

THE LOWER AUTOCENTRAL SERIES OF ABELIAN GROUPS

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ABSTRACT. In the present paper we introduce the lower autocentral series of autocommutator subgroups of a given group. Following our previous work on the subject in 2009, it is shown that every finite abelian group is isomorphic with n^{th} -term of the lower autocentral series of some finite abelian group.

1. Introduction

Let $A = \text{Aut}(G)$ denote the group of automorphisms of a given group G . For any element $g \in G$ and $\alpha \in A$ the element $[g, \alpha] = g^{-1}g^\alpha$ is an *autocommutator* of g and α . We define the autocommutator of higher weight inductively as follows:

$$[g, \alpha_1, \alpha_2, \dots, \alpha_i] = [[g, \alpha_1, \alpha_2, \dots, \alpha_{i-1}], \alpha_i]$$

for all $\alpha_1, \alpha_2, \dots, \alpha_i \in A$.

So the *autocommutator subgroup* of weight $i + 1$ is defined in the following way:

$$K_i(G) = [G, \underbrace{A, \dots, A}_{i\text{-times}}] = \langle [g, \alpha_1, \alpha_2, \dots, \alpha_i] \mid g \in G, \alpha_1, \alpha_2, \dots, \alpha_i \in A \rangle.$$

Clearly $K_i(G)$ is a characteristic subgroup of G for all $i \geq 1$. Therefore, one obtains a descending chain of autocommutator subgroups of G as follows:

$$G \supseteq K_1(G) \supseteq K_2(G) \supseteq \dots \supseteq K_i(G) \supseteq \dots,$$

which we may call it the *lower autocentral series* of G . The aim of the present paper is to prove the following main result.

Theorem 1.1. *For any finite abelian group G and every natural number $n \in \mathbb{N}$, there exists a finite abelian group H such that*

$$G \cong K_n(H).$$

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2. Preliminary results

In order to prove our main result, we need to prove some technical lemmas, which are interesting in their own right.

Lemma 2.1. *Let $G = H \times K$ be the direct product of two characteristic subgroups H and K . Then for any natural number n ,*

$$K_n(H \times K) = K_n(H) \times K_n(K).$$

Proof. Clearly every element $g \in G$ can be written as $g = hk$, where $h \in H$ and $k \in K$. By Lemma 2.1 of [5],

$$\text{Aut}(G) \cong \text{Aut}(H) \times \text{Aut}(K).$$

Using induction on n , we show that

$$[hk, \alpha_1, \alpha_2, \dots, \alpha_n] = [h, \alpha_{1|H}, \alpha_{2|H}, \dots, \alpha_{n|H}][k, \alpha_{1|K}, \alpha_{2|K}, \dots, \alpha_{n|K}],$$

for all $\alpha_1, \alpha_2, \dots, \alpha_n \in \text{Aut}(G)$. If $n = 1$, then since the subgroups H and K are characteristic in G and $hk = kh$, we have

$$[hk, \alpha_1] = [h, \alpha_1][k, \alpha_1].$$

Now, assume the result holds for $n - 1$, then

$$\begin{aligned} & [hk, \alpha_1, \alpha_2, \dots, \alpha_n] \\ &= [[hk, \alpha_1, \alpha_2, \dots, \alpha_{n-1}], \alpha_n] \\ &= [[h, \alpha_{1|H}, \alpha_{2|H}, \dots, \alpha_{n-1|H}][k, \alpha_{1|K}, \alpha_{2|K}, \dots, \alpha_{n-1|K}], \alpha_n] \\ &= [h, \alpha_{1|H}, \alpha_{2|H}, \dots, \alpha_{n-1|H}, \alpha_n][k, \alpha_{1|K}, \alpha_{2|K}, \dots, \alpha_{n-1|K}, \alpha_n]. \end{aligned}$$

This implies that $K_n(G) \subseteq K_n(H) \times K_n(K)$.

By Lemma 2.1 of [5], any automorphism μ of H , can be extended to an automorphism $\bar{\mu}$ of G . Hence, for all $\mu_1, \dots, \mu_n \in \text{Aut}(H)$ and $h \in H$

$$[h, \mu_1, \dots, \mu_n] = [h, \bar{\mu}_1, \dots, \bar{\mu}_n] \in K_n(G).$$

Therefore $K_n(H) \subseteq K_n(G)$. Similarly, $K_n(K)$ is contained in $K_n(G)$. Thus

$$K_n(G) = K_n(H) \times K_n(K). \quad \square$$

Using the above notation, we have the following

Lemma 2.2. *If G is a finite cyclic group, then for any natural number n ,*

$$K_n(G) = G^{2^n}.$$

Proof. Let $G = \langle x \mid x^m = 1 \rangle$ be the cyclic group of order m . Clearly $\varphi : x \mapsto x^i$ is an automorphism of G if and only if $(i, m) = 1$. Since G is abelian, it follows that $\alpha : x \mapsto x^{-1}$ is an automorphism. So by an easy induction, for all $g \in G$, if n is even

$$g^{2^n} = [g, \underbrace{\alpha, \dots, \alpha}_{n\text{-times}}] \in G^{2^n},$$

and if n is odd, then

$$g^{2^n} = [g^{-1}, \underbrace{\alpha, \dots, \alpha}_{n\text{-times}}] \in G^{2^n},$$

which implies that G^{2^n} is contained in $K_n(G)$.

