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A CONCEPT UNIFYING THE ARMENDARIZ AND NI CONDITIONS

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Abstract. We study the structure of the set of nilpotent elements in various kinds of ring and introduce the concept of *NR ring* as a generalization of Armendariz rings and NI rings. We determine the precise relationships between NR rings and related ring-theoretic conditions. The Köthe's conjecture is true for the class of NR rings. We examined whether several kinds of extensions preserve the NR condition. The classical right quotient ring of an NR ring is also studied under some conditions on the subset of nilpotent elements.

1. NR rings

Throughout this note every ring is associative with identity unless otherwise stated. Given a ring R, $N_*(R)$, $N^*(R)$, $J(R)$, and $N(R)$ denote the prime radical, the upper nilradical (i.e., sum of nil ideals), the Jacobson radical, and the set of all nilpotent elements in R, respectively. Note $N_*(R) \subseteq N^*(R) \subseteq N(R)$ and $N^*(R) \subseteq J(R)$.

A ring is called *reduced* if it has no nonzero nilpotent elements. In the following we consider two kinds of generalizations of commutative rings. Due to Marks [20], a ring R is called NI if $N^*(R) = N(R)$. Note that R is NI if and only if $N(R)$ forms an ideal if and only if $R/N^*(R)$ is reduced. According to Birkenmeier et al. [5], a ring R is called *2-primal* if $N_*(R) = N(R)$. It is obvious that R is 2-primal if and only if $R/N_*(R)$ is reduced. Marks [21] gave almost complete characterizations for 2-primal rings, with constructive delimiting examples. Note that a ring R is reduced if and only if R is nil-semisimple (i.e., R has no nonzero nil ideals) and NI if and only if R is semiprime and 2-primal. It is obvious that 2-primal rings are NI, but the converse need not hold by Birkenmeier et al. [6, Example 3.3], Hwang et al. [13, Example 1.2], or Marks [20, Example 2.2]. If a ring R is of bounded index of nilpotency, then

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R is NI if and only if R is 2-primal by [13, Proposition 1.4]. The upper triangular matrix rings over non-commutative reduced rings are typical examples of non-commutative 2-primal (NI) rings.

Given a ring R, $R[x]$ denotes the polynomial ring with an indeterminate x over R. For $f(x) \in R[x]$, let $C_{f(x)}$ denote the set of all coefficients of $f(x)$. Denote the n by n full matrix ring over R by $\mathrm{Mat}_n(R)$ and the n by n upper (resp. lower) triangular matrix ring over R by $U_n(R)$ (resp. $L_n(R)$). Use e_{ij} for the matrix with (i,j)-entry 1 and elsewhere 0.

When given a ring R is reduced, Armendariz [3, Lemma 1] proved that $a_ib_j = 0$ for all i,j whenever $f(x)g(x) = 0$, where $f(x) = \sum_{i=0}^m a_ix^i$, $g(x) = \sum_{j=0}^n b_jx^j$ are in $R[x]$. Rege et al. [23] called a ring (not necessarily reduced) *Armendariz* if it satisfies the condition "$a_ib_j = 0$ for all i,j whenever $(\sum_{i=0}^m a_ix^i)(\sum_{j=0}^n b_jx^j) = 0$. So reduced rings are Armendariz. If R is an Armendariz ring, then $N(R)$ is a subring of R by [2, Proposition 2.7 and Theorem 3.2]. But here we prove that directly, applying the method in the proof of [2, Lemma 2.6].

Proposition 1.1. (1) *Let R be an Armendariz ring and $a^m = 0, b^m = 0$ for $a,b \in R$. Then $a^\alpha b^\beta$ $(1 \le \alpha, \beta \le m-1)$, $a - b \in N(R)$.*

(2) *If R is an Armendariz ring, then $N(R)$ is a subring of R.*

Proof. (1) Let R be an Armendariz ring and $a,b \in N(R)$ with $a^m = 0, b^m = 0$. Let

$$f(x) = (1+ax+\cdots+a^{m-1}x^{m-1})(1-ax)(1-bx)(1+bx+b^2x^2+\cdots+b^{m-1}x^{m-1}).$$

Since $a^m = 0 = b^m$, $f(x) = 1$ and $(af(x))^m = a^m = 0$, i.e.,

$$((a+a^2x+\cdots+a^{m-1}x^{m-1})(1-ax)(1-bx)(1+bx+b^2x^2+\cdots+b^{m-1}x^{m-1}))^m = 0.$$

Since R is Armendariz, we get $(a^\alpha b^\beta)^m = 0$ for $\alpha, \beta \in \{1, \ldots, m-1\}$.

