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NORMALITY OF FUZZY TOPOLOGICAL SPACES IN KUBIAK-ŠOSTAK'S SENSE

M. Azab Abd-Allah

Abstract. The aim of this paper is to study the normality of fuzzy topological spaces in Kubiak-Šostak's sense. Also, some characterizations and the effects of some types of functions on these types of normality are studied.

1. Introduction

The study of fuzzy sets was initiated with the famous paper of Zadeh [8], and thereafter Chang [1] paved the way for the subsequent tremendous growth of the numerous fuzzy topological concepts. In Chang's definition, a fuzzy topology is a crisp subfamily of family of fuzzy subsets, and fuzziness in the openness of a fuzzy subset has not been considered. An essentially more general notion of fuzzy topology, in which each fuzzy subset has a certain degree of openness, was introduced by Kubiak [5] and Šostak [8].

In [4], Krsteska and Kim defined the concepts of fuzzy generalized α-closed sets and fuzzy generalized regular α-closed sets in Chang's fuzzy topological space. By using the above mentioned classes of generalized fuzzy closed sets, they introduced and studied the concepts of fuzzy normal space, fuzzy almost normal space, and fuzzy mildly normal space.

In this paper, we introduce the concepts of fuzzy almost normal, fuzzy normal, fuzzy mildly normal spaces in fuzzy topological spaces in Kubiak-Šostak's sense and then, we investigate some of their characteristic properties. Our results here represent a generalization of Krsteska and Kim's study. We can simply obtain their results by taking the r-cut to min.

2. Preliminaries

Throughout this paper, let X be a nonempty set and I is the closed unit interval $[0,1]$, $I_\circ = (0,1]$ and $I_1 = [0,1)$. The family of all fuzzy subsets on X denoted by I^X. $\underline{0}$ and $\underline{1}$ denote the smallest and the greatest fuzzy

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subsets on X, respectively. For a fuzzy subset $\lambda \in I^X$, $\underline{1} - \lambda$ denotes its complement. Given a function $f : X \to Y$, $f(\lambda)$ and $f^{-1}(\nu)$ define the direct image and the inverse image of f, defined by $f(\lambda)(y) = \bigvee_{f(x)=y} \lambda(x)$ and $f^{-1}(\nu)(x) = \nu(f(x))$, $\forall\ \nu \in I^Y$, $x \in X$, respectively. For fuzzy subsets λ and μ in X, we write $\lambda q \mu$ to mean that λ is quasi-coincident (q-coincident, for short) with μ, i.e., there exists at least one point $x \in X$ such that $\lambda(x) + \mu(x) > 1$. Negation of such a statement is denoted as $\lambda \bar{q} \mu$. Notions and notations not described in this paper are standard and usual.

Definition ([5, 7])**.** A function $\tau : I^X \to I$ is called a fuzzy topology on X if it satisfies the following conditions:

(O1) $\tau(\underline{0}) = \tau(\underline{1}) = 1$.
(O2) $\tau(\lambda_1 \wedge \lambda_2) \geq \tau(\lambda_1) \wedge \tau(\lambda_2)$ for each $\lambda_1, \lambda_2 \in I^X$.
(O3) $\tau(\vee_{i\in\Gamma} \lambda_i) \geq \wedge_{i\in\Gamma} \tau(\lambda_i)$ for any $\{\lambda_i\}_{i\in\Gamma} \subset I^X$.

The pair (X, τ) is called a fuzzy topological spaces (fts, for short). $\tau(\lambda)$ may be interpreted as a gradation of openness for λ. A function $f : (X, \tau_1) \to (Y, \tau_2)$ is said to be a fuzzy continuous if $\tau_1(f^{-1}(\nu)) \geq \tau_2(\nu)$ for each $\nu \in I^Y$.

Theorem 2.1 ([2])**.** *Let (X, τ) be an fts. Then for each $r \in I_0$, $\lambda \in I^X$, one defines an operator $C_\tau : I^X \times I_0 \to I^X$ as follows*:

$$C_\tau(\lambda, r) = \bigwedge\{\mu \in I^X | \lambda \leq \mu,\ \tau(\mu') \geq r\}.$$

For λ, $\mu \in I^X$ and $r, s \in I_0$, the operator C_τ satisfies the following statements:

(C1) $C_\tau(\underline{0}, r) = \underline{0}$.
(C2) $\lambda \leq C_\tau(\lambda, r)$.
(C3) $C_\tau(\lambda, r) \vee C_\tau(\mu, r) = C_\tau(\lambda \vee \mu, r)$.
(C4) $C_\tau(\lambda, r) \leq C_\tau(\lambda, s)$ *if* $r \leq s$.
(C5) $C_\tau(C_\tau(\lambda, r), r) = C_\tau(\lambda, r)$.

