Bull. Korean Math. Soc. **48** (2011), No. 1, pp. 35–50
DOI 10.4134/BKMS.2011.48.1.035

STABLE INDEX PAIRS FOR DISPERSIVE DYNAMICAL SYSTEMS

Yoon Hoe Goo and Jong-Suh Park

Abstract. We construct the index pairs of an isolated neighborhood for a dispersive dynamical system and investigate the existence of an index pair which is stable under small perturbations of the dispersive dynamical systems.

1. Introduction

The Conley index theory, the origin of which goes back to the famous Wazewski Retract Theorem [9], has become an important tool in the qualitative study of differential equations. The theory provides cohomological or homotopic invariants of isolated invariant sets of flows and yields existence results in differential equation. The foundations of the theory, in a locally compact setting, were established by R. Churchil [1], J. Montgomery [11], C. Conley [2], and by H. Kurland [9]. K. Rybakowski [13] extended the theory to the case of non-locally compact spaces. The theory is now designated as the Conley index theory because of the significant role played by C. Conley in its development.

There is a formal similarity between the above indices of the isolated invariant set and the fixed point index [4] of a continuous map. Recently the fixed point index has been extended to the case of a multi-valued admissible map [8]. Thus there is a question if a similar generalization is also possible in case of a general dynamical system.

Such a generalization would be useful in direct applications to differential equation without uniqueness as well as in situations where the classical Conley index is used in course of a proof but extra assumptions or extra verifications are needed to ensure uniqueness [10].

Received May 7, 2009.

2010 *Mathematics Subject Classification.* Primary 37B30; Secondary 54H20.

Key words and phrases. index pair, dispersive dynamical systems, I-solution, isolating neighborhood.

This work was supported by the research fund from Hanseo University in 2008.

E. Dancer mentioned in [3] that an approximation technique could provide a partial extension of the Conley theory to the case of general dynamical systems defined by differential equations without uniqueness.

In this paper we construct the index pairs of an isolated invariant set of a dispersive dynamical system on a metric space. Such a topological approach, contrary to the approximation technique, seems to be closer in spirit to the original Conley index. Additionally it allows applications to differential inclusions.

Our work is organized as follows. In Section 2, we study some properties of I-solution for dispersive dynamical systems. In the following section we investigate the existence and properties of index pairs. In the last section we provide that the index pairs are stable under small perturbations of the dispersive dynamical system.

2. General dynamical systems

Let X be a topological space. We will denote by 2^X the set of all nonempty compact subsets of X.

Definition 2.1. Let (X, d) and (Y, ρ) be metric spaces and let $f : X \to 2^Y$. The map f is said to be (1) *upper semicontinuous* (usc) at $x \in X$ if for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$d(x, z) < \delta \text{ implies } f(z) \subseteq B_\rho(f(x), \varepsilon),$$

(2) *lower semicontinuous* (lsc) at x if for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$d(x, z) < \delta \text{ implies } f(x) \subseteq B_\rho(f(z), \varepsilon).$$

The map f is said to be *continuous* at x if f is upper and lower semicontinuous at x.

It is easy to prove the following proposition.

Proposition 2.2. *Let (X, d) and (Y, ρ) be metric spaces and let $f : X \to 2^Y$. Then*

(1) *f is usc at $x \in X$ if and only if for any open neighborhood U of $f(x)$, there exists an open neighborhood V of x such that $f(z) \subset U$ for all $z \in X$.*

(2) *f is lsc at $x \in X$ if and only if for any open set U with $f(x) \cap U \neq \emptyset$, there exists an open neighborhood V of x such that $f(z) \cap U \neq \emptyset$ for all $z \in V$.*

Proposition 2.3. *Let (X, d) and (Y, ρ) be metric spaces and let $f : X \to 2^Y$.*

(1) *If f is usc at $x \in X$, then $x_n \to x, y_n \in f(x_n)$ and $y_n \to y$ implies $y \in f(x)$.*

(2) *If X is compact, then the inverse of* (1) *holds.*

Proof. (1) Suppose f is usc at $x \in X$ and let $\varepsilon > 0$. Then there exists $\delta > 0$ such that $d(x, z) < \delta$ implies $f(z) \subset B_\rho(f(x), \frac{\varepsilon}{2})$. Also, there exists a positive integer n such that $d(x, x_n) < \delta$ and $d(y, y_n) < \frac{\varepsilon}{2}$. Therefore we have $y_n \in$

$f(x_n) \subset B_\rho(f(x), \frac{\varepsilon}{2})$ and so there exists $z \in f(x)$ such that $\rho(y_n, z) < \frac{\varepsilon}{2}$. Since $\rho(y, z) \leq \rho(y, y_n) + \rho(y_n, z) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$, we have $z \in B_\rho(y, \varepsilon) \cap f(x)$. Hence it follows that $y \in \overline{f(x)} = f(x)$.