Next we claim that $(a+b)^{2m} = 0$ by showing that every term is zero. Each term of $(a+b)^{2m}$ is of the form $c_{i_1}c_{i_2}\cdots c_{i_{2m}}$ with $c_{i_j} = a$ or b. Consider $g(x) = (a+bx)^{2m}$. Then every coefficient of $g(x)$ is a sum of $c = c_{i_1}c_{i_2}\cdots c_{i_{2m}}$'s. Note that c contains at least m occurrences of a or b, say $c_{i_{j_1}} = c_{i_{j_2}} = \cdots = c_{i_{j_m}} = a$ with $1 \le j_1 < j_2 < \cdots < j_m \le 2m$. Then

$$(*) \qquad c = c_{i_1}\cdots c_{i_{j_1-1}}ac_{i_{j_1+1}}\cdots c_{i_{j_m-1}}ac_{i_{j_m+1}}\cdots c_{i_{2m}}.$$

In the equation $(*)$, replacing each c_{i_j} (for $i_j \notin \{j_1, j_2, \cdots, j_m\}$) by $f(x)$, we obtain

$$f(x)\cdots f(x)af(x)\cdots f(x)af(x) = a^m = 0.$$

Since R is Armendariz, we get $c_{i_1}c_{i_2}\cdots c_{i_{2m}} = 0$ by taking proper coefficients of $f(x)$'s. Use $-b$ in place of b to obtain $(a-b)^{2m} = 0$.

Therefore $a^\alpha b^\beta$ $(1 \le \alpha, \beta \le m-1)$, $a - b \in N(R)$, entailing that $N(R)$ forms a subring of R.

(2) follows from (1) immediately. □

Commutative (hence NI) rings need not be Armendariz by [23, Example 3.2], and Armendariz rings need not be NI by [2, Example 4.8].

In this note we will study the structure of rings whose nilpotent elements form subrings. Unifying the concepts of NI rings and Armendariz rings, a ring R (possibly without identity) will be called an *NR ring* if $N(R)$ forms a subring of R. NI rings are clearly NR, and Armendariz rings are NR by Proposition 1.1. But the converses are both irreversible by [23, Example 3.2] and [2, Example 4.8].

Considering the definition of NR rings, one may suspect that $N(R) \subseteq J(R)$ for an NR ring R. However the following example erases the possibility.

Example 1.2. Let $S = \mathbb{C}\{a, b\}$ be the free algebra with non-commuting indeterminates a, b over $\mathbb{C}$, where $\mathbb{C}$ is the field of complex numbers. Let I be an ideal of S generated by a^2. Set $R = S/I$. Then R is Armendariz (hence NR) by [2, Example 4.8] or [7, Example 9.3]. We coincide a, b with their images in R for simplicity. Notice that $N(R)$ is the subring of R generated by

$$\{\alpha a, \beta ara \mid \alpha, \beta \in \mathbb{C}, r \in R\}.$$

In spite of $a \in N(R)$, $1 - ba$ is not invertible in R, entailing $a \notin J(R)$. This implies $N(R) \nsubseteq J(R)$.

Given a ring R and an ideal I of R, we denote $\{r \in R \mid r+I \text{ is regular in } R/I\}$ by $C(I)$.

Lemma 1.3. (1) *The class of NR rings is closed under subrings (possibly without identity).*

(2) *For any ring A, $\mathrm{Mat}_n(A)$ $(n \geq 2)$ cannot be NR.*

(3) *Let R be an NR ring. If $C(0) = R \backslash N(R)$, then R is NI.*

(4) *The class of NI rings is closed under subrings (possibly without identity).*

(5) *Let R be a ring and I a nil ideal of R. If R/I is NR, then so is R.*

(6) *If R is NR, then $er - re \in N(R)$ for $e^2 = e, r \in R$.*

Proof. (1) Let S be a subring of given a ring R. Then $N(S) = N(R) \cap S$. If R is NR, $N(S)$ forms a subring of S. This implies that S is NR.

(2) Since $n \geq 2$, $\mathrm{Mat}_n(R)$ contains two nilpotent elements e_{12}, e_{21}. But $e_{12} + e_{21} \notin N(\mathrm{Mat}_n(R))$.

(3) Assume $C(0) = R \backslash N(R)$. Let $a \in N(R), r \in R$. By observing that ar, ra are not regular, $ar, ra \in N(R)$ by assumption and so this implies that $N(R)$ is an ideal of R. Thus R is NI.

(4) By [13, Proposition 2.4(2)].

(5) Let $\bar{R} = R/I$. If $a + I \in N(\bar{R})$, then $a \in N(R)$ since I is nil, entailing $N(\bar{R}) \subseteq \{a + I \mid a \in N(R)\}$. The converse inclusion is evident, obtaining $N(\bar{R}) = \{a + I \mid a \in N(R)\}$. Let $a, b \in N(R)$. Then $a + I, b + I \in N(\bar{R})$, and since $\bar{R}$ is NR we get $(a - b) + I, ab + I \in N(\bar{R})$. The preceding equality yields $a - b, ab \in N(R)$.

(6) $a = er(1 - e), b = (1 - e)re \in N(R)$ implies $er - re = a - b \in N(R)$ when R is NR. $\square$

The class of NR rings is not closed under factor rings. For example, let R be the ring of quaternions with integer coefficients. Then R is a domain and so NR. However for any odd prime integer q, the ring R/qR is isomorphic to $\mathrm{Mat}_2(\mathbb{Z}_q)$ by the argument in [10, Exercise 2A]. Thus R/qR is not NR.

Note. In Lemma 1.3(3) we consider only the condition "$C(0) = R\backslash N(R)$" without the NR condition. Then $ar, ra \in N(R)$ for $a \in N(R), r \in R$ by the proof of Lemma 1.3(3). Assume that $N(R)$ is of bounded index (of nilpotency) 2. Then for $a, b \in N(R)$, $(a-b)^4 = (ab+ba)^2 = ab^2a + ba^2b = 0$ since $ab, ba \in N(R)$. Thus R is NI. Note $(a-b)^2 = 0$ with $ab + ba = 0$.

