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A LOWER BOUND FOR THE GENUS OF SELF-AMALGAMATION OF HEEGAARD SPLITTINGS

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Abstract. Let M be a compact orientable closed 3-manifold, and F a non-separating incompressible closed surface in M. Let $M^{'} = M - \eta(F)$, where $\eta(F)$ is an open regular neighborhood of F in M. In the paper, we give a lower bound of genus of self-amalgamation of minimal Heegaard splitting $V^{'} \cup_{S^{'}} W^{'}$ of $M^{'}$ under some conditions on the distance of the Heegaard splitting.

1. Introduction

A Heegaard splitting of a 3-manifold M is a decomposition $M = V \cup_S W$ of M in which V and W are compression bodies such that $V \cap W = \partial_+ V = \partial_+ W = S$ and $M = V \cup W$. S is called a Heegaard surface of M. The genus $g(S)$ of S is called the genus of the splitting $V \cup_S W$. We use $g(M)$ to denote the Heegaard genus of M, which is equal to the minimal genus of all Heegaard splittings of M. A Heegaard splitting $V \cup_S W$ for M is minimal if $g(S) = g(M)$. $V \cup_S W$ is said to be weakly reducible if there are essential disks $D_1 \subset V$ and $D_2 \subset W$ with $\partial D_1 \cap \partial D_2 = \emptyset$. Otherwise, $V \cup_S W$ is strongly irreducible. Specially, let M be a 3-manifold with boundary, and $\mathcal{F}$ a collection of boundary components of M. If $V \cup_S W$ is a Heegaard splitting of M such that $\mathcal{F} \subset \partial_- V$ or $\mathcal{F} \subset \partial_- W$, then $M = V \cup_S W$ is called a Heegaard splitting relative to $\mathcal{F}$. In this case, if $g(S)$ is minimal among all the Heegaard splittings of M relative to $\mathcal{F}$, then $g(S)$ is called the minimal genus of M relative to $\mathcal{F}$, and is denoted by $g(M, \mathcal{F})$.

Let M_i be a connected compact orientable 3-manifold, F_i an incompressible boundary component of M_i with $g(F_i) \geq 1$, $i = 1, 2$, and $F_1 \cong F_2$. Let $\varphi : F_1 \to F_2$ be a homeomorphism, and $M = M_1 \cup_\varphi M_2$. Suppose $V_i \cup_{S_i} W_i$ is a Heegaard splitting of M_i $(i = 1, 2)$. Then $V_1 \cup_{S_1} W_1$ and $V_2 \cup_{S_2} W_2$ induce a natural Heegaard splitting $V \cup_S W$ of M with $g(S) = g(S_1) + g(S_2) - g(F)$, which is called the amalgamation of $V_1 \cup_{S_1} W_1$ and $V_2 \cup_{S_2} W_2$ along F_1 and F_2. Thus we have that $g(M) \leq g(M_1) + g(M_2) - g(F)$.

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There exist examples which show that an amalgamation of two minimal genus Heegaard splittings of M_1 and M_2 is stabilized, see [1], [8] and [19] etc. On the other hand, it has been shown that under some conditions on the manifolds, the gluing maps, or the distances of the factor manifolds, the equality $g(M) = g(M_1) + g(M_2) - g(F)$ holds, see [9], [10], [20], [7] and [21] etc.

Suppose now that F is a non-separating incompressible surface in M. Let $\eta(F)$ and $N(F)$ be the open and closed regular neighborhood of F in M. We denote by F_1 and F_2 the two boundary components of $N(F)$. Let $M^{'} = M - \eta(F)$ and $M^{'} = V^{'} \cup_{S^{'}} W^{'}$ be a Heegaard splitting relative to $\partial N(F)$. Then M has a natural Heegaard splitting $V \cup_S W$ called the self-amalgamation of $V^{'} \cup_{S^{'}} W^{'}$ as follows:

Assume that $F_1 \cup F_2 \subset \partial_- W^{'}$ and let α_i be an unknotted arc in $W^{'}$ such that $\partial_1 \alpha_i \subset \partial_+ W^{'}$ and $\partial_2 \alpha_i \subset F_i$ for $i = 1, 2$.