Theorem 2.2 ([3])**.** *Let (X, τ) be an fts. Then for each $r \in I_0$ and $\lambda \in I^X$, one defines an operator $I_\tau : I^X \times I_1 \to I^X$ as follows*:

$$I_\tau(\lambda, r) = \bigvee\{\mu \in I^X | \mu \leq \lambda,\ \tau(\mu) \geq r\}.$$

For λ, $\mu \in I^X$ and r, $s \in I_0$, the operator I_τ satisfies the following statements:

(I1) $I_\tau(\underline{1} - \lambda, r) = \underline{1} - C_\tau(\lambda, r)$.
(I2) $I_\tau(\underline{1}, r) = \underline{1}$.
(I3) $I_\tau(\lambda, r) \leq \lambda$.
(I4) $I_\tau(\lambda, r) \wedge I_\tau(\mu, r) = I_\tau(\lambda \wedge \mu, r)$.
(I5) $I_\tau(\lambda, s) \leq I_\tau(\lambda, r)$ *if* $r \leq s$.
(I6) $I_\tau(I_\tau(\lambda, r), r) = I_\tau(\lambda, r)$.
(I7) *If* $I_\tau(C_\tau(\lambda, r), r) = \lambda$, *then* $C_\tau(I_\tau(\underline{1} - \lambda, r), r) = \underline{1} - \lambda$.

Definition ([6])**.** Let (X, τ) be an fts, $\lambda \in I^X$ and $r \in I_0$.

(1) A fuzzy set λ is called r-regular fuzzy open (for short, r-rfo) if $\lambda = I_\tau(C_\tau(\lambda, r), r)$.

(2) A fuzzy set λ is called r-regular fuzzy closed (for short, r-rfc) if $\lambda = C_\tau(I_\tau(\lambda, r), r)$.
(3) A fuzzy set λ is called r-fuzzy α-open (for short, r-fαo) if

$$\lambda \leq I_\tau(C_\tau(I_\tau(\lambda, r), r), r).$$

λ is called r-fuzzy α-closed (for short, r-fαc) if and only if $\underline{1}-\lambda$ is r-fαo. The r-fuzzy α-closure and r-fuzzy α-interior of λ is defined by

$$\alpha C_\tau(\lambda, r) = \bigwedge\{\mu \in I^X \mid \lambda \leq \mu,\ \mu \text{ is } r\text{-}\alpha\text{fc}\},$$

$$\alpha I_\tau(\lambda, r) = \bigvee\{\mu \in I^X \mid \mu \leq \lambda,\ \mu \text{ is } r\text{-}\alpha\text{fo}\}.$$

Definition. Let (X, τ) be an fts, λ, $\mu \in I^X$ and $r \in I_0$.

(1) A fuzzy set λ is called r-generalized fuzzy closed (for short, r-gfc) if $C_\tau(\lambda, r) \leq \mu$ whenever $\lambda \leq \mu$ and $\tau(\mu) \geq r$. λ is called r-generalized fuzzy open (for short, r-gfo) if $\underline{1} - \lambda$ is r-generalized fuzzy closed.
(2) A fuzzy set λ is called r-generalized regular fuzzy closed (for short, r-grfc) if $C_\tau(\lambda, r) \leq \mu$ whenever $\lambda \leq \mu$ and μ is r-rfo. λ is called r-generalized regular fuzzy open (for short, r-grfo) if and only if $\underline{1} - \lambda$ is r-grfc.
(3) A fuzzy set λ is called r-generalized fuzzy α-closed (for short, r-gfαc) if $\alpha C_\tau(\lambda, r) \leq \mu$ whenever $\lambda \leq \mu$ and $\tau(\mu) \geq r$. λ is called r-generalized fuzzy α-open (for short, r-gfαo) if and only if $\underline{1} - \lambda$ is r-gfαc.
(4) A fuzzy set λ is called r-generalized regular fuzzy α-closed (for short, r-grfαc) if $\alpha C_\tau(\lambda, r) \leq \mu$ whenever $\lambda \leq \mu$ and μ is r-rfo. λ is called r-generalized regular fuzzy α-open (for short, r-grfαo) if and only if $\underline{1} - \lambda$ is r-grfαc.

Remark 2.3. From the above definitions it is not difficult to conclude that the following diagram of implications is true:

$$\begin{array}{ccc} r\text{-gfc set} & \Rightarrow & r\text{-grfc set} \\ \Downarrow & & \Downarrow \\ r\text{-gf}\alpha\text{c set} & \Rightarrow & r\text{-grf}\alpha\text{c set} \end{array}$$

Counter example 2.1. Let $X = \{a, b\}$ and let λ_1, μ_1 and ν_1 are fuzzy sets defined by

$$\lambda_1(a) = 0.2, \quad \lambda_1(b) = 0.4;$$
$$\mu_1(a) = 0.9, \quad \mu_1(b) = 0.4;$$
$$\nu_1(a) = 0.1, \quad \nu_1(b) = 0.4.$$

Define τ_1 and τ_2 on X as follows:

$$\tau_1(\lambda) = \begin{cases} 1, & \text{if } \lambda = \underline{0}, \underline{1}; \\ \frac{1}{4}, & \text{if } \lambda = \lambda_1 ; \\ \frac{1}{3}, & \text{if } \lambda = \mu_1 ; \\ 0, & \text{otherwise}, \end{cases} \qquad \tau_2(\lambda) = \begin{cases} 1, & \text{if } \lambda = \underline{0}, \underline{1}; \\ \frac{1}{2}, & \text{if } \lambda = \mu_1; \\ 0, & \text{otherwise.} \end{cases}$$

(1) In τ_1, it can be shown that ν_1 is an $\frac{1}{4}$-gfαc set, but it is not an $\frac{1}{4}$-gfc set. Also, ν_1 is an $\frac{1}{4}$-grfαc set, but it is not an $\frac{1}{4}$-grfc set.
(2) In τ_2, ν_1 is an $\frac{1}{2}$-grfαc set, but it is not an $\frac{1}{2}$-gfαc set. Also, ν_1 is an $\frac{1}{2}$-grfc set, but it is not an $\frac{1}{2}$-gfc set.