Next assume that $N(R)$ is of bounded index (of nilpotency) ≥ 3. Consider a sequence $(a, b, ba, baab, baababba, \ldots)$ that is generated by $a, b \in N(R)$. The constructing rule is that a term is obtained from all that go before by interchanging the a's and the b's. Then by [22], there is no product U of a's and b's such that U^3 occurs in the sequence. So in this case we do not know whether the condition "$C(0) = R\backslash N(R)$" implies the NI (or NR) condition.

Question. Let R be a ring. If $N(R)$ is of bounded index (of nilpotency) ≥ 3 and $C(0) = R\backslash N(R)$, then is R an NI (or NR) ring?

A ring R is called *directly finite* if $ab = 1$ implies $ba = 1$ for $a, b \in R$. NI rings and Armendariz rings are directly finite by [13, Proposition 2.7] and [18, Lemma 3.4(3)], respectively. So one may conjecture that NR rings are directly finite.

Proposition 1.4. *NR rings are directly finite.*

Proof. We apply the proof of [13, Proposition 2.7]. Let R be an NR ring and assume on the contrary that R is not directly finite. Then R contains an infinite set of matrix units, say $\{e_{11}, e_{12}, e_{13}, \ldots, e_{21}, e_{22}, e_{23}, \ldots\}$, by [9, Proposition 5.5]. Since $N(R)$ forms a subring in R and $e_{12}, e_{21} \in N(R)$, $e_{11} = e_{12}e_{21} \in N(R)$, a contradiction. □

By Propositions 1.1 and 1.4, Armendariz rings and NI rings are directly finite.

A ring is called *Abelian* if every idempotent is central. Abelian rings are directly finite by a simple computation. Armendariz rings are Abelian by the proof of [1, Theorem 6] or [12, Corollary 8]. Armendariz rings are NR by Proposition 1.1. So it is necessary to check the implications between NR rings and Abelian rings. But they are independent of each other as follows.

Example 1.5. (1) Let K be a field and $S = K\{a_0, a_1, a_2, a_3\}$ the free algebra with non-commuting indeterminates a_0, a_1, a_2, a_3 over K. Let I be the ideal of S generated by a_0a_1, a_2a_3. Set $R = S/I$ and we coincide a_0, a_1, a_2, a_3 with their images in R for simplicity. It is easily checked that R has only two idempotents 0 and 1, entailing that R is Abelian. Now consider a_1a_0 and a_3a_2. By the construction of R, a_1a_0 and a_3a_2 are nilpotent. But $a_1a_0 + a_3a_2 \notin N(R)$. So R is not NR.

(2) There exists an NR ring but not Abelian as can be seen by $U_2(F)$ over a field F.

It is obvious that Köthe's conjecture (i.e., the sum of two nil left ideals is nil) holds for NI rings.

Proposition 1.6. *Let R be an NR ring. Then the Köthe's conjecture holds. Especially $N^*(\mathrm{Mat}_n(R)) = \mathrm{Mat}_n(N^*(R))$ and $J(R[x]) = N^*(R)[x]$.*

Proof. Let R be an NR ring and I, J be nil left ideals of R. Then $a+b \in N(R)$ for $a \in I, b \in J$ and so $I + J$ is nil. The other results are obtained by [24, Theorem 2.6.35]. □

One may suspect that if the Köthe's conjecture holds, then R is NR. But it's not true. For example, let F be a field and let $\mathrm{Mat}_2(F)$. Then by [24, Theorem 2.6.35], if the Köthe's conjecture holds, then $N^*(\mathrm{Mat}_2(F)) = \mathrm{Mat}_2(N^*(F)) = 0$. But clearly $N(\mathrm{Mat}_2(F)) \neq 0$, and $\mathrm{Mat}_2(F)$ is not NR by Lemma 1.3(2).

Denote the center of a ring R by $Z(R)$. If $N(R) \subseteq Z(R)$, then $N(R)$ is an ideal of R, obtaining that R is NI (hence NR). The following is compared with this result.

Proposition 1.7. *If R is a ring such that $N(R)$ is a commutative subset of R, then R is NR.*

Proof. Let $a, b \in N(R)$. Then if $N(R)$ is commutative, $a - b \in N(R)$ and $ab \in N(R)$ and so R is NR. □

Consider $R = U_2(A)$ over a reduced ring A. Then $N(R) = \left(\begin{smallmatrix} 0 & A \\ 0 & 0 \end{smallmatrix}\right)$ is a commutative subset of R since $xy = 0 = yx$ for $x, y \in N(R)$. So R is NR by the Proposition 1.7. Letting $I = \left(\begin{smallmatrix} 0 & A \\ 0 & 0 \end{smallmatrix}\right)$, I is a nil ideal of R such that $R/I \cong A \oplus A$ is reduced (hence NR). R is NR also by Lemma 1.3(5).

A ring R is called (*von Neumann*) *regular* if for each $a \in R$ there exists $x \in R$ such that $a = axa$.