Let β be an unknotted arc in $N(F)$ such that $\partial_1 \beta = \partial_2 \alpha_1$ and $\partial_2 \beta = \partial_2 \alpha_2$. Now let $N(\alpha_1 \cup \beta \cup \alpha_2)$ be a closed regular neighborhood of $\alpha_1 \cup \beta \cup \alpha_2$ in $W^{'} \cup N(F)$, and $\eta(\alpha_1 \cup \beta \cup \alpha_2)$ be an open regular neighborhood of $\alpha_1 \cup \beta \cup \alpha_2$ in $W^{'} \cup N(F)$. Let $V = V^{'} \cup N(\alpha_1 \cup \beta \cup \alpha_2)$, and $W = W^{'} \cup N(F) - \eta(\alpha_1 \cup \beta \cup \alpha_2)$. Then $V \cup_S W$ is a Heegaard splitting of M. We call $V \cup_S W$ the self-amalgamation of $V^{'} \cup_{S^{'}} W^{'}$, and M the self-amalgamation of M'. It is clear $g(S) = g(S^{'}) + 1$. Therefore, $g(M) \leq g(M'; \partial N(F)) + 1$.

Qiu and Lei [13] and Du, Lei, and Ma [4] have given lower bounds of Heegaard genera of the self-amalgamation of 3-manifolds under some circumstances.

Theorem 1.1. *Let M be an orientable closed 3-manifold, and F a non-separating incompressible closed surface. Let $M^{'} = M - \eta(F)$. If $M^{'}$ has a Heegaard splitting $V^{'} \cup_{S^{'}} W^{'}$ with $d(S^{'}) > 2g(M^{'})$, then $g(M) \geq g(M^{'}) - g(F)$. Furthermore, if F is a torus, then $g(M) \geq g(M^{'}) + 1$.*

Theorem 1.2. *Let M be an orientable closed 3-manifold, and F a non-separating incompressible closed surface. Let $M^{'} = M - \eta(F)$. If $M^{'}$ has a Heegaard splitting $V^{'} \cup_{S^{'}} W^{'}$ relative to $\partial N(F)$ such that $d(S^{'}) > 2(g(M^{'}, \partial N(F)) + 2g(F))$, then M has a unique minimal Heegaard splitting up to isotopy, i.e., the self-amalgamation of $V^{'} \cup_{S^{'}} W^{'}$.*

In this paper we give a lower bound for genera of self-amalgamations of Heegaard splittings under some condition on the distances of the Heegaard splittings as follows:

Theorem 1.3. *Let M be a compact orientable closed 3-manifold and F a non-separating incompressible closed surface in M. Let $M^{'} = M - \eta(F)$. Suppose $M^{'}$ has a Heegaard splitting $V^{'} \cup_{S^{'}} W^{'}$ with $d(S^{'}) > 2(t + 2g(F))$, where t is an integer with $1 \leq t \leq g(M^{'})$. Then $g(M) \geq t + 1$.*

As a direct consequence of Theorem 1.3, we have:

Corollary 1.4. *Let M be a compact orientable closed 3-manifold and F a non-separating incompressible closed surface in M. Let $M^{'} = M - \eta(F)$. Suppose $M^{'}$ has a Heegaard splitting $V^{'} \cup_{S'} W^{'}$ with $d(S^{'}) > 2(g(M^{'}) + 2g(F))$. Then $g(M) \geq g(M^{'}) + 1$. In particular, if $V^{'} \cup_{S'} W^{'}$ is a Heegaard splitting relative to $\partial N(F)$, then the self-amalgamation of $V^{'} \cup_{S'} W^{'}$ for M is minimal.*

In Section 2, we review some preliminaries which will be used in Section 3. The proof of Theorem 1.3 is given in Section 3.

2. Preliminaries

In this section, we will review some fundamental facts on surfaces in 3-manifolds.

Let M be a 3-manifold. Suppose F is a surface properly embedded in M. If F is incompressible and not parallel to a sub-surface of ∂M, then F is said to be an *essential* surface in M.

Let $M = V \cup_S W$ be a Heegaard splitting, α and β be two essential simple closed curves in S. The *distance* $d(\alpha, \beta)$ of α and β is the smallest integer $n \geq 0$ such that there is a sequence of essential simple closed curves $\alpha = \alpha_0, \alpha_1, \ldots, \alpha_n = \beta$ in S with $\alpha_{i-1} \cap \alpha_i = \emptyset$ for $1 \leq i \leq n$. The *distance* of the Heegaard splitting $V \cup_S W$ is defined to be $d(S) = \min\{d(\alpha, \beta)\}$, where α bounds an essential disk in V and β bounds an essential disk in W. $d(S)$ was first defined by Hempel in [6]. It is clear that $V \cup_S W$ is reducible if and only if $d(S) = 0$, $V \cup_S W$ is weakly reducible if and only if $d(S) \leq 1$.

