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CLASS-MAPPING PROPERTIES OF THE HOHLOV OPERATOR

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Abstract. In the present paper sufficient conditions, in terms of hypergeometric inequalities, are found so that the Hohlov operator preserves a certain subclass of close-to-convex functions (denoted by $\mathcal{R}^\tau(A,B)$) and transforms the classes consisting of k-uniformly convex functions, k-starlike functions and univalent starlike functions into $\mathcal{R}^\tau(A,B)$.

1. Introduction and definitions

Let $\mathcal{A}_0$ be the class of analytic functions in the *open* unit disc

$$\mathcal{U} := \{z \in \mathbb{C} : |z| < 1\}$$

and having the normalized power series expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \ (z \in \mathcal{U}). \tag{1.1}$$

The class $\mathcal{S}$ consists of univalent functions in $\mathcal{A}_0$. The function $f \in \mathcal{A}_0$ is said to be in $k-\mathcal{UCV}$, the class of k-uniformly convex functions ($0 \le k < \infty$), if $f \in \mathcal{S}$ along with the property that for every circular arc γ contained in $\mathcal{U}$, with center ζ where $|\zeta| \le k$, the image curve $f(\gamma)$ is a convex arc (cf. [10]). It is well known that (see [10]) $f \in k-\mathcal{UCV}$ if and only if the image of the function p, where

$$p(z) = 1 + \frac{zf''(z)}{f'(z)} \ (z \in \mathcal{U}),$$

is a subset of the conic region

$$\Omega_k := \{w = u + iv : u^2 > k^2(u-1)^2 + k^2v^2, \ 0 \le k < \infty\}. \tag{1.2}$$

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The class $k-\mathcal{ST}$, consisting of k-starlike functions, is defined via $k-\mathcal{UCV}$ by the usual Alexander's relation, i.e.,

$$f \in k-\mathcal{ST} \Longleftrightarrow g \in k-\mathcal{UCV}, \text{ where } g(z)=\int_0^z \left(\frac{f(t)}{t}\right) dt \text{ (see e.g. [11]).}$$

In particular, if $k=0$ and $k=1$, we get

$$0-\mathcal{UCV} \equiv \mathcal{CV},\ 0-\mathcal{ST} \equiv \mathcal{ST},\ 1-\mathcal{UCV} \equiv \mathcal{UCV} \text{ and } 1-\mathcal{ST} \equiv \mathcal{SP},$$

where $\mathcal{CV}$, $\mathcal{ST}$, $\mathcal{UCV}$, and $\mathcal{SP}$ are respectively the familiar classes of univalent convex functions, univalent starlike functions [4], uniformly convex functions ([7], also see [12], [15]) and parabolic starlike functions [15]. For a unified and systematic study of these classes with the aid of fractional calculus, see e.g. [17, 18, 19, 20, 21].

The function $f \in \mathcal{A}_0$ is said to be in the class $\mathcal{R}^\tau(A,B)$ (see [3]) if

$$\left|\frac{f'(z)-1}{(A-B)\tau-B(f'(z)-1)}\right| < 1 \ (z \in \mathcal{U}, \tau \in \mathbb{C}\setminus\{0\}, -1 \le B < A \le 1). \tag{1.3}$$

For particular values of A, B and τ the class $\mathcal{R}^\tau(A,B)$ includes certain interesting subclasses of $\mathcal{S}$. For example, by taking

$$\tau = e^{-i\eta}\cos\eta \ (-\frac{\pi}{2} < \eta < \frac{\pi}{2}),\ A = 1-2\beta \ (0 \le \beta < 1) \text{ and } B=-1$$

we get the class $\mathcal{R}_\eta(\beta)$, studied by Ponnusamy and Ronning [14], where

$$\mathcal{R}_\eta(\beta) = \Big\{f \in \mathcal{A}_0 : \Re(e^{i\eta}(f'(z)-\beta)) > 0, z \in \mathcal{U}, -\frac{\pi}{2} < \eta < \frac{\pi}{2}, 0 \le \beta < 1\Big\}.$$

Similarly, if we set $\tau = 1,\ A=\beta,\ B=-\beta$ $(0 < \beta \le 1)$ we obtain the class of functions $f \in \mathcal{A}_0$ satisfying the inequality

$$\Big|\frac{f'(z)-1}{f'(z)+1}\Big| < \beta \ (z \in \mathcal{U},\ 0 < \beta \le 1)$$

studied earlier by Padmanabhan [13], Caplinger and Causey [2] and others. Note that the functions in the class $\mathcal{R}^\tau(A,B)$ are univalent and close-to-convex.

The generalized hypergeometric function ${}_pF_q$ $(p, q \in \mathbb{N}_0 := \{0,1,2,\ldots\})$ with p numerator parameters $\alpha_j \in \mathbb{C}$ $(j=1,\ldots,p)$ and q denominator parameters $\beta_k \in \mathbb{C}\setminus\mathbb{Z}_0^-$ $(\mathbb{Z}_0^- := \{0,-1,-2,\ldots\},\ k=1,\ldots,q)$; is defined by (cf. [16])

$${}_pF_q(z) = {}_pF_q(\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q;z) := \sum_{n=0}^{\infty} \frac{(\alpha_1)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n}\frac{z^n}{n!},$$

where $(\lambda)_n$ is the Pochhammer symbol (or shifted factorial), defined in terms of the gamma function by

$$(\lambda)_n := \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1 \ (n=0) \\ \lambda\cdots(\lambda+n-1) \ (n \in \mathbb{N} := \{1,2,\ldots\}). \end{cases}$$

Note that ${}_pF_q(z)$ is an entire function if $p < q+1$. However, if $p = q+1$, then ${}_pF_q(z)$ is analytic in $\mathcal{U}$. Also, if

$$p = q+1 \text{ and } \Re\Big(\sum_{j=1}^{q} \beta_j - \sum_{j=1}^{p} \alpha_j\Big) > 0,$$

then ${}_pF_q(z)$ converges on $\partial\mathcal{U}$. In particular, the function

$$(1.4) \qquad {}_2F_1(a,b;c;z) := \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n(1)_n} z^n$$

is the familiar Gaussian hypergeometric function. Furthermore, the evaluation ${}_2F_1(a,b;c;1)$ is related to the gamma function by

$$(1.5) \qquad {}_2F_1(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \; \Big(\Re(c-a-b) > 0,\ c \notin \mathbb{Z}_0^-\Big).$$