Proposition 1.8. *Given a regular ring R the following conditions are equivalent*:
(1) *R is Armendariz*;
(2) *R is NI*;
(3) *R is NR*;
(4) *R is reduced*;
(5) *R is 2-primal*;
(6) *R is Abelian.*

Proof. Since R is regular, $J(R) = 0$ by [9, Corollary 1.2(c)] and so the equivalences of the conditions (2), (4), (5) are obtained. [9, Theorem 3.2] gives the equivalences of the conditions (4) and (6). (1) implies (6) by [12, Corollary 8]. (4) implies (1) by [3, Lemma 1]. (2) implies (3) trivially. (3) implies (6) by [15, Theorem 1.8]. □

A ring R is called *π-regular* if for each $a \in R$ there exist a positive integer n, depending on a, and $b \in R$ such that $a^n = a^n b a^n$. The Jacobson radicals of π-regular rings are nil, comparing with that regular rings are semiprimitive. Regular rings are clearly π-regular. However the preceding results need not hold on π-regular rings. $U_n(D)$ ($n \geq 2$ and D is a division ring) is π-regular and NR, but it is neither regular nor reduced.

2. Examples of NR rings

In this section we consider several kinds of ring extensions over NR rings, and examine them to be NR. We see typical examples of NR rings in the following.

Theorem 2.1. *For a ring R and $n \geq 2$, the following conditions are equivalent*:
(1) *R is an NR ring*;
(2) *$U_n(R)$ is an NR ring*;
(3) *$L_n(R)$ is an NR ring.*

Proof. (1)$\Rightarrow$(2): Suppose that R is NR. By observing that

$$N(U_n(R)) = \begin{pmatrix} N(R) & R & \cdots & R \\ 0 & N(R) & \cdots & R \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & N(R) \end{pmatrix},$$

the ring $U_n(R)$ is also NR.

(2)$\Rightarrow$(1): Suppose that $U_n(R)$ is an NR ring for any $n \geq 2$. Since R is isomorphic to $\{aI_n \in U_n(R) \mid a \in R\}$, R is NR by Lemma 1.3(1) when $U_n(R)$ is NR, where I_n is the n by n identity matrix.

The proofs of (1)$\Rightarrow$(3) and (3)$\Rightarrow$(1) are similar to those of (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (1). □

However $\mathrm{Mat}_n(A)$ ($n \geq 2$) is not NR by Lemma 1.3(2) for any ring A.

Proposition 2.2. *Let R, S be rings and ${}_RM_S$ an (R,S)-bimodule. Let $E = \left(\begin{smallmatrix} R & M \\ 0 & S \end{smallmatrix}\right)$. Then E is NR if and only if R and S are both NR.*

Proof. By observing $N(E) = \begin{pmatrix} N(R) & M \\ 0 & N(S) \end{pmatrix}$, a similar proof to Theorem 2.1 yields the proposition. □

From now on, $\mathbb{Z}$ denotes the ring of integers.

Proposition 2.3. (1) *The class of NR rings is closed under direct limits.*

(2) *A direct sum of NR rings R_i ($i \in I$) is NR if and only if R_i is NR for all $i \in I$.*

(3) *Let S be an NR ring (possibly without identity) and attach an identity to S, obtaining $R = S \oplus \mathbb{Z}$. Then R is NR.*

Proof. (1) Let $D = \{R_i, \alpha_{ij}\}$ be a direct system of NR rings R_i for $i \in I$ and ring homomorphisms $\alpha_{ij} : R_i \to R_j$ for each $i \leq j$ satisfying $\alpha_{ij}(1) = 1$, where I is a directed partially ordered set. Set $R = \varinjlim R_i$ the direct limit of D with $\iota_i : R_i \to R$ and $\iota_j \alpha_{ij} = \iota_i$. Let $a, b \in R$. Then $a = \iota_i(a_i)$, $b = \iota_j(b_j)$ for some $i, j \in I$ and there is $k \in I$ such that $i \leq k$, $j \leq k$. Define $a + b = \iota_k(\alpha_{ik}(a_i) + \alpha_{jk}(b_j))$ and $ab = \iota_k(\alpha_{ik}(a_i)\alpha_{jk}(b_j))$, where $\alpha_{ik}(a_i)$ and $\alpha_{jk}(b_j)$ are in R_k. Then R forms a ring with $0 = \iota_i(0)$ and $1 = \iota_i(1)$.

Next let $a, b \in N(R)$. There is $k \in I$ such that $a, b \in N(R_k)$ via ι_i's and α_{ij}'s. Since R_k is NR, $a - b \in N(R_k)$ and $ab \in N(R_k)$, concluding R being NR.

(2) Let $R = \sum_{i \in I} R_i$ and $a, b \in N(R)$ with $a = (a_i)$ and $b = (b_i)$. Here we can assume a, b are both nonzero. Put $J = \{j \in I \mid a_j \neq 0 \text{ or } b_j \neq 0\}$. Then clearly J is finite. If all R_i's are NR, then $a_j - b_j, a_j b_j \in N(R_j)$ for all $j \in J$. Thus $a - b, ab \in N(R)$ since J is finite, obtaining that R is NR.