The following are some basic facts and results on Heegaard splittings.

Lemma 2.1 ([18])**.** *Let V be a compression body and F an incompressible surface in V with $\partial F \subset \partial_+ V$. Then each component of $\overline{V - F}$ is a compression body.*

Lemma 2.2 ([5])**.** *Let $V \cup_S W$ be a Heegaard splitting of M and F a properly embedded incompressible surface (possibly disconnected) in M. Then any component of F is parallel to ∂M or $d(S) \leq 2 - \chi(F)$.*

Let $M = V \cup_S W$ be a Heegaard splitting, and F a boundary component of M. By gluing a $F \times I$ to F and then amalgamating the standard Heegaard splitting of genus $2g(F)$ of $F \times I$ (see [16]) with the given Heegaard splitting (V, W) of M, we get a new Heegaard splitting of M. The construction above is called a *boundary stabilization* on the boundary component F. This was defined by Moriah in [11].

Lemma 2.3 ([17])**.** *Suppose P and Q are Heegaard splitting surfaces for the compact orientable 3-manifold M. Then either $d(P) \leq 2genus(Q)$ or Q is isotopic to P or to a stabilization or boundary-stabilization of P.*

Let $M = V \cup_S W$ be a strongly irreducible Heegaard splitting, and $\mathcal{F}$ a collection of essential surfaces in M. $\mathcal{F}$ is called a *minimal separating system* if

$M-\mathcal{F}$ contains two components M_1 and M_2 and for any proper subset $\mathcal{F}'$ of $\mathcal{F}$, $M-\mathcal{F}'$ contains only one component. The following lemma is an extension of Schultens's lemma [18]. Bachman, Schleimer and Sedgwick [2] first proved Lemma 2.4 when F is connected and closed.

Lemma 2.4 ([13])**.** *Let $M = V\cup_S W$ be a strongly irreducible Heegaard splitting and $\mathcal{F}$ a minimal separating system in M which cuts M into two manifolds M_1 and M_2. Then S can be isotoped such that*

(1) *each of $S\cap M_1$ and $S\cap M_2$ is incompressible*; *or*

(2) *one of $S\cap M_1$ and $S\cap M_2$, say $S\cap M_1$, is incompressible while all components of $S\cap M_2$ are incompressible except one bicompressible component*; *or*

(3) *one of $S\cap M_1$ and $S\cap M_2$, say $S\cap M_1$, is incompressible while $S\cap M_2$ is compressible. Furthermore, there is a Heegaard surface S' isotopic to S such that*

(i) *at most one component of $S'\cap M_1$ is compressible while $S'\cap M_2$ is incompressible, and*

(ii) *S' is obtained by ∂-compressing S in M_2 only one time.*

Proof. Let $\{H_1, H_2\} = \{W, V\}$. If each component of $S\cap M_1$ and $S\cap M_2$ is incompressible, then Lemma 2.4(1) holds. If one of $S\cap M_1$ and $S\cap M_2$ is bicompressible, then, since $V\cup_S W$ is strongly irreducible, Lemma 2.4(2) holds. We may assume that

Assumption (1) one or both of $S\cap M_1$ and $S\cap M_2$ are compressible in $M_1\cap H_1$ and $M_2\cap H_1$, respectively.

Assumption (2) $S\cap M_i$ is incompressible in $M_i\cap H_2$ for $i=1,2$.

Since $\mathcal{F}$ is a collection of essential surfaces in M, H_1 and H_2 are non-trivial compression bodies. Let D be an essential disk of H_2 such that $|D\cap\mathcal{F}|$ is minimal among all essential disks in H_2. By Assumption (2), $|D\cap\mathcal{F}|>0$. Furthermore, we may assume that

Assumption (3) S is a strongly irreducible Heegaard surface such that $|D\cap\mathcal{F}|$ is minimal among all Heegaard surfaces isotopic to S and satisfying Assumptions (1) and (2).

Let a be an outermost component of $D\cap\mathcal{F}$ on D. This means that a, together with an arc b on $\partial D(\subset S)$, bounds a disk B in D which lies in either $M_1\cap H_2$ or $M_2\cap H_2$ such that $B\cap\mathcal{F}=a$, and we may assume that $B\subset M_2\cap H_2$. By the minimality of $|D\cap\mathcal{F}|$, B is a ∂-compressing disk of $S\cap M_2$.