Definition. Let $f : (X, \tau_1) \to (Y, \tau_2)$ be a function between fts's (X, τ_1) and (Y, τ_2). Then the function f is called:

(1) fuzzy regular continuous if $f^{-1}(\nu)$ is r-rfo for each $\nu \in I^Y$ and $r \in I_0$ such that $\tau_2(\nu) \geq r$.
(2) fuzzy regular irresolute if $f^{-1}(\nu)$ is r-rfo for each $\nu \in I^Y$, $r \in I_0$ such that ν is r-rfo.
(3) fuzzy almost regular generalized continuous if $f^{-1}(\nu)$ is r-grfo for each $\nu \in I^Y$ and $r \in I_0$ such that $\tau_2(\nu) \geq r$.
(4) fuzzy almost continuous if $\tau_1(f^{-1}(\nu)) \geq r$ for each $\nu \in I^Y$, $r \in I_0$ such that ν is r-rfo.

Definition. Let $f : (X, \tau_1) \to (Y, \tau_2)$ be a function between fts's (X, τ_1) and (Y, τ_2). Then the function f is called:

(1) fuzzy regular closed if $f(\lambda)$ is r-rfc for each $\lambda \in I^X$, $r \in I_0$ such that $\tau_1(\lambda') \geq r$.
(2) fuzzy almost closed if $\tau_2(f(\lambda)') \geq r$ for each $\lambda \in I^X$, $r \in I_0$ such that λ is r-rfc.
(3) fuzzy almost generalized closed if $f(\lambda)$ is r-gfc for each $\lambda \in I^X$, $r \in I_0$ such that λ is r-rfc.
(4) fuzzy almost regular generalized closed if $f(\lambda)$ is r-grfc for each $\lambda \in I^X$, $r \in I_0$ such that λ is r-rfc.

3. Fuzzy normal spaces

Definition. An fts (X, τ) is said to be:

(1) fuzzy almost normal space if for each λ_1, $\lambda_2 \in I^X$ and $r \in I_0$ such that $\tau(\lambda_1') \geq r$, λ_2 is r-rfc set and $\lambda_1 \bar{q} \lambda_2$, there exist μ_1, $\mu_2 \in I^X$ such that $\lambda_1 \leq \mu_1$, $\lambda_2 \leq \mu_2$ with $\mu_1 \bar{q} \mu_2$.
(2) fuzzy normal space if for each λ_1, $\lambda_2 \in I^X$ and $r \in I_0$ such that $\tau(\lambda_1') \geq r$, $\tau(\lambda_2') \geq r$ and $\lambda_1 \bar{q} \lambda_2$, there exists μ_1, $\mu_2 \in I^X$ such that $\tau(\mu_1) \geq r$, $\tau(\mu_2) \geq r$, $\lambda_1 \leq \mu_1$, $\lambda_2 \leq \mu_2$ and $\mu_1 \bar{q} \mu_2$.
(3) fuzzy mildly normal space if for each λ_1, $\lambda_2 \in I^X$ and $r \in I_0$ such that λ_1 and λ_2 are r-rfc sets and $\lambda_1 \bar{q} \lambda_2$, there exists μ_1, $\mu_2 \in I^X$ such that $\tau(\mu_1) \geq r$, $\tau(\mu_2) \geq r$, $\lambda_1 \leq \mu_1$, $\lambda_2 \leq \mu_2$ and $\mu_1 \bar{q} \mu_2$.

Clearly, every fuzzy normal space is almost fuzzy normal space and every fuzzy almost normal space is fuzzy mildly normal space, but the converse is not true in general as the following example shows:

Example 3.1. Let $X = \{x\}$ and define the fuzzy sets μ, ν, ρ, γ and ω as follows:

$$\mu(x) = 0.4, \quad \nu(x) = 0.7, \quad \rho(x) = 0.8, \quad \gamma(x) = 0.6, \quad \omega(x) = 0.2.$$

Define the fuzzy topologies τ_1, τ_2 as follows:

$$\tau_1(\lambda) = \begin{cases} 1, & \text{if } \lambda = \underline{0}, \underline{1}; \\ \frac{1}{4}, & \text{if } \lambda = \mu\ ; \\ \frac{1}{3}, & \text{if } \lambda = \nu\ ; \\ 0, & \text{otherwise.} \end{cases}$$

$$\tau_2(\lambda) = \begin{cases} 1, & \text{if } \lambda = \underline{0}, \underline{1}; \\ \frac{1}{4}, & \text{if } \lambda = \nu\ ; \\ \frac{1}{2}, & \text{if } \lambda = \rho\ ; \\ \frac{1}{5}, & \text{if } \lambda = \gamma\ ; \\ \frac{1}{6}, & \text{if } \lambda = \omega\ ; \\ 0, & \text{otherwise.} \end{cases}$$

(1) (X, τ_1) is fuzzy mildly normal space but not fuzzy almost normal space.
(2) (X, τ_2) is fuzzy almost normal space but not fuzzy normal space.