Conversely let R be NR and assume on the contrary that R_{i_0} is not NR for some $i_0 \in I$. Then for some $x, y \in N(R_{i_0})$, $x - y \notin N(R_{i_0})$ or $xy \notin N(R_{i_0})$. Taking $a = (a_i), b = (b_i) \in R$ such that $x = a_{i_0}, y = b_{i_0}$ and $a_k = 0, b_k = 0$ for all $k \in I \backslash \{i_o\}$. Then since $a, b \in N(R)$ and R is NR, $a - b \in N(R)$ and $ab \in N(R)$ and so $a_{i_0} - b_{i_0} \in N(R_{i_0})$ and $a_{i_0} b_{i_0} \in N(R_{i_0})$, a contradiction.

(3) Since S is a subring of R and $N(S) \oplus 0 = N(R)$, R is NR. □

Letting $I = \{M \in U_n(R) \mid \text{ the diagonal entries of } M \text{ are all zero}\}$ in Theorem 2.1, I is a nil ideal of $U_n(R)$ such that $U_n(R)/I$ is isomorphic to the direct sum of n-copies of R. So if R is NR, then $U_n(R)/I$ in NR by Proposition 2.3(2). Thus Lemma 1.3(5) implies that $U_n(R)$ is NR.

By Proposition 2.3(2), one may suspect that the direct product of NR rings is NR. However the direct product of NR rings need not be NR as follows.

Example 2.4. The construction and computation are according to Huh et al. [19, Examples 1.6], [13, Example 2.5], and Marks [21, Remark, p. 508]. Let K be a field and define $D_n = K\{x_n\}$, a free algebra generated by x_n, with a relation $x_n^{n+2} = 0$ for each nonnegative integer n. Then clearly $D_n \cong K[x]/(x^{n+2})$, where (x^{n+2}) is the ideal of $K[x]$ generated by x^{n+2}. Next let $R_n = \left(\begin{smallmatrix} D_n & x_n D_n \\ x_n D_n & D_n \end{smallmatrix}\right)$ be a subring of the 2 by 2 matrix ring over D_n. Then every R_n is 2-primal (hence NR) by the computation in [19, Example 1.6]. Set $R = \prod_{n=0}^{\infty} R_n$. However the sum of two nilpotent elements $\left(\begin{smallmatrix} 0 & x_n \\ 0 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 0 \\ x_n & 0 \end{smallmatrix}\right)$ is not nilpotent, entailing that R is not NR.

Proposition 2.5. *Let e be a central idempotent of a ring R. Then the following statements are equivalent*:

(1) *R is NR*;

(2) *eR and $(1 - e)R$ are both NR.*

Proof. (1)⇒(2) comes from Lemma 1.3(1) since eR and $(1 - e)R$ are subrings of R.

(2)⇒(1): Suppose that eR and $(1-e)R$ are both NR. Then by observing $N(R) = N(eR) \oplus N((1-e)R)$, Proposition 2.3(2) implies that R is NR. □

The polynomial rings need not preserve the NR condition as follows.

Example 2.6. The following construction is due to Smoktunowicz [25]. Let K be a countable field and $\bar{A}$ the algebra of polynomials with zero constant terms in non-commuting indeterminates x, y, z over K. Then $\bar{A}$ can be enumerated, say $\bar{A} = \{f_1, f_2, \ldots\}$. By the argument in the proof of [25, Theorem 12], there are natural numbers $m_1, m_2, \ldots$ such that (i) $m_1 > 10^8$, $m_{i+1} > m_i 2^{i+101}$ for $i \geq 1$, (ii) each m_i divides m_{i+1} and (iii) $m_i > 3^{2\deg(f_i)}(\deg(f_i))^2 40^2$ for $i \geq 1$. Let I be the ideal of $\bar{A}$ generated by $\{f_i^{10m_{i+1}} \mid i = 1, 2, \ldots\}$ and $S = \bar{A}/I$. Then S is nil. Attach an identity to S, obtaining $R = S \oplus \mathbb{Z}$. $N(R) = S \oplus 0$ implies that R is NI (hence NR).

Next assume that the polynomial ring over an NR ring is also NR. Then the polynomial ring $R[X, Y]$ in two commuting indeterminates X, Y over R is also NR. However $\bar{x} + \bar{y}X + \bar{z}Y$ is not nilpotent by the proof of [25, Theorem 12], in spite of $\bar{x}, \bar{y}X, \bar{z}Y \in N(R[X, Y])$. This is contrary to that $R[X, Y]$ is NR.

Suppose that an NI ring R is of bounded index of nilpotency. Then R is 2-primal by [13, Proposition 1.4] and moreover $R[x]$ is 2-primal (hence NR) by [5, Proposition 2.6].

Next we consider the NR condition of Ore extensions. For a ring R, a ring endomorphism $\sigma : R \to R$ and a σ-derivation $\delta : R \to R$, the *Ore extension* $R[x; \sigma, \delta]$ of R is the ring obtained by giving $R[x]$ the multiplication $xr = \sigma(r)x + \delta(r)$ for all $r \in R$. If $\delta = 0$, we write $R[x; \sigma]$ for $R[x; \sigma, 0]$ and is called an *Ore extension of endomorphism type* (also called a *skew* polynomial ring). While if $\sigma = 1$, we write $R[x; \delta]$ for $R[x; 1, \delta]$ and is called an *Ore extension of derivation type* (also called a *differential* polynomial ring). Use $\mathbb{Z}_n$ to denote the ring of integers modulo n.