Now there are two cases:

Case 1. $S\cap M_1$ is compressible in $M_1\cap H_1$ ($S\cap M_2$ is compressible or incompressible in $M_2\cap H_1$).

Now let S' be the Heegaard surface of M obtained by ∂-compressing S along B. In fact, S' is isotopic to S. We denote by H_1' and H_2' the two components of $M-S'$. We may assume that $H_1\subset H_1'$. Since the ∂-compression is done in $M_2\cap H_2$, $M_1\cap H_1\subset M_1\cap H_1'$ and $S\cap M_1\subset S'\cap M_1$. Since $S\cap M_1$ is compressible in $M_1 \cap H_1$, $S^{'} \cap M_1$ is compressible in $M_1 \cap H_1^{'}$. Now if $S^{'} \cap M_1$ is compressible in $M_1 \cap H_2^{'}$, then Lemma 2.4(2) holds.

Suppose that $S^{'} \cap M_1$ is incompressible in $M_1 \cap H_2^{'}$. If $S^{'} \cap M_2$ is compressible $M_2 \cap H_2^{'}$, this contradicts Assumption (2). Hence $S^{'} \cap M_2$ is incompressible in $M_2 \cap H_2^{'}$. Now $D \cap H_2^{'}$ is an essential disk in $H_2^{'}$. But $|D \cap H_2^{'} \cap \mathcal{F}| = |D \cap \mathcal{F}| - 1$. This contradicts Assumption (3) regardless of compressibility or incompressibility of $S^{'} \cap M_2$ in $M_2 \cap H_1^{'}$.

Case 2. $S \cap M_2$ is compressible in $M_2 \cap H_1$, and $S \cap M_1$ is incompressible in $M_1 \cap H_1$.

Similarly, let $S^{'}$ be the Heegaard surface of M obtained by ∂-compressing S along B. We denote by $H_1^{'}$ and $H_2^{'}$ the two components of $M - S^{'}$. We may assume that $H_1 \subset H_1^{'}$. Since the ∂-compression is done in $M_2 \cap H_2$, $S \cap M_1 \subset S^{'} \cap M_1$. By observation we can see that $S^{'} \cap M_1$ is incompressible in $M_1 \cap H_1^{'}$ since new component of $S^{'} \cap M_1$ is obtained by attaching a band and new component of $M_1 \cap H_1^{'}$ is obtained by attaching a 1-handle incident to the band. At most one component of $S^{'} \cap M_1$ is compressible in $M_1 \cap H_2^{'}$ since $S \cap M_1$ is incompressible in $M_1 \cap H_2$ by Assumption (2).

Now if $S^{'} \cap M_2$ is compressible $M_2 \cap H_2^{'}$, this contradicts Assumption (2). Hence $S^{'} \cap M_2$ is incompressible $M_2 \cap H_2^{'}$. If $S^{'} \cap M_2$ is incompressible in $M_2 \cap H_1^{'}$, then Lemma 2.4(3) holds.

Suppose that $S^{'} \cap M_2$ is compressible in $M_2 \cap H_1^{'}$. If it is the case that one component of $S^{'} \cap M_1$ is compressible in $M_1 \cap H_2^{'}$, this contradicts the strong irreducibility of $S^{'}$. Hence the remaining case is that $S^{'} \cap M_1$ is incompressible in $M_1 \cap H_2^{'}$, while $|D \cap H_2^{'} \cap \mathcal{F}| = |D \cap \mathcal{F}| - 1$. This contradicts Assumption (3). □

Lemma 2.5 ([4])**.** *Let S, S_1, S_2 be three Heegaard surfaces of M such that $S_1 \cap S_2 = \emptyset$ and the component of $M - S_1 \cup S_2$ containing S_1 and S_2 contains at least one component of ∂M. Then at least one S_i is not obtained by doing stabilizations on S.*

Proof. Suppose both S_1 and S_2 are obtained by doing stabilizations on S.

Since each Heegaard surface separates M, S_1 and S_2 are disjoint, $M - S_1 \cup S_2$ has three components M_1, M_*, M_2, $M = M_1 \cup_{S_1} M_* \cup_{S_2} M_2$. By the assumption, we have $\partial M_* = S_1 \cup S_2 \cup S_*$, where S_* is a non-empty union of components of ∂M.