We now recall the Hohlov operator $\mathcal{I}_c^{a,b} : \mathcal{A}_0 \to \mathcal{A}_0$, defined in terms of the Hadamard product (or convolution) by (cf. [8])

$$(1.6) \qquad (\mathcal{I}_c^{a,b}(f))(z) = z\ {}_2F_1(a,b;c;z) * f(z)\ (f \in \mathcal{A}_0, z \in \mathcal{U}).$$

Thus from (1.1) and (1.4) we have

$$(1.7) \qquad (\mathcal{I}_c^{a,b}(f))(z) = z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} a_n z^n \ (z \in \mathcal{U}).$$

It is well known that the class $\mathcal{S}$ and many of its important subclasses are not closed under the ring operations of usual addition and multiplication of functions. Therefore, the study of class-preserving and class-transforming operators is an interesting problem in geometric function theory. The Hohlov operator unifies several such previously well studied operators, namely the Alexander, Libera, Bernardi and Carlson-Shaffer operators (denoted here by $\mathcal{A}$, $\mathcal{L}$, $\mathcal{B}$ and $\mathcal{L}(a,c)$ respectively). Thus

$$\mathcal{A}(f) = \mathcal{I}_2^{1,1}(f),\ \mathcal{L}(f) = \mathcal{I}_3^{1,2}(f),\ \mathcal{B}(f) = \mathcal{I}_{\gamma+2}^{1,\gamma+1}(f),\ \mathcal{L}(a,c)(f) = \mathcal{I}_c^{a,1}(f).$$

Kanas and Srivastava [9] and Ponnusamy and Ronning [14] (also see Gangadharan et al. [5]) obtained coefficient inequalities so that the operator $I_c^{a,b}$ preserves the class $k-\mathcal{UCV}$ and transforms the classes

$$\mathcal{R}_\eta(\beta) \text{ into } k-\mathcal{UCV};\ R_\eta(\beta) \text{ into } k-\mathcal{ST};$$
$$\mathcal{ST} \text{ into } k-\mathcal{UCV};\ \mathcal{ST} \text{ into } k-\mathcal{ST} \text{ and } k-\mathcal{UCV} \text{ into } k-\mathcal{ST}.$$

The main object of the present paper is to consider the more general class $\mathcal{R}^\tau(A,B)$ (instead of $\mathcal{R}_\eta(\beta)$) and find sufficient conditions in terms of hypergeometric inequalities for the *reverse* of some of the transformations considered in [9] and [14]. More specifically sufficient conditions are obtained here to ensure that the Hohlov operator $\mathcal{I}_c^{a,b}$ maps the classes

$$k-\mathcal{UCV} \text{ into } \mathcal{R}^\tau(A,B),\ k-\mathcal{ST} \text{ into } \mathcal{R}^\tau(A,B) \text{ and } \mathcal{ST} \text{ into } \mathcal{R}^\tau(A,B).$$

Furthermore, the invariance of the class $\mathcal{R}^\tau(A,B)$ under the operator $\mathcal{I}_c^{a,b}$ is discussed. Lastly, a sufficient condition is obtained so that the function $z_2F_1(a,b;c;z)$ belongs to $\mathcal{R}^\tau(A,B)$. Sufficient conditions for the particular cases of $\mathcal{I}_c^{a,b}$ are also emphasized in the form of corollaries to the main theorems.

2. Some preliminary lemmas

We need each of the following results in our investigation.

Lemma 1 (see [10], [11]). *Let*

$$P_k(z) = 1 + p_1(k)z + p_2(k)z^2 + \cdots \ (z \in \mathcal{U}, p_1(k) > 0) \tag{2.1}$$

be the Riemann map of $\mathcal{U}$ *onto* Ω_k *where the region* Ω_k *is defined as in* (1.2) *and let the function* f *be given by* (1.1). *If* $f \in k - \mathcal{UCV}$, *then*

$$|a_n| \leq \frac{(p_1(k))_{n-1}}{n!} \ (n \in \mathbb{N} \setminus \{1\}). \tag{2.2}$$

Further if $f \in k - \mathcal{ST}$, *then*

$$|a_n| \leq \frac{(p_1(k))_{n-1}}{(n-1)!} \ (n \in \mathbb{N} \setminus \{1\}). \tag{2.3}$$

The estimates (2.2) *and* (2.3) *are sharp.*

Lemma 2 (see [3]). *Let the function f, given by* (1.1), *be a member of* $\mathcal{R}^\tau(A,B)$. *Then*

$$|a_n| \leq (A-B)\frac{|\tau|}{n} \ (n \in \mathbb{N} \setminus \{1\}). \tag{2.4}$$

The estimate in (2.4) *is sharp for the function*

$$f(z) = \int_0^1 \Big(1 + \frac{(A-B)\tau t^{n-1}}{1+Bt^{n-1}}\Big) dt \ (z \in \mathcal{U}, n \in \mathbb{N} \setminus \{1\}).$$

Lemma 3 (see [3]). *Let the function* $f \in \mathcal{A}_0$ *be of the form* (1.1). *If*

$$\sum_{n=2}^{\infty}(1+|B|)n|a_n| \leq (A-B)|\tau| \ (-1 \leq B < A \leq 1,\ \tau \in \mathbb{C} \setminus \{0\}), \tag{2.5}$$

then $f \in \mathcal{R}^\tau(A,B)$. *The result is sharp for the function*

$$f(z) = z + \frac{(A-B)\tau}{(1+|B|)n} z^n \ (z \in \mathcal{U}, n \in \mathbb{N} \setminus \{1\}).$$

Lemma 4 (see [1]). *Let the function f of the form* (1.1) *be a member of* $\mathcal{S}$ (*or* $\mathcal{ST}$). *Then the sharp estimate*

$$|a_n| \leq n \ (n \in \mathbb{N} \setminus \{1\}) \tag{2.6}$$

holds true.

