounded.*

Proof. (1) We note that $\|H_u^\alpha U_z^\alpha 1\|_{2,\alpha} = \|H_u^\alpha k_z^\alpha\|_{2,\alpha}$ and hence $\|(H_u^\alpha)_z 1\|_{2,\alpha} = \|H_u^\alpha k_z^\alpha\|_{2,\alpha}$. Since $\|(H_u^\alpha)_z\|1_{2,\alpha} = \|U_z^\alpha H_u^\alpha U_z^\alpha 1\|_{2,\alpha} = \|H_u^\alpha k_z^\alpha\|_{2,\alpha} \le \|H_u^\alpha\|$, we have the results.

(2) By Lemma 2.1, $(T_u^\alpha)_z = T_{u\circ\varphi_z}^\alpha$. Then $(H_u^\alpha)_z = (I - T_u^\alpha)_z = I - T_{u\circ\varphi_z}^\alpha = H_{u\circ\varphi_z}^\alpha$. For $f \in A_\alpha^2$, $\|H_{u\circ\varphi_z}^\alpha(f)\|_{2,\alpha} = \|(H_u^\alpha)_z(f)\|_{2,\alpha} = \|U_z^\alpha H_u^\alpha U_z^\alpha(f)\|_{2,\alpha} = \|H_u^\alpha U_z^\alpha(f)\|_{2,\alpha} \le \|H_u^\alpha\|\|U_z(f)\|_{2,\alpha} = \|H_u^\alpha\|\|f\|_{2,\alpha}$. Thus $\|H_{u\circ\varphi_z}^\alpha\| \le \|H_u^\alpha\|$. Since H_u^α is bounded, $H_{u\circ\varphi_z}^\alpha$ is bounded. $\square$

Consider some products of Toeplitz operators and Hankel operators. The simple calculation implies that $H_u^{\alpha *} H_u^\alpha = T_{|u|^2}^\alpha - T_{\overline{u}}^\alpha T_u^\alpha$. Suppose $u \in L^1(\mathbb{D}, dA)$ and $f \in A_\alpha^2$. Since $H_u^\alpha f(z) = u(z)f(z) - P_\alpha(uf)(z) = u(z)\langle f, K_z^\alpha\rangle - \langle uf, K_z^\alpha\rangle = \langle (u(z)-u)f, K_z^\alpha\rangle$ for $g \in (A_\alpha^2)^\perp$,

$$\begin{aligned}\langle H_u^\alpha f, g\rangle &= \int_{\mathbb{D}}\int_{\mathbb{D}} (u(z)-u(w))\overline{K_z^\alpha(w)} f(w)\,dA_\alpha(w)\overline{g(z)}\,dA_\alpha(z) \\ &= \int_{\mathbb{D}} f(w)\int_{\mathbb{D}} (u(z)-u(w))\overline{g(z)} K_w^\alpha(z)\,dA_\alpha(z)\,dA_\alpha(w) \\ &= \int_{\mathbb{D}} f(w)\overline{(-H_{\overline{u}}^\alpha)g(w)}\,dA_\alpha(w) = \langle f, -H_{\overline{u}}^\alpha g\rangle\end{aligned}$$

and hence $H_u^{\alpha *} = -H_{\overline{u}}^\alpha$.

Suppose u,v,u^2,v^2 are in MK. If H_u^α is compact, then the following are compact:

(i) $T_{|u|^2}^\alpha - T_{\overline{u}}^\alpha T_u^\alpha$ (ii) $T_{\overline{v}u}^\alpha - T_{\overline{u}}^\alpha T_v^\alpha$ (iii) $H_u^\alpha T_v^\alpha$ (iv) $H_u^\alpha H_v^{\alpha *} = H_u^\alpha H_{\overline{v}}^\alpha$.

Corollary 3.11. *Suppose $u_1, \ldots, u_n \in MK$ and $z \in \mathbb{D}$. Then*

$$U_z^\alpha H_{u_1}^\alpha \cdots H_{u_n}^\alpha U_z^\alpha = H_{u_1\circ\varphi_z}^\alpha \cdots H_{u_1\circ\varphi_z}^\alpha.$$

Proof. It follows from the fact that $U_z^\alpha U_z^\alpha$ is the identity operator. $\square$

Corollary 3.12. *Suppose u_1, $u_2 \in L^1(\mathbb{D}, dA)$. If $u_1 = u_2 \circ \varphi_z$ for some $z \in \mathbb{D}$, then $H_{u_1}^\alpha$ and $H_{u_2}^\alpha$ are unitary equivalent.*

Proof. By Corollary 3.10, $H_{u\circ\varphi_z}^\alpha = (H_u^\alpha)_z$ and hence $H_{u_1}^\alpha = (H_{u_2}^\alpha)_z$. Thus $H_{u_1}^\alpha$ and $H_{u_2}^\alpha$ are unitary equivalent. $\square$

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