Theorem 3.1. *Let (X, τ) be an fts. Then the following statements are equivalent*:

(1) *(X, τ) is a fuzzy mildly normal space*;
(2) *For each pair of r-rfc sets λ_1, $\lambda_2 \in I^X$ and $r \in I_0$ such that $\lambda_1 \bar{q} \lambda_2$, there exist r-rfo sets μ_1, μ_2 such that $\lambda_1 \leq \mu_1$, $\lambda_2 \leq \mu_2$ and $\mu_1 \bar{q} \mu_2$;*
(3) *For each pair of r-rfc sets λ_1, $\lambda_2 \in I^X$ and $r \in I_0$ such that $\lambda_1 \bar{q} \lambda_2$, there exist r-gfo sets μ_1, μ_2 such that $\lambda_1 \leq \mu_1$, $\lambda_2 \leq \mu_2$ and $\mu_1 \bar{q} \mu_2$;*
(4) *For each pair of r-rfc sets λ_1, $\lambda_2 \in I^X$ and $r \in I_0$ such that $\lambda_1 \bar{q} \lambda_2$, there exist r-gf$\alpha$o sets μ_1 and μ_2, such that $\lambda_1 \leq \mu_1$, $\lambda_2 \leq \mu_2$ and $\mu_1 \bar{q} \mu_2$;*
(5) *For each r-rfc set $\lambda \in I^X$ and each r-rfo set $\mu \in I^X$ such that $\lambda \leq \mu$, there exists $\rho \in I^X$ such that $\tau(\rho) \geq r$ and*

$$\lambda \leq \rho \leq C_\tau(\rho, r) \leq \mu;$$

(6) *For each r-rfc set $\lambda \in I^X$ and each r-rfo set $\mu \in I^X$ such that $\lambda \leq \mu$, there an r-rfo set $\rho \in I^X$ such that*

$$\lambda \leq \rho \leq C_\tau(\rho, r) \leq \mu;$$

(7) *For each r-rfc set $\lambda \in I^X$ and r-rfo set $\mu \in I^X$ such that $\lambda \leq \mu$, there exists an r-gfo set $\rho \in I^X$ such that*

$$\lambda \leq \rho \leq C_\tau(\rho, r) \leq \mu;$$

(8) *For each r-rfc set $\lambda \in I^X$ and each r-rfo set $\mu \in I^X$ such that $\lambda \leq \mu$, there exists an r-gfαo set $\rho \in I^X$ such that*

$$\lambda \leq \rho \leq \alpha C_\tau(\rho, r) \leq \mu;$$

(9) *For each r-rfc set $\lambda \in I^X$ and each r-rfo set $\mu \in I^X$ such that $\lambda \leq \mu$, there exists an r-fαo set $\rho \in I^X$ such that*

$$\lambda \leq \rho \leq \alpha C_\tau(\rho, r) \leq \mu;$$

(10) *For each r-rfc sets λ_1, $\lambda_2 \in I^X$ and $r \in I_0$ such that $\lambda_1 \bar{q} \lambda_2$, there exist r-fαo sets μ_1 and μ_2, such that $\lambda_1 \leq \mu_1$, $\lambda_2 \leq \mu_2$ and $\mu_1 \bar{q} \mu_2$.*

Proof. (1) $\Rightarrow$ (5) For each $r \in I_0$, let λ be r-rfc set and let μ be an r-rfo set such that $\lambda \leq \mu$. Then $\lambda \bar{q} \underline{1} - \mu$. By (1), there exist an r-rfo sets λ_1 and μ_1, such that $\lambda \leq \lambda_1$, $\underline{1} - \mu \leq \lambda_1$ such that $\lambda_1 \bar{q} \mu_1$. Thus

$$\lambda \leq \lambda_1 \leq C_\tau(\lambda_1, r) \leq \underline{1} - \mu_1 \leq \mu.$$

(5) $\Rightarrow$ (2) Let λ_1 and λ_2 be an r-rfc set such that $\lambda_1 \bar{q} \lambda_2$. Then $\lambda_1 \leq \underline{1} - \lambda_2$. By (5), there exists $\rho \in I^X$ such that $\tau(\rho) \geq r$ and

$$\lambda_1 \leq \rho \leq C_\tau(\rho, r) \leq \underline{1} - \lambda_2.$$

It follows that

$$\lambda_1 \leq I_\tau(C_\tau(\rho, r), r) \leq \underline{1} - \lambda_2$$

and

$$\lambda_2 \leq \underline{1} - C_\tau(\rho, r) = I_\tau(\underline{1} - \rho, r).$$

Then $\mu_1 = I_\tau(C_\tau(\rho, r), r)$ and $\mu_2 = I_\tau(\underline{1} - \rho, r)$ are r-rfo sets such that $\lambda_1 \leq \mu_1$, $\lambda_2 \leq \mu_2$ and $\mu_1 \bar{q} \mu_2$.