Lemma 5 (see [6]). *Let the function $f \in \mathcal{A}_0$ be of the form* (1.1). *If*

$$\sum_{n=2}^{\infty} n|a_n| \leq 1, \tag{2.7}$$

then $f \in \mathcal{ST}$.

3. Mapping properties of the Hohlov operator

Throughout in the present section we shall take

$$-1 \leq B < A \leq 1,\ \frac{-\pi}{2} < \eta < \frac{\pi}{2}.$$

Theorem 1. *Let $a, b \in \mathbb{C} \setminus \{0\}$ and $c \in \mathbb{C}$ satisfy*

$$\Re c > \max\{0, 2\Re a + 2, 2\Re b + 2\}. \tag{3.1}$$

If the hypergeometric inequality

$$\begin{aligned} &\frac{\Gamma(\Re c)\{\Gamma(\Re c - 2\ \Re a - 2)\Gamma(\Re c - 2\ \Re b - 2)\}^{\frac{1}{2}}}{|\Gamma(\Re c - a)||\Gamma(\Re c - b)|}\Big[|(a)_2||(b)_2| \\ &+ 3|ab|\{(\Re c - 2\ \Re a - 2)(\Re c - 2\Re b - 2)\}^{\frac{1}{2}} \\ &+ \{(\Re c - 2\Re a - 2)_2(\Re c - 2\Re b - 2)_2\}^{\frac{1}{2}}\Big] \\ \leq\ & \frac{(A-B)|\tau|}{(1+|B|)} + 1 \end{aligned} \tag{3.2}$$

is satisfied, then $\mathcal{I}_c^{a,b}$ maps the class $\mathcal{S}$ (or $\mathcal{ST}$) into $\mathcal{R}^{\tau}(A, B)$.

Proof. Let the function f given by (1.1) be a member of $\mathcal{S}$ or $\mathcal{ST}$. By (1.7)

$$\mathcal{I}_c^{a,b}(f) = z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} a_n z^n\ (z \in \mathcal{U}).$$

In view of Lemma 3, it is thus sufficient to show that

$$\sum_{n=2}^{\infty}(1+|B|)n\Big|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} a_n\Big| \leq (A-B)|\tau|.$$

By making use of Lemma 4 and the elementary inequality

$$|(c)_p| > (\Re c)_p\ (p \in \mathbb{N}) \tag{3.3}$$

it is again sufficient to prove that

$$S_1 = \sum_{n=2}^{\infty} n^2 \frac{|(a)_{n-1}(b)_{n-1}|}{(\Re c)_{n-1}(1)_{n-1}} \leq \frac{(A-B)|\tau|}{(1+|B|)}. \tag{3.4}$$

The term S_1 above is equivalently written as

$$S_1 = \sum_{n=1}^{\infty}(n+1)^2 \frac{|(a)_n(b)_n|}{(\Re c)_n(1)_n} = \sum_{n=1}^{\infty}\{n(n-1)+3n+1\}\frac{|(a)_n(b)_n|}{(\Re c)_n(1)_n}$$

$$= \sum_{n=2}^{\infty} \frac{|(a)_n(b)_n|}{(\Re c)_n(1)_{n-2}} + 3\sum_{n=1}^{\infty} \frac{|(a)_n(b)_n|}{(\Re c)_n(1)_{n-1}} + \sum_{n=1}^{\infty} \frac{|(a)_n(b)_n|}{(\Re c)_n(1)_n}$$

$$= \sum_{n=0}^{\infty} \frac{|(a)_{n+2}||(b)_{n+2}|}{(\Re c)_{n+2}(1)_n} + 3\sum_{n=0}^{\infty} \frac{|(a)_{n+1}||(b)_{n+1}|}{(\Re c)_{n+1}(1)_n} + \sum_{n=0}^{\infty} \frac{|(a)_n||(b)_n|}{(\Re c)_n(1)_n} - 1.$$

The repeated applications of the relation

$$(d)_m = d(d+1)_{m-1} \ (d \in \mathbb{C},\ m \in \mathbb{N})$$

yield

$$S_1 = \frac{|(a)_2||(b)_2|}{(\Re c)_2} \sum_{n=0}^{\infty} \frac{|(a+2)_n||(b+2)_n|}{(\Re c+2)_n(1)_n} + \frac{3|ab|}{\Re c} \sum_{n=0}^{\infty} \frac{|(a+1)_n||(b+1)_n|}{(\Re c+1)_n(1)_n} + \sum_{n=0}^{\infty} \frac{|(a)_n||(b)_n|}{(\Re c)_n(1)_n} - 1. \tag{3.5}$$

Applying Cauchy's inequality to individual sums in (3.5) we get

$$S_1 \le \frac{|(a)_2||(b)_2|}{(\Re c)_2}\Big[\Big\{\sum_{n=0}^{\infty} \frac{(a+2)_n(\bar a+2)_n}{(\Re c+2)_n(1)_n}\Big\}^{\frac12}\Big\{\sum_{n=0}^{\infty} \frac{(b+2)_n(\bar b+2)_n}{(\Re c+2)_n(1)_n}\Big\}^{\frac12}\Big]$$
$$+ \frac{3|ab|}{\Re c}\Big[\Big\{\sum_{n=0}^{\infty} \frac{(a+1)_n(\bar a+1)_n}{(\Re c+1)_n(1)_n}\Big\}^{\frac12}\Big\{\sum_{n=0}^{\infty} \frac{(b+1)_n(\bar b+1)_n}{(\Re c+1)_n(1)_n}\Big\}^{\frac12}\Big]$$
$$+ \Big[\Big\{\sum_{n=0}^{\infty} \frac{(a)_n(\bar a)_n}{(\Re c)_n(1)_n}\Big\}^{\frac12}\Big\{\sum_{n=0}^{\infty} \frac{(b)_n(\bar b)_n}{(\Re c)_n(1)_n}\Big\}^{\frac12}\Big] - 1$$
$$= \frac{|(a)_2||(b)_2|}{(\Re c)_2}\Big[\{{}_2F_1(a+2,\bar a+2;\Re c+2;1)\}^{\frac12}\{{}_2F_1(b+2,\bar b+2;\Re c+2;1)\}^{\frac12}\Big]$$
$$+ \frac{3|ab|}{\Re c}\{\,{}_2F_1(a+1,\bar a+1;\Re c+1;1)\}^{\frac12}\{\,{}_2F_1(b+1,\bar b+1;\Re c+1;1)\}^{\frac12}$$
$$+ \{\,{}_2F_1(a,\bar a;\Re c;1)\}^{\frac12}\{\,{}_2F_1(b,\bar b;\Re c;1)\}^{\frac12} - 1.$$