Example 2.7. (1) There exists an NR ring over which the skew polynomial ring need not be NR. For a domain D let $R = D \oplus D$, then R is reduced (hence NR). Consider the automorphism σ of R defined by $\sigma(s, t) = (t, s)$. Let $R[x; \sigma]$ be the skew polynomial ring over R by σ. Consider $(1, 0)x$ and $(0, 1)x$. Then $((1, 0)x(1, 0)x) = 0$ and $((0, 1)x(0, 1)x) = 0$, whence $(1, 0)x, (1, 0)x \in N(R[x; \sigma])$. But $(1, 0)x + (0, 1)x = (1, 1)x$ is not nilpotent, entailing that $R[x; \sigma]$ is not NR.

(2) There exists an NR ring over which the differential polynomial ring need not be NR. We use the ring and argument in [4, Example 11]. Let $\mathbb{Z}_2[t]$ be the polynomial ring with an indeterminate t over $\mathbb{Z}_2$. Then $R = \mathbb{Z}_2[t]/(t^2)$ is commutative (hence NR), where (t^2) is the ideal of $\mathbb{Z}_2[t]$ generated by t^2. Define a derivation δ on R by $\delta(t + (t^2)) = 1 + (t^2)$. Then $R[x; \delta] \cong \mathrm{Mat}_2(\mathbb{Z}_2[y^2])$ which is not NR by Lemma 1.3(2).

In the following we see two cases in which $R[x;\sigma,\delta]$ can be NR.

Let σ be an endomorphism of a ring R. Due to Krempa [16], σ is called *rigid* if $r\sigma(r)=0$ implies $r=0$ for $r\in R$. It is easily checked that a rigid endomorphism is injective. A ring R is called *σ-rigid* if there exists a rigid endomorphism σ of R. Hong et al. [11, Proposition 5] proved that a ring R is σ-rigid if and only if $R[x;\sigma,\delta]$ is a reduced ring and σ is a monomorphism of R. In Example 2.7(1), σ is not rigid since $(1,0)\sigma(1,0)=0$.

Let R be a local ring with an endomorphism σ and a σ-derivation δ, and suppose that the maximal ideal M of R is nilpotent. In this situation, Mark [20, Theorem 3.4] proved that $R[x;\sigma,\delta]$ is 2-primal if $\delta(M)\subseteq M$. In Example 2.7(2), R is a local ring with the maximal ideal $M=\mathbb{Z}_2t+(t^2)$, but $\delta(M)\nsubseteq M$ since $\delta(M)$ contains $1+(t^2)$.

Next we study the NR condition of classical quotient rings over NR rings. A ring R is called right *Ore* if given $a,b\in R$ with $b\in C(0)$ there exist $a_1,b_1\in R$ with $b_1\in C(0)$ such that $ab_1=ba_1$. Left Ore rings can be defined by symmetry. It is well-known that R is a right Ore ring if and only if the classical right quotient ring of R exists. If both right and left quotient rings exist, then they are equal. Let F be a field and R the free algebra in two indeterminates over F. Then R is a domain but cannot be right (left) Ore. It is also well-known that R is a right Ore domain if and only if the classical right quotient ring of R is a division ring. A subset I of a ring R is called *right* (resp. *left*) *T-nilpotent* provided that for every sequence $a_1,a_2,\dots$ in I there is a positive integer n such that $a_n\cdots a_2a_1=0$ (resp. $a_1a_2\cdots a_n=0$). Nilpotent subsets of a ring are obviously both right and left T-nilpotent but the converse does not hold in general, and that right or left T-nilpotent subsets are clearly nil but nil ideals need not be right (or left) T-nilpotent. The T-nilpotence is not left-right symmetric by [24, Example 2.7.38].

Theorem 2.8. *Let R be a right Ore ring and Q the classical right quotient ring of R. Suppose that $N(R)$ is left T-nilpotent. Then the following conditions are equivalent*:

(1) *Q is an NI local ring with $N(Q)=J(Q)=\{ab^{-1}\in Q\,|\,a\in N(R)$ and $b\in C(0)\}$*;

(2) *R is NR and $C(0)=C(N^*(R))=R\backslash N(R)$.*

Proof. (1)⇒(2): We will apply the proof of [8, Proposition 2.1(1)]. Suppose that the condition (1) holds. Then R is NI (hence NR) by Lemma 1.3(4) and $Q/N(Q)$ is a division ring. So if $a,b\in R\backslash N(R)$, then they are invertible in $Q/N(Q)$ and so $ab\notin N(Q)$. Thus $ab\notin N(R)$ since $N(R)\subseteq N(Q)$. This yields that $R/N(R)$ is a domain, and hence $C(N(R))=R\backslash N(R)$ and $C(0)\subseteq C(N(R))$. Conversely let $u\in C(N(R))$. Then u is invertible in Q since $Q/N(Q)$ is a division ring with $N(Q)=J(Q)=\{ab^{-1}\in Q\mid a\in N(R)$ and $b\in C(0)\}$, entailing $u\in C(0)$. This yields $C(N(R))\subseteq C(0)$.