(2) $\Rightarrow$ (6) Let λ_1 be any r-rfc set and let λ_2 be r-rfo set such that $\lambda_1 \leq \lambda_2$. Then $\lambda_1 \bar{q} \underline{1} - \lambda_2$. According to (2), there exist r-rfo sets μ_1 and μ_2 such that $\lambda_1 \leq \mu_1$, $\underline{1} - \lambda_2 \leq \mu_2$ and $\mu_1 \bar{q} \mu_2$. Let $\rho = I_\tau(C_\tau(\mu_1, r), r)$. Then ρ is an r-rfo set such that

$$\lambda_1 \leq \rho \leq C_\tau(\rho, r) \leq \underline{1} - \mu_2 \leq \lambda_2.$$

(6) $\Rightarrow$ (2) Let λ_1 and λ_2 be r-rfc sets such that $\lambda_1 \bar{q} \lambda_2$. Then $\lambda_1 \leq \underline{1} - \lambda_2$. By (6), there exists r-rfo set ρ such that

$$\lambda_1 \leq \rho \leq C_\tau(\rho, r) \leq \underline{1} - \lambda_2.$$

For $C_\tau(\rho, r) \leq \underline{1} - \lambda_2$, there exists r-rfo set μ such that

$$C_\tau(\rho, r) \leq \mu \leq C_\tau(\mu, r) \leq \underline{1} - \lambda_2.$$

Then $\lambda_2 \leq \underline{1} - C_\tau(\mu, r) = I_\tau(\underline{1} - \mu, r)$, $I_\tau(\underline{1} - \mu, r)$ is r-rfo set and $\rho \bar{q} I_\tau(\underline{1} - \mu, r)$.

(4) $\Rightarrow$ (8) Let λ_1 be an r-rfc set and let λ_2 be r-rfo set such that $\lambda_1 \leq \lambda_2$. Then $\lambda_1 \bar{q} \underline{1} - \lambda_2$. By (4), there exist r-gfαo sets μ_1 and μ_2 such that $\lambda_1 \leq \mu_1$, $\underline{1} - \lambda_2 \leq \mu_2$ and $\mu_1 \bar{q} \mu_2$. Since μ_2 is r-gfαo set and $\underline{1} - \lambda_2$ is r-rfc set, from $\underline{1} - \lambda_2 \leq \mu_2$ follows that $\underline{1} - \lambda_2 \leq \alpha I_\tau(\mu_2, r)$. Thus

$$\underline{1} - \lambda_2 \leq \alpha I_\tau(\mu_2, r) \leq \mu_2 \leq \underline{1} - \mu_1.$$

Since $\underline{1} - I_\tau(\mu_2)$ is an r-gfαc set, from $\mu_1 \leq \underline{1} - \alpha I_\tau(\mu_2, r)$, we obtain that

$$\alpha C_\tau(\mu_1, r) \leq \underline{1} - \alpha I_\tau(\mu_2, r).$$

Hence

$$\lambda_1 \leq \mu_1 \leq \alpha C_\tau(\mu_1, r) \leq \lambda_2.$$

(8) ⇒ (9) Let λ_1 be an r-rfc set and let λ_2 be an r-rfo set such that $\lambda_1 \leq \lambda_2$. Then $\lambda_1 \bar{q} \underline{1} - \mu$. By (8), there exists an r-gfαo set μ_1 such that

$$\lambda_1 \leq \mu_1 \leq \alpha C_\tau(\mu_1, r) \leq \lambda_2.$$

Since μ_1 is an r-gfαo set, from $\lambda_1 \leq \mu_1$ follows that $\lambda_1 \leq \alpha I_\tau(\mu_1, r)$. Then, $\mu_2 = \alpha I_\tau(\mu_1, r)$ is an r-fαo set and

$$\lambda_1 \leq \mu_2 \leq \alpha C_\tau(\mu_2, r) \leq \alpha C_\tau(\mu_1, r) \leq \lambda_2.$$

(9) ⇒ (10) Let λ_1 and λ_2 be are r-rfc sets such that $\lambda_1 \bar{q} \lambda_2$. Then $\lambda_1 \leq \underline{1} - \lambda_2$. By (9), there exists r-fαo set μ_1 such that

$$\lambda_1 \leq \mu_1 \leq \alpha C_\tau(\mu_1, r) \leq \underline{1} - \lambda_2.$$

Then $\mu_2 = \underline{1} - \alpha C_\tau(\mu_1, r)$ is an r-fαo set and $\mu_1 \bar{q} \mu_2$.

(10) ⇒ (1) Let λ_1 and λ_2 be are r-rfc sets such that $\lambda_1 \bar{q} \lambda_2$. Then $\lambda_1 \leq \underline{1} - \lambda_2$. By (10), there exist r-fαo sets μ_1 and μ_2 such that $\lambda_1 \leq \mu_1$, $\lambda_2 \leq \mu_2$ and $\mu_1 \bar{q} \mu_2$. Let $\rho_1 = I_\tau(C_\tau(I_\tau(\mu_1, r), r), r)$ and $\rho_2 = I_\tau(C_\tau(I_\tau(\mu_2, r), r), r)$. Then $\tau(\rho_1) \geq r$ and $\tau(\rho_2) \geq r$ and $\rho_1 \bar{q} \rho_2$. Hence (X, τ) is a fuzzy normal space.