Since the condition (3.1) holds we use the Gauss summation formula (1.5) and get

$$S_1 \le \frac{|(a)_2||(b)_2|}{(\Re c)_2}\Big\{\frac{\Gamma(\Re c+2)\Gamma(\Re c-2\ \Re a-2)}{\Gamma(\Re c-a)\Gamma(\Re c-\bar a)}\Big\}^{\frac12}\Big\{\frac{\Gamma(\Re c+2)\Gamma(\Re c-2\ \Re b-2)}{\Gamma(\Re c-b)\Gamma(\Re c-\bar b)}\Big\}^{\frac12}$$
$$+ \frac{3|ab|}{\Re c}\Big\{\frac{\Gamma(\Re c+1)\Gamma(\Re c-2\ \Re a-1)}{\Gamma(\Re c-a)\Gamma(\Re c-\bar a)}\Big\}^{\frac12}\Big\{\frac{\Gamma(\Re c+1)\Gamma(\Re c-2\ \Re b-1)}{\Gamma(\Re c-b)\Gamma(\Re c-\bar b)}\Big\}^{\frac12}$$
$$+ \Big\{\frac{\Gamma(\Re c)\Gamma(\Re c-2\ \Re a)}{\Gamma(\Re c-a)\Gamma(\Re c-\bar a)}\Big\}^{\frac12}\Big\{\frac{\Gamma(\Re c)\Gamma(\Re c-2\ \Re b)}{\Gamma(\Re c-b)\Gamma(\Re c-\bar b)}\Big\}^{\frac12} - 1.$$

Moreover, the gamma function is symmetric about real axis, i.e., $\overline{\Gamma(z)} = \Gamma(\bar{z})$. Therefore,

$$S_1 \le \frac{\Gamma(\Re c)\{\Gamma(\Re c - 2\Re a - 2)\Gamma(\Re c - 2\Re b - 2)\}^{\frac{1}{2}}}{|\Gamma(\Re c - a)||\Gamma(\Re c - b)|}\Big[|(a)_2||(b)_2| + 3|ab|\{(\Re c - 2\Re a - 2)(\Re c - 2\Re b - 2)\}^{\frac{1}{2}} + \{(\Re c - 2\Re a - 2)_2(\Re c - 2\Re b - 2)_2\}^{\frac{1}{2}}\Big] - 1.$$

Thus in view of (3.4) if the hypergeometric inequality (3.2) is satisfied, then $\mathcal{I}_c^{a,b}(f) \in \mathcal{R}^\tau(A, B)$ as asserted. The proof of Theorem 1 is complete. □

Corollary 1. *Let $a \in \mathbb{C} \setminus \{0\}$ and $c \in \mathbb{C}$ satisfy*

$$\Re c > \max\{0,\ 2\Re a + 2\}.$$

If the hypergeometric inequality

$$\frac{\Gamma(\Re c)\Gamma(\Re c - 2\Re a - 2)}{|\Gamma(\Re c - a)|^2}\Big[|(a)_2|^2 + 3|a|^2(\Re c - 2\Re a - 2) + (\Re c - 2\Re a - 2)_2\Big] \le \frac{(A - B)|\tau|}{1 + |B|} + 1$$

is satisfied, then $\mathcal{I}_c^{a,\bar{a}}$ maps the class $\mathcal{S}$ or $\mathcal{ST}$ into $\mathcal{R}^\tau(A, B)$.

Proof. Take $b = \bar{a}$ in Theorem 1. □

Corollary 2. *Let $a \in \mathbb{C} \setminus \{0\}$ and $c \in \mathbb{C}$ satisfy*

$$\Re c > \max\{4,\ 2\Re a + 2\}.$$

If the hypergeometric inequality

$$\frac{\Gamma(\Re c)\{\Gamma(\Re c - 2\Re a - 2)\Gamma(\Re c - 4)\}^{\frac{1}{2}}}{|\Gamma(\Re c - a)|\Gamma(\Re c - 1)}\Big[2|(a)_2| + 3|a|\{(\Re c - 2\Re a - 2)(\Re c - 4)\}^{\frac{1}{2}} + \{(\Re c - 2\Re a - 2)_2(\Re c - 4)_2\}^{\frac{1}{2}}\Big] \le \frac{(A - B)|\tau|}{1 + |B|} + 1$$

is satisfied, then $\mathcal{L}(a, c)$ maps the class of $\mathcal{S}$ (or $\mathcal{S}T$) into $\mathcal{R}^\tau(A, B)$.

Proof. Take $b = 1$ in Theorem 1. □

Theorem 2. *Let $a, b \in \mathbb{C} \setminus \{0\}$, $p_1 = p_1(k)$ be defined by* (2.1) *and $c \in \mathbb{C}$ satisfy*

$$\Re c > \max\{0, 2\Re a + p_1, 2\Re b + p_1\}. \tag{3.6}$$

(i) *If the hypergeometric inequality*

$$\frac{|ab|p_1}{\Re c}\{{}_3F_2(a + 1, \bar{a} + 1, p_1 + 1; \Re c + 1, 2; 1)\}^{\frac{1}{2}}\{{}_3F_2(b + 1, \bar{b} + 1, p_1 + 1; \tag{3.7}$$

$$\Re c+1,2;1)\}^{\frac{1}{2}}+\{{}_3F_2(a,\bar{a},p_1;\Re c,1;1)\}^{\frac{1}{2}}\{{}_3F_2(b,\bar{b},p_1;\Re c,1;1)\}^{\frac{1}{2}}$$
$$\leq \frac{(A-B)|\tau|}{1+|B|}+1$$

is satisfied, then $\mathcal{I}_c^{a,b}$ *maps the class* $k-\mathcal{ST}$ *into* $\mathcal{R}^\tau(A,B)$.