(2) Suppose f is not usc at $x \in X$. Then there exists $\varepsilon > 0$ such that for any $\delta > 0$, there exists $z \in B_d(x, \delta)$ such that $f(z) \not\subseteq B_\rho(f(x), \varepsilon)$. Therefore for any positive integer n, there exists $x_n \in B_d(x, \frac{1}{n})$ such that $f(x_n) \not\subseteq B_\rho(f(x), \varepsilon)$. Also there exists $y_n \in f(x_n) - B_\rho(f(x), \varepsilon)$. Since X is compact, (y_n) has a convergent subsequence. Let $y_n \to y$. Since $x_n \to x$, we have $y \in f(x)$. But since $\rho(y_n, f(x)) \geq \varepsilon$ for all n, we have $\rho(y, f(x)) \geq \varepsilon$. This is a contradiction. Hence f is usc at $x \in X$. This completes the proof. □

Continuing on in the same vein, we state another condition that is equivalent to upper semicontinuity.

Proposition 2.4. *Let (X, d) and (Y, ρ) be metric spaces and let $f : X \to 2^Y$. Then f is usc if and only if for any open set $U \subset Y$, $f^{-1}(U) \equiv \{x \in X \mid f(x) \subset U\}$ is open in X.*

Proof. Suppose f is usc. Then for any $x \in f^{-1}(U)$, we have $f(x) \subset U$. By Proposition 2.2, there exists a neighborhood V of x such that $z \in V$ implies $f(z) \subset U$. Therefore we have $V \subset f^{-1}(U)$ and so $f^{-1}(U)$ is open in X.

Conversely, let $x \in X$ and let U be any open neighborhood of $f(x)$. Then $V \equiv f^{-1}(U)$ is a neighborhood of x and for any $z \in V$ we have $f(z) \subset U$. By Proposition 2.2, f is usc at x. Hence f is usc and so the proof is complete. □

Proposition 2.5. *Let $f : X \to 2^Y$ be usc. If K is a compact subset of X, then $f(K)$ is a compact subset of Y.*

Proof. Let $\{U_\alpha \mid \alpha \in A\}$ be an open cover of $f(K)$. For each $x \in K$, since $f(x)$ is compact, there exists a finite subset A_x of A such that $f(x) \subset \bigcup_{\alpha \in A_x} U_\alpha$. Since f is usc, there exists a neighborhood V_x of x such that $f(V_x) \subset \bigcup_{\alpha \in A_x} U_\alpha$. Since K is compact, there are finitely many $x_1, \ldots, x_n \in K$ such that $K \subset \bigcup_{i=1}^n V_{x_i}$. Then

$$f(K) \subset f(\bigcup_{i=1}^n V_{x_i}) = \bigcup_{i=1}^n f(V_{x_i}) \subset \bigcup_{i=1}^n \bigcup_{\alpha \in A_{x_i}} U_\alpha.$$

Thus $f(K)$ is compact. □

We denote the set of all real numbers, nonnegative real numbers, and non-positive real numbers by $\mathbb{R}, \mathbb{R}^+$ and $\mathbb{R}^-$, respectively.

Definition 2.6. An usc mapping $f : X \times \mathbb{R} \to 2^X$ is called a *dispersive dynamical system* if the following conditions are satisfied:

(1) For all $x \in X$, $f(x, 0) = \{x\}$;

(2) For all $s, t \in \mathbb{R}$ with $st \geq 0$ and all $x \in X$, $f(f(x, s), t) = f(x, s + t)$;

(3) For all $x, y \in X$, if $y \in f(x, t)$, then $x \in f(y, -t)$.

The sets $f(\{x\} \times \mathbb{R}), f(\{x\} \times \mathbb{R}^+), f(\{x\} \times \mathbb{R}^-)$ will be called the *trajectory*, the positive trajectory and the negative trajectory of x and denoted by $f(x), f^+(x), f^-(x)$, respectively.

In the following argument, all of the notations and results are stated with respect to a given dispersive dynamical system f defined on a metric space X.

Definition 2.7. Let $I \subset \mathbb{R}$ be an interval. By a *I-solution* of f in $N \subset X$ we mean a continuous map $\sigma : I \to N$ such that

$$\sigma(t) \in f(\sigma(s), t-s) \quad \text{for all } \ s, t \in I.$$

We will say that the solution σ *originates* at $x \in X$ if $0 \in I$ and $\sigma(0) = x$. A $[0, t]$-solution ($[t, 0]$-solution in case $t \leq 0$) σ of f in N will be called a *t-connection* from x to y provided $\sigma(0) = x, \sigma(t) = y$. The set of all I-solution of f in N and the set of all I-solutions originating at x will be denoted by $sol_N(I), sol_N(I, x)$, respectively. We will also write $conn_N(t, x, y)$ for the set of all t-connections from x to y in N. In case $N = X$ the subscripts N in sol_N and $conn_N$ will be omitted.