The implications (1) ⇒ (3) ⇒ (4), (3) ⇒ (7) ⇒ (8) and (2) ⇒ (1) are trivial. □

Theorem 3.2. *Let (X, τ) be an fts. Then the following statements are equivalent*:

(1) *(X, τ) is a fuzzy normal space (resp. fuzzy almost normal space)*;
(2) *For each λ_1, $\lambda_2 \in I^X$ and $r \in I_0$ such that $\tau(\lambda_1') \geq r$, $\tau(\lambda_2') \geq r$ and $\lambda_1 \bar{q} \lambda_2$, there exist r-rfo sets μ_1, μ_2 such that $\lambda_1 \leq \mu_1$, $\lambda_2 \leq \mu_2$ and $\mu_1 \bar{q} \mu_2$*;
(3) *For each λ_1, $\lambda_2 \in I^X$ and $r \in I_0$ such that $\tau(\lambda_1') \geq r$, $\tau(\lambda_2') \geq r$ and $\lambda_1 \bar{q} \lambda_2$, there exist r-gfo sets μ_1, μ_2 such that $\lambda_1 \leq \mu_1$, $\lambda_2 \leq \mu_2$ and $\mu_1 \bar{q} \mu_2$*;
(4) *For each λ_1, $\lambda_2 \in I^X$ and $r \in I_0$ such that $\tau(\lambda_1') \geq r$, $\tau(\lambda_2') \geq r$ and $\lambda_1 \bar{q} \lambda_2$, there exist r-g f$\alpha$o sets μ_1 and μ_2, such that $\lambda_1 \leq \mu_1$, $\lambda_2 \leq \mu_2$ and $\mu_1 \bar{q} \mu_2$*;
(5) *For each λ_1, $\lambda_2 \in I^X$ and $r \in I_0$ such that $\tau(\lambda_1') \geq r$, $\tau(\lambda_2) \geq r$ and $\lambda_1 \leq \lambda_2$, there exists $\rho \in I^X$ such that $\tau(\rho) \geq r$ and*

$$\lambda_1 \leq \rho \leq C_\tau(\rho, r) \leq \lambda_2;$$

(6) *For each λ_1, $\lambda_2 \in I^X$ and $r \in I_0$ such that $\tau(\lambda_1') \geq r$, $\tau(\lambda_2) \geq r$ and $\lambda_1 \leq \lambda_2$, there an r-rfo set $\rho \in I^X$ such that*

$$\lambda_1 \leq \rho \leq C_\tau(\rho, r) \leq \lambda_2;$$

(7) *For each λ_1, $\lambda_2 \in I^X$ and $r \in I_0$ such that $\tau(\lambda_1') \geq r$, $\tau(\lambda_2) \geq r$ and $\lambda_1 \leq \lambda_2$, there exists an r-gfo set $\rho \in I^X$ such that*

$$\lambda_1 \leq \rho \leq C_\tau(\rho, r) \leq \lambda_2;$$

(8) *For each* λ_1, $\lambda_2 \in I^X$ *and* $r \in I_0$ *such that* $\tau(\lambda_1') \geq r$, $\tau(\lambda_2) \geq r$ *and* $\lambda_1 \leq \lambda_2$, *there exists an* r-*gfαo set* $\rho \in I^X$ *such that*

$$\lambda_1 \leq \rho \leq \alpha C_\tau(\rho, r) \leq \lambda_2;$$

(9) *For each* λ_1, $\lambda_2 \in I^X$ *and* $r \in I_0$ *such that* $\tau(\lambda_1') \geq r$, $\tau(\lambda_2) \geq r$ *and* $\lambda_1 \leq \lambda_2$, *there exists an* r-*fαo set* $\rho \in I^X$ *such that*

$$\lambda_1 \leq \rho \leq \alpha C_\tau(\rho, r) \leq \lambda_2;$$

(10) *For each* λ_1, $\lambda_2 \in I^X$ *and* $r \in I_0$ *such that* $\tau(\lambda_1') \geq r$, $\tau(\lambda_2') \geq r$ *and* $\lambda_1 \bar{q} \lambda_2$, *there exist* r-*fαo sets* μ_1 *and* μ_2, *such that* $\lambda_1 \leq \mu_1$, $\lambda_2 \leq \mu_2$ *and* $\mu_1 \bar{q} \mu_2$.