(ii) *Furthermore, if*

$$\frac{|ab|p_1}{\Re c}\{{}_3F_2(a+1,\bar{a}+1,p_1+1;\Re c+1,2;1)\}^{\frac{1}{2}}\{{}_3F_2(b+1,\bar{b}+1,p_1+1;\Re c+1,2;1)\}^{\frac{1}{2}}+\{{}_3F_2(a,\bar{a},p_1;\Re c,1;1)\}^{\frac{1}{2}}\{{}_3F_2(b,\bar{b},p_1;\Re c,1;1)\}^{\frac{1}{2}}\leq 2 \tag{3.8}$$

is satisfied, then $\mathcal{I}_c^{a,b}$ *maps the class* $k-\mathcal{ST}$ *into* $\mathcal{ST}$.

Proof. (i) Let the function f given by (1.1) be a member of $k-\mathcal{ST}$. As in the proof of Theorem1, it is sufficient to show that

$$\sum_{n=2}^{\infty}(1+|B|)n\Big|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}a_n\Big|\leq (A-B)|\tau|.$$

Using the coefficient estimate (2.3) it is again sufficient to show that

$$S_2=\sum_{n=2}^{\infty}n\frac{|(a)_{n-1}(b)_{n-1}|(p_1)_{n-1}}{(\Re c)_{n-1}(1)_{n-1}(1)_{n-1}}\leq\frac{(A-B)|\tau|}{(1+|B|)}. \tag{3.9}$$

Now

$$S_2=\sum_{n=1}^{\infty}(n+1)\frac{|(a)_n(b)_n|(p_1)_n}{(\Re c)_n(1)_n(1)_n}=\sum_{n=1}^{\infty}\frac{|(a)_n(b)_n|(p_1)_n}{(\Re c)_n(1)_{n-1}(1)_n}+\sum_{n=1}^{\infty}\frac{|(a)_n(b)_n|(p_1)_n}{(\Re c)_n(1)_n(1)_n}$$
$$=\frac{|ab|p_1}{\Re c}\sum_{n=0}^{\infty}\frac{|(a+1)_n(b+1)_n|(p_1+1)_n}{(\Re c+1)_n(1)_n(2)_n}+\sum_{n=0}^{\infty}\frac{|(a)_n||(b)_n|(p_1)_n}{(\Re c)_n(1)_n(1)_n}-1.$$

An application of Cauchy's inequality gives

$$S_2\leq\frac{|ab|p_1}{\Re c}\Big[\Big\{\sum_{n=0}^{\infty}\frac{(a+1)_n(\bar{a}+1)_n(p_1+1)_n}{(\Re c+1)_n(2)_n(1)_n}\Big\}^{\frac{1}{2}}\Big\{\sum_{n=0}^{\infty}\frac{(b+1)_n(\bar{b}+1)_n(p_1+1)_n}{(\Re c+1)_n(2)_n(1)_n}\Big\}^{\frac{1}{2}}\Big]$$
$$+\Big\{\sum_{n=0}^{\infty}\frac{(a)_n(\bar{a})_n(p_1)_n}{(\Re c)_n(1)_n(1)_n}\Big\}^{\frac{1}{2}}\Big\{\sum_{n=0}^{\infty}\frac{(b)_n(\bar{b})_n(p_1)_n}{(\Re c)_n(1)_n(1)_n}\Big\}^{\frac{1}{2}}-1.$$

Since the condition (3.6) is satisfied, the above summations can be written as evaluations of generalized hypergeometric functions and we get

$$S_2\leq\frac{|ab|p_1}{\Re c}\Big[\{{}_3F_2(a+1,\bar{a}+1,p_1+1;\Re c+1,2;1)\}^{\frac{1}{2}}\{{}_3F_2(b+1,\bar{b}+1,p_1+1;\Re c+1,2;1)\}^{\frac{1}{2}}\Big]$$
$$+\{{}_3F_2(a,\bar{a},p_1;\Re c,1;1)\}^{\frac{1}{2}}\{{}_3F_2(b,\bar{b},p_1;\Re c,1;1)\}^{\frac{1}{2}}-1.$$

Therefore, in view of (3.9), if the condition (3.7) is satisfied, then $\mathcal{I}_c^{a,b}(f)\in\mathcal{R}^\tau(A,B)$.

(ii) We follow the lines of proof of (i). In this case we use Lemma 5 (instead of Lemma 3). The proof of Theorem 2 is complete. □

Corollary 3. *Let $a \in \mathbb{C} \setminus \{0\}, p_1 = p_1(k)$ be defined by* (2.1) *and $c \in \mathbb{C}$ satisfy*

$$\Re c > \max\{0, 2\Re a + p_1\}.$$

If the hypergeometric inequality

$$\frac{|a|^2 p_1}{\Re c}\{{}_3F_2(a+1, \bar{a}+1, p_1+1; \Re c+1, 2; 1)\} + \{{}_3F_2(a, \bar{a}, p_1; \Re c, 1; 1)\} \\ \leq \frac{(A-B)|\tau|}{1+|B|} + 1$$

is satisfied, then $\mathcal{I}_c^{a,\bar{a}}$ maps the class $k-\mathcal{ST}$ into $\mathcal{R}^\tau(A,B)$. Further, if

$$\frac{|a|^2 p_1}{\Re c}\{\ {}_3F_2(a+1, \bar{a}+1, p_1+1; \Re c+1, 2; 1)\} + \{\ {}_3F_2(a, \bar{a}, p_1; \Re c, 1; 1)\} \leq 2$$

is satisfied, then $\mathcal{I}_c^{a,\bar{a}}$ maps the class $k-\mathcal{ST}$ into $\mathcal{ST}$.