Suppose $S^{'}$ is a stabilization of S. We describe $S^{'}$ in a slightly different way. Let $N(S)$ be a closed regular neighborhood of S in M. Identify a suitable component of $N(S) - S$ with $S \times [0,1]$ so that $S = S \times \{0\}$. Then $S^{'} = \partial(S \times [0,1] \cup N(\alpha)) - S$, where α is an arc in M with $\alpha \cap S \times [0,1] = \partial\alpha \subset S \times \{1\}$ and $N(\alpha)$ is a 1-handle attached to $S \times [0,1]$. Now the 3-manifold $S \times [0,1] \cup N(\alpha)$ provides a homology from S to $S^{'}$, and moreover this homology is carried in a regular neighborhood of a 2-complex $S \times \{1\} \cup \alpha$ in M.

Since S_i is obtained by a sequence of stabilizations of S, by induction we have that S and S_i are homological, and moreover the homology is carried in a regular neighborhood of a 2-complex in M. Hence S_1 and S_2 are homological in M and the homology is carried in a regular neighborhood $N(X)$ of a 2-complex X in M, $S_1, S_2 \subset X$.

Claim. Either $\partial M_1 \neq S_1$ or M_1 is not a subset of $N(X)$. The similar is true for M_2.

Proof of Claim. We are going to prove the claim by contradiction. Suppose $\partial M_1 = S_1$ and $M_1 \subset N(X)$.

Note $N(X) \cap M_1$ is a regular neighborhood of $X \cap M_1$ in M_1. Let $D(M_1)$ be the double of M_1, which is obtained by gluing two copies of M_1 along their boundaries via the identity. Let $D(X \cap M_1)$ (resp. $D(N(X) \cap M_1)$) be the union of two copies of $X \cap M_1$ (resp. $N(X) \cap M_1$) in $D(M_1)$. Then $D(N(X) \cap M_1)$ is a regular neighborhood of $D(X \cap M_1)$ in $D(M_1)$.

Now $D(M_1)$ is a closed 3-manifold and $D(N(X) \cap M_1)$=$D(M_1)$. This is not possible, since $D(N(X) \cap M_1)$ has the 2-complex $D(X \cap M_1)$ as a deformation retract, which cannot be a closed 3-manifold. So the claim is proved. □

Let $M_i^{'} = M_i$ if $\partial M_i \neq S_i$, and otherwise $M_i^{'} = M_i - B_i^3$ where $B_i^3 \subset \text{int}M_i$ is a small 3-ball disjoint from $N(X)$. Let $M^{'} = M - B_1^3 - B_2^3$. Clearly

(1) $N(X) \subset M^{'}$,

(2) $M^{'} = M_1^{'} \cup_{S_1} M_* \cup_{S_2} M_2^{'}$, $\partial M_* = S_1 \cup S_2 \cup S_*$, $\partial M_1^{'} = S_1 \cup S_1^{'}$, $\partial M_2^{'} = S_2 \cup S_2^{'}$. Each one of S_*, $S_1^{'}$, $S_2^{'}$ is non-empty.

Since $N(X)$ carries the homology from S_1 to S_2, S_1 and S_2 are still homologous in $M^{'}$.

On the other hand, S_1 and S_2 are two closed disjoint orientable surfaces in orientable 3-manifold $M^{'}$, S_1 and S_2 are homological in $M^{'}$ if and only if $S_1 \cup S_2$ cobounds a submanifold in $M^{'}$ or each of S_1 and S_2 bounds a submanifold and homologically trivial, which is not possible by (2). □

3. The proof of the main theorem

Now we come to the proof of Theorem 1.3.

Proof of Theorem 1.3. By assumption, $M^{'} = M - \eta(F)$, and $V^{'} \cup_{S^{'}} W^{'}$ is a Heegaard splitting of $M^{'}$ with $d(S^{'}) > 2(t + 2g(F)) > 0$, then by Haken's lemma (refer to [3]), $M^{'}$ and M are irreducible.

Suppose that the inequality $g(M) \geq t + 1$ does not hold, then there exists a minimal Heegaard splitting $V \cup_S W$ of M with $g(S) < t + 1$.

We divide it into the following two cases to discuss.

Case 1. The Heegaard splitting $V \cup_S W$ is strongly irreducible.

Claim 1. S can be isotoped so that $S \cap M^{'}$ is bicompressible while $S \cap N(F)$ is incompressible.