We have the following basic facts [3, 4, 6, 7, 8, 9, 10].

Proposition 2.8. *If σ_1 and σ_2 are solutions of f on $[a, b]$ and $[b, c]$, respectively, with $\sigma_1(b) = \sigma_2(b)$, then the concatenation σ of σ_1 and σ_2 defined by*

$$\sigma(t) = \begin{cases} \sigma_1(t), & t \in [a, b] \\ \sigma_2(t), & t \in [b, c] \end{cases}$$

is a solution of f on $[a, c]$.

Proposition 2.9. *Let $y \in f(x, t_1 - t_0)$, where $t_0 \leq t_1$. Then there exists a solution $\sigma : [t_0, t_1] \to X$ such that $\sigma(t_0) = x, \sigma(t_1) = y$.*

Proposition 2.10 (Barbashin's Theorem)**.** *Let (σ_n) be a sequence of solutions $\sigma_n : [t_0, t_1] \to X$ and let $\sigma_n(t_0) \to x$. Then there exist a subsequence (σ_{n_i}) of (σ_n) and a solution $\tau : [t_0, t_1] \to X$ such that (σ_{n_i}) converges uniformly to τ on $[t_0, t_1]$.*

We shall investigative some properties of I-solution for a dispersive dynamical system f which will be used in the next section.

Proposition 2.11. *If I, J are compact intervals such that $I \subset J$, then for every I-solution σ there exists a J-solution τ being an extension of σ.*

Proof. Let $I = [b, c]$ and $J = [a, d]$. Put $x_0 = \sigma(b)$ and $x_1 = \sigma(c)$. Choose $y_0 \in f(x_0, b-a)$ and $y_1 \in f(x_1, d-c)$. Since $x_0 \in f(y_0, b-a)$, by Proposition 2.9, there exist solutions $\gamma_1 : [0, b-a] \to X$ and $\gamma_2 : [0, d-c] \to X$ such that $\gamma_1(0) = y_0, \gamma_1(b-a) = x_0, \gamma_2(0) = x_1, \gamma_2(d-c) = y_1$. Define $\tau : [a, d] \to X$ by

$$\tau(t) = \begin{cases} \gamma_1(t-a), & t \leq b \\ \sigma(t), & b \leq t \leq c \\ \gamma_2(t-c), & t \geq c\ . \end{cases}$$

Then γ is a solution being an extension of σ. □

Proposition 2.12. *Assume $N \subset X$ is compact, and sequences $(x_n), (y_n) \subset N$, $(t_n) \subset \mathbb{R}$ are convergent: $x_n \to x$, $y_n \to y$, $t_n \to t$. If $conn_N(t_n, x_n, y_n) \neq \emptyset$, then $conn_N(t, x, y) \neq \emptyset$.*

Proof. First consider the case $t \geq 0$. There exists a sequence (σ_n) of solutions $\sigma_n : [0, t_n] \to N$ such that $\sigma_n(0) = x_n, \sigma_n(t_n) = y_n$ for all n. Choose a number $T > t$. We may assume that $t_n < T$ for all n. By Proposition 2.11, there exists a sequence (τ_n) of solutions $\tau_n : [0, T] \to X$ being an extension of σ_n for all n. By Proposition 2.10, there exist a subsequence (τ_{n_i}) of (τ_n) and a solution $\gamma : [0, T] \to X$ such that (τ_{n_i}) converges uniformly to γ on $[0, T]$.

Let $0 \leq s < t$. We may assume that $s < t_{n_i}$ for all i. Since

$$\gamma(s) = \lim_{i \to \infty} \tau_{n_i}(s) = \lim_{i \to \infty} \sigma_{n_i}(s),$$

we have $\gamma(s) \in N$. Since γ is continuous, $\gamma(t) = \lim_{s \to t^-} \gamma(s) \in N$. Thus we have

$$\gamma|_{[0,t]} : [0, t] \to N.$$

We have $\gamma(0) = \lim_{i \to \infty} \tau_{n_i}(0) = \lim_{i \to \infty} \sigma_{n_i}(0) = \lim_{i \to \infty} x_{n_i} = x$. Given any $\varepsilon > 0$, there exists $\delta > 0$ such that if $|t - s| < \delta$, then $d(\gamma(t), \gamma(s)) < \frac{\varepsilon}{3}$. Since (τ_{n_i}) converges uniformly to γ on $[0, T], y_{n_i} \to y$ and $t_{n_i} \to t$, there exists k such that

$$d(\tau_{n_k}(s), \gamma(s)) < \frac{\varepsilon}{3} \;\text{ for all }\; s \in [0, T], \;\; d(y_{n_k}, y) < \frac{\varepsilon}{3}, \;\;\text{and}\;\; |t - t_{n_k}| < \delta.$$