Proof. Taking $b = \bar{a}$ in Theorem 2 we get the result. □

Corollary 4. *Let $a \in \mathbb{C} \setminus \{0\}, p_1 = p_1(k)$ be defined by* (2.1) *and $c \in \mathbb{C}$ satisfy*

$$\Re c > \max\{2 + p_1, 2\Re a + p_1\}.$$

If the hypergeometric inequality

(3.10)

$$\frac{(\Re c - 1)^{\frac{1}{2}}}{(\Re c - p_1 - 1)^{\frac{1}{2}}}\Big[\frac{|a|p_1}{\{(\Re c - p_1 - 2)\Re c\}^{\frac{1}{2}}}\{{}_3F_2(a+1,\ \bar{a}+1, p_1+1; \Re c+1, 2; 1)\}^{\frac{1}{2}} \\ + \{{}_3F_2(a, \bar{a}, p_1; \Re c, 1; 1)\}^{\frac{1}{2}}\Big] \\ \leq \frac{(A-B)|\tau|}{1+|B|} + 1$$

is satisfied, then $\mathcal{L}(a,c)$ maps the class $k-\mathcal{ST}$ into $\mathcal{R}^\tau(A,B)$. Further, if

(3.11)

$$\frac{(\Re c - 1)^{\frac{1}{2}}}{(\Re c - p_1 - 1)^{\frac{1}{2}}}\Big[\frac{|a|p_1}{\{(\Re c - p_1 - 2)\Re c\}^{\frac{1}{2}}}\{{}_3F_2(a+1,\ \bar{a}+1, p_1+1; \Re c+1, 2; 1)\}^{\frac{1}{2}} \\ + \{{}_3F_2(a, \bar{a}, p_1; \Re c, 1; 1)\}^{\frac{1}{2}}\Big] \leq 2$$

is satisfied, then $\mathcal{L}(a,c)$ maps the class $k-\mathcal{ST}$ into $\mathcal{ST}$.

Proof. We take $b = 1$ in Theorem 2. The Gauss summation formula (1.5) provides the following simplification

$$\begin{aligned} {}_3F_2(2, 2, p_1+1; \Re c+1, 2; 1) &= {}_2F_1(2, p_1+1; \Re c+1; 1) \\ &= \frac{\Gamma(\Re c+1)\Gamma(\Re c - p_1 - 2)}{\Gamma(\Re c - 1)\Gamma(\Re c - p_1)} \\ &= \frac{(\Re c)(\Re c - 1)}{(\Re c - p_1 - 1)(\Re c - p_1 - 2)} \end{aligned}$$

and

$$\begin{aligned}{}_3F_2(1,1,p_1;\Re c,1;1) &= {}_2F_1(1,p_1;\Re c;1)\\ &= \frac{\Gamma(\Re c)\Gamma(\Re c-p_1-1)}{\Gamma(\Re c-1)\Gamma(\Re c-p_1)} = \frac{(\Re c-1)}{(\Re c-p_1-1)}.\end{aligned}$$

Thus, the conditions (3.7) and (3.8) of Theorem 2 simplify to (3.10) and (3.11) respectively. Therefore, the assertions of Corollary 4 follows from Theorem 2. □

Theorem 3. *Let* $a,b \in \mathbb{C}\setminus\{0\}$, $p_1 = p_1(k)$ *be defined by* (2.1) *and* $c \in \mathbb{C}$ *satisfy*

$$\Re c > \max\{0, 2\Re a + p_1 - 1, 2\Re b + p_1 - 1\}. \tag{3.12}$$

(i) *If the hypergeometric inequality*

$$\Big[\,{}_3F_2(a,\bar{a},p_1;\Re c,1;1)\Big]^{\frac{1}{2}}\Big[\,{}_3F_2(b,\bar{b},p_1;\Re c,1;1)\Big]^{\frac{1}{2}} \le \frac{(A-B)|\tau|}{1+|B|}+1 \tag{3.13}$$

is satisfied, then $\mathcal{I}_c^{a,b}$ *maps the class* $k-\mathcal{UCV}$ *into the class* $\mathcal{R}^\tau(A,B)$.

(ii) *Furthermore, if*

$$\Big[\,{}_3F_2(a,\bar{a},p_1;\Re c,1;1)\Big]^{\frac{1}{2}}\Big[\,{}_3F_2(b,\bar{b},p_1;\Re c,1;1)\Big]^{\frac{1}{2}} \le 2 \tag{3.14}$$

is satisfied, then $\mathcal{I}_c^{a,b}$ *maps the class of* $k-\mathcal{UCV}$ *into the class* $\mathcal{ST}$.

Proof. (i) Let the function f given by (1.1) be a member of $k-\mathcal{UCV}$. We follow the lines of proof of Theorem1. Taking into account the estimates (2.2) for a_n and the elementary inequality (3.3), we show that

$$S_3 = \sum_{n=2}^{\infty} n\frac{|(a)_{n-1}(b)_{n-1}|(p_1)_{n-1}}{(\Re c)_{n-1}(1)_{n-1}(1)_n} \le \frac{(A-B)|\tau|}{1+|B|}. \tag{3.15}$$

The term S_3 can be equivalently written as

$$S_3 = \sum_{n=1}^{\infty}\frac{|(a)_n(b)_n|(p_1)_n}{(\Re c)_n(1)_n(1)_n} = \sum_{n=0}^{\infty}\frac{|(a)_n||(b)_n|(p_1)_n}{(\Re c)_n(1)_n(1)_n} - 1.$$

An application of Cauchy's inequality and the relation

$$\overline{(d)_n} = (\,\bar{d}\,)_n \;\; (n \in \mathbb{N}_0)$$

for any complex number d give

$$S_3 \le \Big\{\sum_{n=0}^{\infty}\frac{(a)_n(\bar{a})_n(p_1)_n}{(\Re c)_n(1)_n(1)_n}\Big\}^{\frac{1}{2}}\Big\{\sum_{n=0}^{\infty}\frac{(b)_n(\bar{b})_n(p_1)_n}{(\Re c)_n(1)_n(1)_n}\Big\}^{\frac{1}{2}} - 1. \tag{3.16}$$

The conditions $\Re c > 2\Re a + p_1 - 1$ and $\Re c > 2\Re b + p_1 - 1$ given in (3.12) ensure that the sums in the r.h.s of (3.16) are convergent hypergeometric series; so that

$$S_3 \le \{{}_3F_2(a,\bar{a},p_1;\Re c,1;1)\}^{\frac{1}{2}}\{{}_3F_2(b,\bar{b},p_1;\Re c,1;1)\}^{\frac{1}{2}} - 1.$$

Therefore, in view of (3.15) if the inequality (3.13) is satisfied, then $\mathcal{I}_c^{a,b}(f) \in \mathcal{R}^\tau(A,B)$ as asserted.