Proof of Claim 1. In this case, $\partial N(F)$ is a minimal separating system in M which cuts M into $M^{'}$ and $N(F)$. By Lemma 2.4, S can be isotoped to one of the following three cases:

(1) $S \cap M^{'}$ and $S \cap N(F)$ are incompressible.

Since $g(S) < t+1$ and $d(S^{'}) > 2(t+2g(F))$, by Lemma 2.2, $S \cap M^{'}$ is ∂-parallel in $M^{'}$, then S is isotopic to F, a contradiction.

(2) one of $S \cap M^{'}$ and $S \cap N(F)$ is bicompressible while the other is incompressible.

By the arguments in (1), $S \cap M^{'}$ is bicompressible while $S \cap N(F)$ is incompressible.

(3) $S \cap M^{'}$ is compressible while $S \cap N(F)$ is incompressible. Furthermore, there is a Heegaard surface S^{*} isotopic to S such that $S^{*} \cap M^{'}$ is incompressible and at most one component of $S^{*} \cap N(F)$ is compressible. By the same arguments as (1), this is impossible. This completes the proof of Claim 1.

By Claim 1, we may assume that $S \cap M^{'}$ is bicompressible while $S \cap N(F)$ is incompressible. Furthermore, we assume that $|S \cap N(F)|$ is minimal among all Heegaard surfaces isotopic to S and satisfying the above conditions.

Since $V \cup_S W$ is strongly irreducible, there is only one component, say P, of $S \cap M^{'}$ which is bicompressible. And any other component of $S \cap M^{'}$ is incompressible. Suppose that there is a component of $S \cap M^{'}$ besides P, say Q, which is incompressible, then by Lemma 2.2, Q is ∂-parallel in $M^{'}$, then Q can be isotoped to be disjoint from $M^{'}$. This contradicts the minimality of $|S \cap N(F)|$. Thus $S \cap M^{'}$ has only one component, it is connected.

Obviously, any component of $\partial N(F) \cap V$ is incompressible in V, and any component of $\partial N(F) \cap W$ is incompressible in W. Then by Lemma 2.1, any component of $V \cap M^{'}$ and $W \cap M^{'}$ is a compression body. Since $S \cap M^{'}$ is connected, $V \cap M^{'}$ is one compression body, and so is $W \cap M^{'}$.

By the above arguments, $S \cap M^{'}$ is connected and bicompressible. Let S_V be the surface obtained by maximally compressing $S \cap M^{'}$ in $V \cap M^{'}$. We may assume that $S \cap M^{'}$ is compressed to S_V in $V \cap M^{'}$ by cutting $S \cap M^{'}$ open along a collection $\mathcal{D} = \{D_1, \ldots, D_n\}$ of pairwise disjoint compressing disks in $V \cap M^{'}$. Since $V \cup_S W$ is strongly irreducible, by the No nesting Lemma [14], S_V is incompressible in $M^{'}$. Then by Lemma 2.2, we know that any component of S_V is ∂-parallel in $M^{'}$.

Let $A_1, \ldots, A_r$ be all the components of S_V with boundary, $\partial A_i \subset \partial N(F)$ for $1 \leq i \leq r$. Suppose that each A_i is parallel to a subsurface A_i' of $\partial N(F)$ for $1 \leq i \leq r$.

Claim 2. For any components A_i, A_j of S_V, $A_i^{'} \cap A_j^{'} = \emptyset$.

Proof of Claim 2. Suppose that there are two components of S_V, say A_{i_0} and A_{j_0}, such that $A_{i_0}^{'} \cap A_{j_0}^{'} \neq \emptyset$, we may further assume that $A_{i_0}^{'} \subset A_{j_0}^{'}$. Then set $\mathcal{A}_1 = \{A_i : A_i^{'} \subset A_{j_0}', 1 \leq i \leq r, i \neq j_0\}$ and $\mathcal{A}_2 = \{A_i : A_i^{'} \cap A_{j_0}^{'} = \emptyset, 1 \leq i \leq r\}$. We claim that $\mathcal{A}_2 = \emptyset$. Otherwise, since $S \cap M^{'}$ is connected, there must exist

$A_{i_1} \in \mathcal{A}_1$, $A_{i_2} \in \mathcal{A}_2$, and $D_{p_1}, D_{p_2} \in \mathcal{D}$ such that $D_{p_1} \cap D_{p_2} = \emptyset$ and in the compression, the two copies of D_{p_k} lie in A_{i_k} and A_{j_0} respectively, $k = 1, 2$. But this contradicts to the assumption that $S \cap M^{'}$ is separating in $M^{'}$. Thus $\mathcal{A}_2 = \emptyset$. We denote by $W_{A_{j_0}}$ the handlebody bounded by A_{j_0} and $A^{'}_{j_0}$ in $M^{'}$. Then all components of S_V lie in $W_{A_{j_0}}$, so S can be isotoped to be disjoint from $M^{'}$ in M, a contradiction. This completes the proof of Claim 2.