Proof. It is clear from Theorem 3.1. □

Theorem 3.3. *Let* (X, τ_1) *and* (Y, τ_2) *be fts's such that* (X, τ_1) *is a fuzzy normal space. If* $f : (X, \tau_1) \to (Y, \tau_2)$ *is a fuzzy almost continuous, fuzzy almost closed and surjective function, then* (Y, τ_2) *is a fuzzy mildly normal space.*

Proof. For each $r \in I_0$, let ν_1, $\nu_2 \in I^Y$ be r-rfc sets such that $\nu_1 \bar{q} \nu_2$. Since f is a fuzzy almost continuous function, $\tau_1(f^{-1}(\nu_1)') \geq r$, $\tau_1(f^{-1}(\nu_2)') \geq r$ and $f^{-1}(\nu_1) \bar{q} f^{-1}(\nu_2)$. Since (X, τ_1) is a fuzzy normal space, there exist μ_1, $\mu_2 \in I^X$ such that $\tau_1(\mu_1) \geq r$, $\tau_1(\mu_2) \geq r$ and $f^{-1}(\nu_1) \leq \mu_1$, $f^{-1}(\nu_2) \leq \mu_2$ with $\mu_1 \bar{q} \mu_2$. Since $I_{\tau_1}(C_{\tau_1}(\mu_1, r), r)$ and $I_{\tau_1}(C_{\tau_1}(\mu_2, r), r)$ are r-rfo sets and

$$I_{\tau_1}(C_{\tau_1}(\mu_1, r), r) \bar{q} I_{\tau_1}(C_{\tau_1}(\mu_2, r), r).$$

Furthermore,

$$f^{-1}(\nu_i) \leq \mu_i \leq I_{\tau_1}(C_{\tau_1}(\mu_i, r), r) \text{ for each } i \in \{1, 2\}.$$

Since f is fuzzy almost closed, there exist γ_1, $\gamma_2 \in I^Y$ such that $\tau_2(\gamma_1) \geq r$, $\tau_2(\gamma_2) \geq r$ and $\nu_1 \leq \gamma_1$, $\nu_2 \leq \gamma_2$ with

$$f^{-1}(\gamma_i) \leq I_{\tau_1}(C_{\tau_1}(\mu_i, r), r) \text{ for each } i \in \{1, 2\}.$$

Moreover $\gamma_1 \bar{q} \gamma_2$. Hence (Y, τ_2) is a fuzzy mildly normal space. □

Corollary 3.4. *Let* (X, τ_1) *and* (Y, τ_2) *be fuzzy topological spaces and let* (X, τ_1) *be a fuzzy normal space. If* $f : (X, \tau_1) \to (Y, \tau_2)$ *is a fuzzy almost continuous and fuzzy closed function, then* (Y, τ_2) *is a fuzzy mildly normal space.*

Corollary 3.5. *Let* (X, τ_1) *and* (Y, τ_2) *be fuzzy topological spaces and let* (X, τ_1) *be fuzzy mildly normal space. If* $f : (X, \tau_1) \to (Y, \tau_2)$ *is a fuzzy almost continuous, fuzzy almost closed and fuzzy open (resp. fuzzy continuous, fuzzy closed, fuzzy open) function, then* (Y, τ_2) *is a fuzzy mildly normal space.*

Theorem 3.6. *Let* (X, τ_1) *and* (Y, τ_2) *be fuzzy topological spaces and let* (Y, τ_2) *be a fuzzy mildly normal space (resp. fuzzy normal space). If* $f : (X, \tau_1) \to (Y, \tau_2)$ *is a fuzzy almost regular generalized continuous, fuzzy regular closed (resp. fuzzy almost closed) injective function, then* (X, τ_1) *is a fuzzy mildly normal space.*

Proof. For each $r \in I_0$, let λ_1, $\lambda_2 \in I^X$ be r-rfc sets such that $\lambda_1 \bar{q} \lambda_2$. Since f is a fuzzy regular closed (resp. fuzzy almost closed) injective function, $f(\lambda_1)$ and $f(\lambda_2)$ are r-rfc sets (resp. $\tau_2(f(\lambda_1)') \geq r$ and $\tau_2(f(\lambda_2)') \geq r$). Since (Y, τ_2) is a fuzzy mildly normal (resp. fuzzy normal) space, there exist ν_1, $\nu_2 \in I^Y$ such that $\tau_2(\nu_1) \geq r$, $\tau_2(\nu_2) \geq r$ and $f(\lambda_1) \leq \nu_1$, $f(\lambda_2) \leq \nu_2$ with $\nu_1 \bar{q} \nu_2$.

Now let $\gamma_1 = I_{\tau_2}(C_{\tau_2}(\nu_1, r), r)$ and let $\gamma_2 = I_{\tau_2}(C_{\tau_2}(\nu_2, r), r)$. Then γ_1, γ_2 are r-rfo sets such that $f(\lambda_1) \leq \gamma_1$, $f(\lambda_2) \leq \gamma_2$ and $\gamma_1 \bar{q} \gamma_2$. Since f is a fuzzy almost regular generalized continuous function, then $f^{-1}(\gamma_1)$ and $f^{-1}(\gamma_2)$ are r-rgfo sets. Furthermore, $\lambda_1 \leq f^{-1}(\gamma_1)$, $\lambda_2 \leq f^{-1}(\gamma_2)$ and $f^{-1}(\gamma_1) \bar{q} f^{-1}(\gamma_2)$. Hence by Theorem 3.1, (X, τ_1) is a fuzzy mildly normal space. □

Theorem 3.7. *Let $f : (X, \tau_1) \to (Y, \tau_2)$ be a fuzzy regular continuous, fuzzy almost generalized closed and surjective function. If (X, τ_1) is a fuzzy mildly normal space, then (Y, τ_2) is a fuzzy normal space.*