(ii) We follow the lines of proof of (i). In this case we use Lemma 5 (instead of Lemma 3).The proof of Theorem 3 is complete. □

Corollary 5. *Let the complex numbers a, b and c be as in Theorem 3 and further satisfy the inequality*

$$\{{}_3F_2(a,\bar{a},p_1;\Re c,1;1)\}^{\frac{1}{2}}\{{}_3F_2(b,\bar{b},p_1;\Re c,1;1)\}^{\frac{1}{2}} \le (1-\beta)\cos\eta + 1.$$

Then the operator $\mathcal{I}_c^{a,b}$ maps $k-\mathcal{UCV}$ into $\mathcal{R}_\eta(\beta)$.

Proof. Taking $A = 1-2\beta$ $(0 \le \beta < 1), B = -1$ and $\tau = e^{-i\eta}\cos\eta$ in Theorem 3(i) we get the result. □

Corollary 6. *Let $a \in \mathbb{C}\setminus\{0\}, p_1 = p_1(k)$ be defined by (2.1) and $c \in \mathbb{C}$ satisfy*

$$\Re c > \max\{0, 2\Re a + p_1 - 1\}.$$

If the hypergeometric inequality

$${}_3F_2(a,\bar{a},p_1;\Re c,1;1) \le \frac{(A-B)|\tau|}{1+|B|} + 1$$

is satisfied, then $\mathcal{I}_c^{a,\bar{a}}$ maps the class $k-\mathcal{UCV}$ into $\mathcal{R}^\tau(A,\ B)$. Further, if

$${}_3F_2(a,\bar{a},p_1;\Re c,1;1) \le 2$$

is satisfied, then $\mathcal{I}_c^{a,\bar{a}}$ maps the class $k-\mathcal{UCV}$ into $\mathcal{ST}$.

Proof. Taking $b = \bar{a}$ in Theorem 3 we get the result. □

Corollary 7. *Let $a \in \mathbb{C}\setminus\{0\}, p_1 = p_1(k)$ be defined by (2.1) and $c \in \mathbb{C}$ satisfy*

$$\Re c > \max\{2\Re a + p_1 - 1, p_1 + 1\}.$$

If the hypergeometric inequality

$$(3.17) \qquad {}_3F_2(a,\bar{a},p_1;\Re c,1;1) \le \left(\frac{(A-B)|\tau|}{1+|B|} + 1\right)^2\left(\frac{\Re c - p_1 - 1}{\Re c - 1}\right)$$

is satisfied, then $\mathcal{L}(a,c)$ maps $k-\mathcal{UCV}$ into $\mathcal{R}^\tau(A,B)$. Further, if

$$(3.18) \qquad {}_3F_2(a,\bar{a},p_1;\Re c,1;1) \le 4\left(\frac{\Re c - p_1 - 1}{\Re c - 1}\right)$$

is satisfied, then $\mathcal{L}(a,c)$ maps $k-\mathcal{UCV}$ into $\mathcal{ST}$.

Proof. We take $b = 1$ in Theorem 3. Note that

$$\begin{aligned} {}_3F_2(1,1,p_1;\Re c,1;1) &= {}_2F_1(1,p_1;\Re c;1) \\ &= \frac{\Gamma(\Re c)\Gamma(\Re c - p_1 - 1)}{\Gamma(\Re c - 1)\Gamma(\Re c - p_1)} = \frac{(\Re c - 1)}{(\Re c - p_1 - 1)}. \end{aligned}$$

Thus, the conditions (3.13) and (3.14) of Theorem 3 simplify to (3.17) and (3.18) respectively. The assertion of Corollary 7 now follows from Theorem 3. □

Theorem 4. *Let $a, b \in \mathbb{C} \setminus \{0\}$ and $c \in \mathbb{C}$ satisfy*

$$\Re c > \max\{0, 2\Re a, 2\Re b\}. \tag{3.19}$$

If the hypergeometric inequality

$$\frac{\Gamma(\Re c)\{\Gamma(\Re c - 2\Re a)\Gamma(\Re c - 2\Re b)\}^{\frac{1}{2}}}{|\Gamma(\Re c - a)||\Gamma(\Re c - b)|} \leq \frac{1}{1+|B|} + 1 \tag{3.20}$$

is satisfied, then $\mathcal{I}_c^{a,b}$ maps the class of $\mathcal{R}^\tau(A,B)$ into $\mathcal{R}^\tau(A,B)$.

Proof. Let the function f given by (1.1) be a member of $\mathcal{R}^\tau(A,B)$. By virtue of Lemma 3 and the coefficient inequality (2.4) it is sufficient to show that

$$(1+|B|)S_4 \leq 1, \tag{3.21}$$

where

$$S_4 = \sum_{n=2}^{\infty} \frac{|(a)_{n-1}(b)_{n-1}|}{(\Re c)_{n-1}(1)_{n-1}} = \sum_{n=0}^{\infty} \frac{|(a)_n||(b)_n|}{(\Re c)_n(1)_n} - 1.$$

Applications of Cauchy's inequality and the Gauss summation formula (1.5) give

$$\begin{aligned} S_4 &\leq \Big\{ \sum_{n=0}^{\infty} \frac{(a)_n(\bar{a})_n}{(\Re c)_n(1)_n} \Big\}^{\frac{1}{2}} \Big\{ \sum_{n=0}^{\infty} \frac{(b)_n(\bar{b})_n}{(\Re c)_n(1)_n} \Big\}^{\frac{1}{2}} - 1 \\ &= \Big\{ {}_2F_1(a, \bar{a}; \Re c; 1) \Big\}^{\frac{1}{2}} \Big\{ {}_2F_1(b, \bar{b}; \Re c; 1) \Big\}^{\frac{1}{2}} - 1 \\ &= \frac{\Gamma(\Re c)\{\Gamma(\Re c - 2\Re a)\Gamma(\Re c - 2\Re b)\}^{\frac{1}{2}}}{|\Gamma(\Re c - a)||\Gamma(\Re c - b)|} - 1. \end{aligned}$$