By Claim 2, for each component of $\partial N(F) \cap V$, say $A^{'}_{i_*}$, there is one and only one component A_{i_*} of S_V which is parallel to $A^{'}_{i_*}$. Let $B_1, \ldots, B_t$ be the components of $\overline{\partial N(F) - \cup_{i=1}^{r} A_i^{'}} = \partial N(F) \cap W$. Take a small regular neighborhood $B_i \times I$ of B_i in $W \cap M^{'}$, where $B_i \times \{0\} = B_i$, $i = 1, 2, \ldots, t$. Set $V_1^{'} = (V \cap M^{'}) \bigcup \cup_{i=1}^{t} B_i \times I$ and $W_1^{'} = \overline{M' - V_1^{'}}$. Then $V_1^{'}$ is obtained from $\partial N(F) \times I$ by adding 1-handles whose co-cores are disks in $\mathcal{D}$, so $V_1^{'}$ is a compression body. Note that $W_1^{'} = \overline{(W \cap M') - \cup_{i=1}^{t} B_i \times I} \cong W \cap M^{'}$, $W_1^{'}$ is a compression body. Let $S_1^{'} = \partial_+ V_1^{'}$. Then it is obvious that $S_1^{'} = \partial_+ W_1^{'}$. Thus, $S_1^{'}$ is a Heegaard surface of $M^{'}$. Since $S \cap M^{'}$ is compressible in $W \cap M^{'}$, there exists a compressing disk D of $S \cap M^{'}$ in $W \cap M^{'}$ with $D \cap (B_i \times I) = \emptyset$ and $D \subset W_1^{'}$. Since $\partial B_i \times I$ are spanning annuli in $V_1^{'}$, there exists an essential disk E in $V_1^{'}$ with $E \cap (\partial B_i \times I) = \emptyset$ (cf. [12] Lemma 2.1). Thus $d(S_1^{'}) \leq 2$. Let $S_1 = (S \cap M^{'}) \cup (\partial N(F) \cap W)$. Then $S_1^{'}$ is the surface obtained from S_1 by pushing $\partial N(F) \cap W$ slightly into $W \cap M^{'}$.

Now let $S_2 = (S \cap M^{'}) \cup (\partial N(F) \cap V)$, we denote by $S_2^{'}$ the surface obtained from S_2 by pushing $\partial N(F) \cap V$ slightly into $M^{'} \cap V$. By similar arguments as above, we know that $S_2^{'}$ is also a Heegaard surface of $M^{'}$ and $d(S_2^{'}) \leq 2$. By a small isotopy of $S_1^{'}$, we may assume that $S_1^{'} \cap S_2^{'} = \emptyset$. By the construction of $S_1^{'}$ and $S_2^{'}$, we know that the component of $M^{'} - S_1^{'} \cup S_2^{'}$ containing $S_1^{'}$ and $S_2^{'}$ also contains $\partial N(F)$.

Since $\chi(S \cap M^{'}) \geq \chi(S) > -2t$, and

$$\begin{aligned} \chi(S_1^{'}) &= \chi(S \cap M^{'}) + \chi(\partial N(F) \cap W) \\ &\geq \chi(S) + \chi(\partial N(F)) \\ &= \chi(S) + 2\chi(F), \end{aligned}$$

we have

(1) $$g(S_1^{'}) < t + 2g(F) - 1.$$

Similarly,

(2) $$g(S_2^{'}) < t + 2g(F) - 1.$$

Since $d(S^{'}) > 2(t + 2g(F)) > 2g(S_i^{'}) \geq 2g(M^{'})$, by Lemma 2.3, $V^{'} \cup_{S^{'}} W^{'}$ is the unique minimal Heegaard splitting of $M^{'}$ up to isotopy, and $S_i^{'}$ is isotopic to $S^{'}$ or to a possible stabilization or boundary-stabilization of $S^{'}$ for $i = 1, 2$. But $d(S^{'}) > 2(t + 2g(F)) > 2$ while $d(S_i^{'}) \leq 2$, $S_i^{'}$ cannot be isotopic to the unique minimal Heegaard surface $S^{'}$ of $M^{'}$ for $i = 1, 2$. Hence $S_i^{'}$ is obtained by doing stabilization or boundary-stabilization on $S^{'}$ for $i = 1, 2$.