Conversely, assume $(2^n, m) = 1$ then m is odd number and hence $G = G^{2^n}$, which follows that $K_n(G) \subseteq G^{2^n}$. Now, assume $(2^n, m) \neq 1$, i.e., m is an even number then using an easy induction for all $\alpha_1, \alpha_2, \dots, \alpha_n \in \text{Aut}(G)$, we obtain that

$$[g, \alpha_1, \alpha_2, \dots, \alpha_n] = g^{(i_1-1)\dots(i_n-1)} \in G^{2^n},$$

where $\alpha_j(g) = g^{i_j}$, $(i_j, m) = 1$ that is to say $i_j - 1$ is even, for all $1 \leq j \leq n$. This completes the proof. \square

Lemma 2.3. *Let G be a finite abelian group of odd order m and Z_2 the cyclic group of order 2. Then $K_n(G)$ and $K_n(G \times Z_2)$ are both isomorphic with G for all natural number n .*

Proof. Clearly, by the assumption $(2^n, m) = 1$ and hence by Lemma 2.2, $K_n(G) = G$.

One notes that G and Z_2 are both characteristic subgroups in the direct product $G \times Z_2$. Therefore, Lemma 2.1 implies that

$$K_n(G \times Z_2) = K_n(G) \times K_n(Z_2).$$

Now, the triviality of $K_n(Z_2)$ gives the result. \square

The following proposition is very useful in our further investigations.

Proposition 2.4. *Let G be a cyclic group of order 2^m and H be an abelian 2-group of exponent 2^n with $n < m$. Then*

$$K_n(G \times H) = G^{2^n} \times H^{2^{n-1}}.$$

Proof. Let $G = \langle g \mid g^{2^m} = 1 \rangle$ be the cyclic group of order 2^m . Then we define the automorphisms α_h and α'_h of the group $G \times H$, given by $g^{\alpha_h} = gh, h^{\alpha_h} = h$ and $g^{\alpha'_h} = gh^{-1}, h^{\alpha'_h} = h^{-1}$ for all $h \in H$.

Now, if n is even, then

$$h^{2^{n-1}} = [g, \underbrace{\alpha'_h, \dots, \alpha'_h}_{n\text{-times}}],$$

and if n is odd we have

$$h^{2^{n-1}} = [g, \alpha_h, \underbrace{\alpha'_h, \dots, \alpha'_h}_{(n-1)\text{-times}}].$$

These imply that $H^{2^{n-1}} \subseteq K_n(G \times H)$ and $G^{2^n} \subseteq K_n(G) \subseteq K_n(G \times H)$. Thus

$$G^{2^n} \times H^{2^{n-1}} \subseteq K_n(G \times H).$$

On the other hand, using the structures of the groups G and H for all $\alpha_1, \alpha_2, \dots, \alpha_n \in \text{Aut}(G \times H)$ and $x \in G \times H$,

$$[x, \alpha_1, \alpha_2, \dots, \alpha_n] \in G^{2^n} \times H^{2^{n-1}},$$

which implies that $K_n(G \times H) \subseteq G^{2^n} \times H^{2^{n-1}}$ and the proof is complete. \square

The following theorem follows from the above proposition, which is interested in its own right.

Theorem 2.5. *For all natural numbers $m \geq n_1 \geq \dots \geq n_r$ and $n \geq 2$,*

$$K_n(Z_{2^m} \times Z_{2^{n_1}} \times \dots \times Z_{2^{n_r}}) = Z_{2^{m-n}} \times Z_{2^{n_1-(n-1)}} \times \dots \times Z_{2^{n_r-(n-1)}}.$$

Proof of Theorem 1.1. Let G be a finite abelian group, which can be written as a product of its Sylow subgroups. Now, if $|G|$ is an odd number, then by Lemma 2.3,

$$G = K_n(G).$$

Assume 2 divides $|G|$ and A is the Sylow 2-subgroup of G , then $G = A \times P_1 \times \dots \times P_s$, where P_i 's are Sylow p_i -subgroups of G ($1 \leq i \leq r$). By Lemma 2.1,

$$K_n(G) = K_n(A) \times P_1 \times \dots \times P_s.$$

As A is an abelian 2-group, we may write A as a direct product of cyclic groups of orders some powers of 2, as follows:

$$A \cong Z_{2^m} \times Z_{2^{n_1}} \times \dots \times Z_{2^{n_r}},$$

where $m \geq n_1 \geq \dots \geq n_r$.

Now, we choose the abelian group

$$H = Z_{2^{m+n}} \times Z_{2^{n_1+n-1}} \times \dots \times Z_{2^{n_r+(n-1)}} \times P_1 \times \dots \times P_s.$$

It can be easily seen that

$$K_n(H) = G,$$

and hence the claim is proved. \square

References

- [1] C. Chiš, M. Chiš, and G. Silberberg, *Abelian groups as autocommutator groups*, Arch. Math. (Basel) **90** (2008), no. 6, 490–492.
- [2] M. Deaconescu and G. L. Walls, *Cyclic groups as autocommutator groups*, Comm. Algebra **35** (2007), no. 1, 215–219.
- [3] P. Hegarty, *The absolute centre of a group*, J. Algebra **169** (1994), no. 3, 929–935.
- [4] ———, *Autocommutator subgroups of finite groups*, J. Algebra **190** (1997), no. 2, 556–562.
- [5] M. Naghshineh, M. R. R. Moghaddam, and F. Parvaneh, *The third term of the lower autocentral series of abelian groups*, Journal of Mathematical Extension, Vol. 4, No. 1 (2009), 1–6.
- [6] D. J. S. Robinson, *A Course in the Theory of Groups*, Graduate Texts in Mathematics, 80. Springer-Verlag, New York-Berlin, 1982.

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