(2)⇒(1): Assume that R is NR and $C(0)=R\backslash N(R)$. Then R is NI by Lemma 1.3(3) and moreover $R/N^*(R)$ is a domain. Next we will apply the proof of [8, Proposition 2.1(1)]. Let $ab^{-1} \in N(Q)$. Then a is not regular, forcing $a \in N(R)$ by the assumption. Conversely let $ab^{-1} \in Q$ with $a \in N(R)$. Then since $C(0) = R \backslash N(R)$ we have

$$(ab^{-1})^2 = ac_1d_1^{-1}, (ab^{-1})^3 = ac_1c_2d_2^{-1}, \ldots, (ab^{-1})^{n+1} = ac_1 \cdots c_nd_n^{-1}$$

with c_i's in $N(R)$ and d_i's in $C(0)$ for $i = 1, \ldots, n$. But $N(R)$ is left T-nilpotent by hypothesis and so $(ab^{-1})^k = ac_1 \cdots c_{k-1}d_{k-1}^{-1} = 0$ for some k. Whence $ab^{-1} \in N(Q)$ when $a \in N(R)$, entailing

$$N(Q) = \{ab^{-1} \in Q \mid a \in N(R) \text{ and } b \in C(0)\}.$$

From $C(0) = R \backslash N(R)$, $N(Q)$ is an ideal of Q and so Q is NI. Consequently $Q/N(Q)$ is a division ring since every element in $Q \backslash N(Q)$ is invertible. So Q is a local ring such that $N(Q) = J(Q) = \{ab^{-1} \in Q \mid a \in N(R) \text{ and } b \in C(0)\}$. □

Considering Theorem 2.8 and [8, Proposition 2.1], one may conjecture $N(R) = N_*(R)$ under the equivalences of Theorem 2.8. However the following provides a counterexample.

Example 2.9. Let $S = \mathbb{Z}_2[x]$ and consider

$$D_n(S) = \{M \in U_n(S) \mid \text{the diagonal entries of } M \text{ are equal}\},$$

a subring of $U_n(S)$ and define a map $\sigma : D_n(S) \to D_{n+1}(S)$ by $A \mapsto \left(\begin{smallmatrix} A & 0 \\ 0 & A \end{smallmatrix}\right)$. Then $D_n(S)$ can be considered as a subring of $D_{n+1}(S)$ via σ (i.e., $A = \sigma(A)$ for $A \in D_n(S)$). Set R be the direct limit of the direct system $(D_n(S), \sigma_{ij})$ with $\sigma_{ij} = \sigma^{j-i}$. Then R is semiprime (i.e., $N_*(R) = 0$) by [14, Theorem 2.2(2)] but $N^*(R) \neq 0$ (hence not 2-primal). Moreover R is NI by [13, Example 1.2] and Lemma 1.3(4). Note

$$N(R) = \{M \in R \mid \text{the diagonal entries of } M \text{ are zero}\} \neq 0$$

and so

$$\begin{aligned} C(0) = C(N(R)) &= R \backslash N(R) \\ &= \{M \in R \mid \text{the diagonal entries of } M \text{ are nonzero}\}. \end{aligned}$$

By [17, Theorem 1.3 and Proposition 1.9], every $D_n(S)$ is both left and right Ore; hence R is also left and right Ore by the construction. It is easily checked that the classical quotient ring Q of R is the direct limit of the direct system $(D_n(\mathbb{Z}_2(x)), \sigma_{ij})$ with $\sigma_{ij} = \sigma^{j-i}$, where $\mathbb{Z}_2(x)$ is the quotient field of $\mathbb{Z}_2[x]$. Note that Q is a π-regular ring with

$$N(Q) = J(Q) = \{N \in Q \mid \text{the diagonal entries of } N \text{ are zero}\} \neq 0.$$

This result yields that Q is NI and local with

$$N(Q) = J(Q) = \{ab^{-1} \in Q \mid a \in N(R) \text{ and } b \in C(0)\}.$$

Notice that $N(R)$ is left T-nilpotent. Indeed, consider a sequence $a_1, a_2, \ldots$ in $N(R)$. We can assume that every $a_i = (a_{st})_i$ is nonzero. Let j be largest such that the j-th row of a_1 contains a nonzero entry. Then since $s < t$ in each a_i, we get $a_{j+1}a_j \cdots a_2 a_1 = 0$.

If given a ring R is π-regular, then every regular element is invertible and so R is itself its right (left) quotient ring. Thus letting S be any division ring in Example 2.9, the direct limit R and its classical right (left) quotient ring coincide since R is π-regular. This provides another counterexample since $N(R) \neq 0$ and $N_*(R) = 0$ by a similar computation.

In Theorem 2.8, each equality in $C(0) = C(N^*(R)) = R \backslash N(R)$ is necessary as the following examples show.