There are two subcases:

Subcase 1. $M^{'} = V^{'} \cup_{S^{'}} W^{'}$ is a Heegaard splitting relative to $\partial N(F)$.

By Lemma 2.5, one of $S_1^{'}$ and $S_2^{'}$, say $S_1^{'}$, is obtained by doing boundary-stabilizations on $S^{'}$ at least one time. Since $S_1^{'}$ and $S^{'}$ are both Heegaard splittings relative to $\partial N(F)$, $S_1^{'}$ is obtained by doing boundary-stabilizations on $S^{'}$ at least two times. Hence $g(S_1^{'}) \geq g(M^{'}) + 2g(F)$, a contradiction.

Subcase 2. $M^{'} = V^{'} \cup_{S^{'}} W^{'}$ is a Heegaard splitting with $F_1 \subset \partial_- V^{'}$ and $F_2 \subset \partial_- W^{'}$.

Since $F_1 \subset \partial_- V^{'}$ and $F_2 \subset \partial_- W^{'}$, and by the construction of $S_1^{'}$ and $S_2^{'}$, $S_1^{'}$ and $S_2^{'}$ are Heegaard surfaces relative to $\partial N(F)$. By Lemma 2.3 and (1), $S_1^{'}$ is obtained from $S^{'}$ by doing boundary-stabilizations at least one time, hence $g(S_1^{'}) \geq g(S^{'}) + g(F)$. By the similar arguments, we have $g(S_2^{'}) \geq g(S^{'}) + g(F)$.

Now

$$\chi(S_1^{'}) = \chi(S \cap M^{'}) + \chi(\partial N(F) \cap W) \leq 2 - 2(g(M^{'}) + g(F)),$$

and

$$\chi(S_2^{'}) = \chi(S \cap M^{'}) + \chi(\partial N(F) \cap V) \leq 2 - 2(g(M^{'}) + g(F)),$$

hence

$$\begin{aligned} 2\chi(S \cap M^{'}) &\leq 4 - 4(g(M^{'}) + g(F)) - \chi(\partial N(F) \cap W) - \chi(\partial N(F) \cap V) \\ &= 4 - 4(g(M^{'}) + g(F)) - 2\chi(F), \end{aligned}$$

then $\chi(S \cap M^{'}) \leq -2g(M^{'})$, and $\chi(S \cap N(F)) \leq 0$, now we have $g(S) \geq g(M^{'}) + 1 \geq t + 1$, a contradiction to our assumption.

Case 2. The Heegaard splitting $V \cup_S W$ is weakly reducible.

Now $M = V \cup_S W$ is irreducible and weakly reducible, then $V \cup_S W$ has an untelescoping [15] as

$$V \cup_S W = (V_1 \cup_{S_1} W_1) \cup_{H_1} \dots \cup_{H_{n-1}} (V_n \cup_{S_n} W_n)$$

where $n \geq 2$, each component of $H_1, \dots, H_{n-1}$ is an incompressible closed surface in M, and $M_i = V_i \cup_{S_i} W_i$ is a non-trivial strongly irreducible Heegaard splitting for $1 \leq i \leq n$. Since $V \cup_S W$ is minimal, $g(S) = g(M) < t + 1$. Note that $g(H_i) < g(S)$. Since $d(S^{'}) > 2(t + 2g(F))$, by Lemma 2.2, any component of $H_i \cap M^{'}$ is ∂-parallel in $M^{'}$ for each i, then H_i can be isotoped to be disjoint from $M^{'}$ for each i. This means that each component of $H_1, \dots, H_{n-1}$ is parallel to F. Now one of the manifolds $M_1, \dots, M_n$ is homeomorphic to $M^{'}$, and each of the other is homeomorphic to $F \times I$.

Suppose some M_{i_0} is homeomorphic to $M^{'}$, $V_{i_0} \cup_{S_{i_0}} W_{i_0}$ is a Heegaard splitting of $M^{'}$, we have $g(S_{i_0}) \leq g(S) - 1 < t \leq g(M^{'})$, a contradiction. This case cannot happen.

This completes the proof of Theorem 1.3. □

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