Thus, in view of (3.21), if the hypergeometric inequality (3.20) is satisfied, then $\mathcal{I}_c^{a,b}(f) \in \mathcal{R}^\tau(A,B)$ as asserted. The proof of Theorem 4 is complete. □

Corollary 8. *Let $a \in \mathbb{C} \setminus \{0\}$ and $c \in \mathbb{C}$ satisfy*

$$\Re c > \max\{0, 2\Re a\}.$$

If the hypergeometric inequality

$$\frac{\Gamma(\Re c)\Gamma(\Re c - 2\Re a)}{|\Gamma(\Re c - a)|^2} \leq \frac{1}{1+|B|} + 1$$

is satisfied, then $\mathcal{I}_c^{a,\bar{a}}$ maps the class of $\mathcal{R}^\tau(A,B)$ into itself.

Proof. Taking $b = \bar{a}$ in Theorem 4 we get the result. □

Corollary 9. *Let $a \in \mathbb{C} \setminus \{0\}$ and $c \in \mathbb{C}$ satisfy*

$$\Re c > \max\{2, 2\Re a\}.$$

If the hypergeometric inequality

$$\frac{\Gamma(\Re c)\{\Gamma(\Re c - 2\Re a)\Gamma(\Re c - 2)\}^{\frac{1}{2}}}{|\Gamma(\Re c - a)|\Gamma(\Re c - 1)} \leq \frac{1}{1+|B|} + 1$$

is satisfied, then $\mathcal{L}(a,c)$ *maps* $\mathcal{R}^{\tau}(A,B)$ *into itself.*

Proof. Take $b=1$ in Theorem 4. □

Theorem 5. *Let* $a,b \in \mathbb{C}\setminus\{0\}$ *and* $c\in\mathbb{C}$ *satisfy*

$$\Re c > \max\{0, 2\Re a+1, 2\Re b+1\}. \tag{3.22}$$

If the hypergeometric inequality

$$\frac{\Gamma(\Re c)\{\Gamma(\Re c-2\Re a-1)\Gamma(\Re c-2\Re b-1)\}^{\frac{1}{2}}}{|\Gamma(\Re c-a)||\Gamma(\Re c-b)|}\Big[|ab| + \{(\Re c-2\Re a-1)(\Re c-2\Re b-1)\}^{\frac{1}{2}}\Big] \le 1+\frac{(A-B)|\tau|}{1+|B|} \tag{3.23}$$

is satisfied, then $z\ {}_2F_1(a,b;c;z)\in\mathcal{R}^{\tau}(A,B)$.

Proof. We know

$$z\ {}_2F_1(a,b;c;z)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}z^{n+1}=z+\sum_{n=2}^{\infty}\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}z^n.$$

Therefore, by Lemma 3, it is sufficient to show that

$$S_5=\sum_{n=2}^{\infty}n\frac{|(a)_{n-1}||(b)_{n-1}|}{(\Re c)_{n-1}(1)_{n-1}}\le\frac{(A-B)|\tau|}{1+|B|}. \tag{3.24}$$

As in our demonstration of Theorem 3 we can write S_5 equivalently as

$$S_5 = \frac{|a||b|}{\Re c}\sum_{n=0}^{\infty}\frac{|(a+1)_n||(b+1)_n|}{(\Re c+1)_n(1)_n}+\sum_{n=0}^{\infty}\frac{|(a)_n||(b)_n|}{(\Re c)_n(1)_n}-1.$$

Applications of Cauchy's inequality followed by the Gauss summation formula (1.5) and the relation $\Gamma(\ \bar{z}\ )=\overline{\Gamma(z)}$ give

$$\begin{aligned}
S_5 &\le \frac{|ab|}{\Re c}\Big[\Big\{\sum_{n=0}^{\infty}\frac{(a+1)_n(\bar a+1)_n}{(\Re c+1)_n(1)_n}\Big\}^{\frac{1}{2}}\Big\{\sum_{n=0}^{\infty}\frac{(b+1)_n(\bar b+1)_n}{(\Re c+1)_n(1)_n}\Big\}^{\frac{1}{2}}\Big]\\
&\quad+\Big\{\sum_{n=0}^{\infty}\frac{(a)_n(\bar a)_n}{(\Re c)_n(1)_n}\Big\}^{\frac{1}{2}}\Big\{\sum_{n=0}^{\infty}\frac{(b)_n(\bar b)_n}{(\Re c)_n(1)_n}\Big\}^{\frac{1}{2}}-1\\
&=\frac{|ab|}{\Re c}\Big[\{\ {}_2F_1(a+1,\bar a+1;\Re c+1;1)\}^{\frac{1}{2}}\{\ {}_2F_1(b+1,\bar b+1;\Re c+1;1)\}^{\frac{1}{2}}\Big]\\
&\quad+\{\ {}_2F_1(a,\bar a;\Re c;1)\}^{\frac{1}{2}}\{\ {}_2F_1(b,\bar b;\Re c;1)\}^{\frac{1}{2}}-1\\
&=\frac{\Gamma(\Re c)\{\Gamma(\Re c-2\Re a-1)\Gamma(\Re c-2\Re b-1)\}^{\frac{1}{2}}}{|\Gamma(\Re c-a)||\Gamma(\Re c-b)|}\\
&\quad\times\Big[|ab|+\{(\Re c-2\Re a-1)(\Re c-2\Re b-1)\}^{\frac{1}{2}}\Big]-1.
\end{aligned}$$

Thus, in view of (3.24), if the inequality (3.23) is satisfied, then $z\,{}_2F_1(a,b;c;z) \in R^\tau(A,B)$. The proof of Theorem 5 is complete. □

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