Example 2.10. The argument is almost all due to [8, Example 2.2]. (1) (The case of $C(0) = C(N^*(R)) \subsetneqq R\backslash N(R)$) We refer the argument in [8, Example 2.2(1)]. Let $R = U_2(\mathbb{Z})$. Then clearly $N(R) = N^*(R) = \left(\begin{smallmatrix} 0 & \mathbb{Z} \\ 0 & 0 \end{smallmatrix}\right)$; and the classical right quotient ring Q of R is $U_2(\mathbb{Q})$, where $\mathbb{Q}$ is the field of rational numbers. Q is NI with $N(Q) = J(Q) = \{ab^{-1} \in Q \mid a \in N(R) \text{ and } b \in C(0)\} = \left(\begin{smallmatrix} 0 & \mathbb{Q} \\ 0 & 0 \end{smallmatrix}\right)$. However Q is not local since $Q/J(Q) \cong \mathbb{Q} \oplus \mathbb{Q}$, and $C(0) = C(N(R)) = \{\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right) \in R \mid a \neq 0, c \neq 0\} \subsetneqq R\backslash N(R) = \{\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right) \in R \mid a \neq 0 \text{ or } c \neq 0\}$.

(2) (The case of $C(0) \subsetneqq C(N^*(R)) = R\backslash N(R)$) We refer the argument in [8, Example 2.2(2)]. Let $R = \{\left(\begin{smallmatrix} n & a \\ 0 & n \end{smallmatrix}\right) \mid n \in \mathbb{Z} \text{ and } a \in \mathbb{Z}/2\mathbb{Z}\}$. Then R is commutative, $C(0) = \{\left(\begin{smallmatrix} m & a \\ 0 & m \end{smallmatrix}\right) \in R \mid m \text{ is odd}\}$, and $C(N(R)) = R\backslash N(R) = \{\left(\begin{smallmatrix} s & a \\ 0 & s \end{smallmatrix}\right) \in R \mid s \neq 0\}$ because $N(R) = \left(\begin{smallmatrix} 0 & \mathbb{Z}/2\mathbb{Z} \\ 0 & 0 \end{smallmatrix}\right)$ and $R/N(R) \cong \mathbb{Z}$. The classical right quotient ring Q of R is

$$\left\{ \begin{pmatrix} q & a \\ 0 & q \end{pmatrix} \mid q \in \mathbb{Z}_{(2)}, a \in \mathbb{Z}_2 \right\},$$

where $\mathbb{Z}_{(2)}$ denotes the localization of $\mathbb{Z}$ at the prime ideal $2\mathbb{Z}$. Then Q is a commutative local ring with $N(Q) = \{ab^{-1} \in Q \mid a \in N(R) \text{ and } b \in C(0)\} = \left(\begin{smallmatrix} 0 & \mathbb{Z}/2\mathbb{Z} \\ 0 & 0 \end{smallmatrix}\right) \subsetneqq J(Q) = \left\{\left(\begin{smallmatrix} q & a \\ 0 & q \end{smallmatrix}\right) \in Q \mid q \in J(\mathbb{Z}_{(2)})\right\}$. Note that $Q/N(Q) \cong \mathbb{Z}_{(2)}$ and $Q/J(Q) \cong \mathbb{Z}/2\mathbb{Z}$.

(3) (The case of $C(0) \subsetneqq C(N^*(R)) \subsetneqq R\backslash N(R)$). We refer the argument in [8, Example 2.2(3)]. We use the ring and the argument in [5, Example 5.10]. Let $R = \left(\begin{smallmatrix} \mathbb{Z} & \mathbb{Z}/2\mathbb{Z} \\ 0 & \mathbb{Z}/2\mathbb{Z} \end{smallmatrix}\right)$. Then R is NI with $N(R) = \left(\begin{smallmatrix} 0 & \mathbb{Z}/2\mathbb{Z} \\ 0 & 0 \end{smallmatrix}\right)$, and each regular element in R is of the form $\left(\begin{smallmatrix} n & a \\ 0 & \bar{1} \end{smallmatrix}\right)$ with n odd and $a \in \mathbb{Z}/2\mathbb{Z}$. So the classical right quotient ring Q of R is $\left(\begin{smallmatrix} \mathbb{Z}_{(2)} & \mathbb{Z}/2\mathbb{Z} \\ 0 & \mathbb{Z}/2\mathbb{Z} \end{smallmatrix}\right)$. Q is clearly NI. However Q is not local since $Q/J(Q) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, and $N(Q) = \{ab^{-1} \in Q \mid a \in N(R) \text{ and } b \in C(0)\} = \left(\begin{smallmatrix} 0 & F \\ 0 & 0 \end{smallmatrix}\right) \subsetneqq J(Q) = \left(\begin{smallmatrix} J(\mathbb{Z}_{(2)}) & \mathbb{Z}/2\mathbb{Z} \\ 0 & 0 \end{smallmatrix}\right)$. Note that every element of R with $(1,1)$-entry even is not regular, and this yields

$$\begin{aligned} C(0) &= \{\left(\begin{smallmatrix} m & a \\ 0 & b \end{smallmatrix}\right) \in R \mid m \text{ is odd and } b = \bar{1}\} \\ &\subsetneqq C(N(R)) = \left\{\left(\begin{smallmatrix} k & a \\ 0 & b \end{smallmatrix}\right) \in R \mid k \neq 0 \text{ and } b \neq 0\right\} \\ &\subsetneqq R\backslash N(R) = \left\{\left(\begin{smallmatrix} h & a \\ 0 & b \end{smallmatrix}\right) \in R \mid h \neq 0 \text{ or } b \neq 0\right\}. \end{aligned}$$

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