Proof. For each $r \in I_0$, let ν_1, $\nu_2 \in I^Y$ such that $\tau_2(\nu_1') \geq r$, $\tau_2(\nu_2') \geq r$ and $\nu_1 \bar{q} \nu_2$. Since f is fuzzy regular continuous, $f^{-1}(\nu_1)$ and $f^{-1}(\nu_2)$ are r-rfc sets with $f^{-1}(\nu_1) \bar{q} f^{-1}(\nu_2)$. Since (X, τ_1) is fuzzy mildly normal, there exist λ_1, $\lambda_2 \in I^X$ such that $\tau_1(\lambda_1) \geq r$, $\tau_1(\lambda_2) \geq r$ and $f^{-1}(\nu_1) \leq \lambda_1$, $f^{-1}(\nu_2) \leq \lambda_2$ with $\lambda_1 \bar{q} \lambda_2$.

Let $\mu_1 = I_{\tau_1}(C_{\tau_1}(\lambda_1, r), r)$ and let $\mu_2 = I_{\tau_1}(C_{\tau_1}(\lambda_2, r), r)$. Then clearly μ_1 and μ_2 are r-rfo sets such that $f^{-1}(\nu_1) \leq \mu_1$, $f^{-1}(\nu_2) \leq \mu_2$ and $\mu_1 \bar{q} \mu_2$. Since f is fuzzy almost generalized closed, there exist γ_1, $\gamma_2 \in I^Y$ such that γ_1, γ_2 are r-gfo sets and $\nu_1 \leq \gamma_1$, $\nu_2 \leq \gamma_2$, $f^{-1}(\gamma_1) \leq \mu_1$ and $f^{-1}(\gamma_2) \leq \mu_2$. Since $\mu_1 \bar{q} \mu_2$, then $\gamma_1 \bar{q} \gamma_2$. But γ_1, γ_2 are r-gfo, $\nu_1 \leq I_{\tau_2}(\gamma_1, r)$ and $\nu_2 \leq I_{\tau_2}(\gamma_2, r)$. Furthermore $I_{\tau_2}(\gamma_1, r) \bar{q} I_{\tau_2}(\gamma_2, r)$. Hence (Y, τ_2) is a fuzzy normal space. □

Corollary 3.8. *Let $f : (X, \tau_1) \to (Y, \tau_2)$ be a fuzzy regular continuous, fuzzy closed and surjective function. If (X, τ_1) is a fuzzy mildly normal space, then (Y, τ_2) is a fuzzy normal space.*

Theorem 3.9. *Let $f : (X, \tau_1) \to (Y, \tau_2)$ be a fuzzy regular irresolute (resp. fuzzy almost continuous), fuzzy almost regular generalized closed and surjective function. If (X, τ_1) is a fuzzy mildly normal space (resp. fuzzy normal space), then (Y, τ_2) is a fuzzy mildly normal space.*

Proof. For each $r \in I_0$, let ν_1, $\nu_2 \in I^Y$ be r-rfc sets such that $\nu_1 \bar{q} \nu_2$. Since f is a fuzzy regular irresolute (resp. fuzzy almost closed) function, $f^{-1}(\nu_1)$ and $f^{-1}(\nu_2)$ are r-rfc sets (resp. $\tau_1(f^{-1}(\nu_1)') \geq r$, $\tau_1(f^{-1}(\nu_2)') \geq r$) such that $f^{-1}(\nu_1) \bar{q} f^{-1}(\nu_2)$. Since (X, τ_1) is a fuzzy mildly normal space (resp. fuzzy normal space), there exist λ_1, $\lambda_2 \in I^X$ such that $f^{-1}(\nu_1) \leq \lambda_1$, $f^{-1}(\nu_2) \leq \lambda_2$ and $\lambda_1 \bar{q} \lambda_2$.

Let $\mu_1 = I_{\tau_1}(C_{\tau_1}(\lambda_1, r), r)$ and let $\mu_2 = I_{\tau_1}(C_{\tau_1}(\lambda_2, r), r)$. Then clearly μ_1, μ_2 are r-rfo sets such that $f^{-1}(\nu_1) \leq \mu_1$, $f^{-1}(\nu_2) \leq \mu_2$ and $\mu_1 \bar{q} \mu_2$. Since f is a fuzzy almost regular generalized closed function, there exist γ_1, $\gamma_2 \in I^Y$ such that γ_1 and γ_2 are r-grfo sets, $\nu_1 \leq \gamma_1$, $\nu_2 \leq \gamma_2$, $f^{-1}(\gamma_1) \leq \mu_1$ and

$f^{-1}(\gamma_2) \leq \mu_2$. Since $\mu_1 \bar{q} \mu_2$, then $\gamma_1 \bar{q} \gamma_2$. Hence (Y, τ_2) is a fuzzy mildly normal space. □

Corollary 3.10. *Let $f : (X, \tau_1) \to (Y, \tau_2)$ be a fuzzy almost continuous, fuzzy almost closed and surjective function. If (X, τ_1) is a fuzzy normal space, then (Y, τ_2) is a fuzzy mildly normal space.*

Proof. The proof is determined straightforward. □

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Department of Mathematics
Faculty of Science
Assuit University
Assuit, Egypt
E-mail address: mazab57@yahoo.com