Then we have

$$\begin{aligned} d(\gamma(t), y) &\leq d(\gamma(t), \gamma(t_{n_k})) + d(\gamma(t_{n_k}), \tau_{n_k}(t_{n_k})) + d(\tau_{n_k}(t_{n_k}), y) \\ &= d(\gamma(t), \gamma(t_{n_k})) + d(\gamma(t_{n_k}), \tau_{n_k}(t_{n_k})) + d(\sigma_{n_k}(t_{n_k}), y) \\ &= d(\gamma(t), \gamma(t_{n_k})) + d(\gamma(t_{n_k}), \tau_{n_k}(t_{n_k})) + d(y_{n_k}, y) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Thus $\gamma(t) = y$ and so $\gamma|_{[0,t]} \in conn_N(t, x, y)$.

The proof of the case $t < 0$ is similar. □

We conclude the section with the following proposition.

Proposition 2.13. *Assume N is compact, and $(x_n) \subset N$ is a sequence such that $x_n \to x$. Then*

(a) *Let $(t_n) \subset \mathbb{R}^+$ be a sequence such that $sol_N([0, t_n], x_n) \neq \emptyset$ for all n. If $t_n \to \infty$, then $sol_N(\mathbb{R}^+, x) \neq \emptyset$.*

(b) *Let $(t_n) \subset \mathbb{R}^-$ be a sequence such that $sol_N([t_n, 0], x_n) \neq \emptyset$ for all n. If $t_n \to -\infty$, then $sol_N(\mathbb{R}^-, x) \neq \emptyset$.*

Proof. (a) We may assume that $t_n < t_{n+1}$ for all n. Let $\sigma_n \in sol_N([0, t_n], x_n)$. Since $\sigma_n(0) = x_n \to x$, by Proposition 2.10, there exist a subsequence (σ_{n_i}) of

(σ_n) and a solution $\tau_1 : [0, t_1] \to X$ such that (σ_{n_i}) converges uniformly to τ_1 on $[0, t_1]$. We have

$$\tau_1(0) = \lim_{i\to\infty} \sigma_{n_i}(0) = \lim_{i\to\infty} x_{n_i} = \lim_{n\to\infty} x_n = x.$$

Since $\tau_1(s) = \lim_{i\to\infty} \sigma_{n_i}(s) \in N$ for all $s \in [0, t_1]$, we have $\tau_1 : [0, t_1] \to N$. We may assume that $n_1 \geq 2$. Since $\sigma_{n_i} = x_{n_i} \to x$, by Proposition 2.10, there exist a subsequence $(\sigma_{n_{i_k}})$ of (σ_{n_i}) and a solution $\tau_2 : [0, t_2] \to X$ such that $(\sigma_{n_{i_k}})$ converges uniformly to τ_2 on $[0, t_2]$. Clearly $\tau_2 : [0, t_2] \to N$. Since $\tau_2(s) = \lim_{k\to\infty} \sigma_{n_{i_k}}(s) = \lim_{i\to\infty} \sigma_{n_i}(s) = \tau_1(s)$ for all $s \in [0, t_1]$, we have $\tau_2 = \tau_1$ on $[0, t_1]$. Repeating this process, we obtain a sequence (τ_n) of solutions $\tau_n : [0, t_n] \to N$ such that $\tau_{n+1} = \tau_n$ on $[0, t_n]$ for all n. Define $\gamma : \mathbb{R}^+ \to N$ by $\gamma|_{[0,t_n]} = \tau_n$. Then $\gamma \in sol_N(\mathbb{R}^+, x)$.

(b) It is similar to the proof of (a). □

3. Existence of index pairs and their properties

In this section we investigate the existence of an index pair for an isolating neighborhood and its properties. Let N be a compact subset of X. Define a map $f_N : N \times \mathbb{R} \to 2^N$ by

$$f_N(x, t) = \{y \in N \mid conn_N(t, x, y) \neq \emptyset\}.$$

Proposition 3.1. *The mapping f_N is a dispersive dynamical system.*

Proof. Let $(x, t) \in N \times \mathbb{R}$. We may assume that $t \geq 0$. Let (y_n) be a sequence in $f_N(x, t)$. For each positive integer n, there exists a solution $\sigma_n : [0, t] \to N$ such that $\sigma_n(0) = x$ and $\sigma_n(t) = y_n$. Since N is compact, (y_n) has a convergent subsequence. Let $y_n \to y \in N$. By Barbashin's Theorem, there exist a subsequence (σ_{n_i}) of (σ_n) and a solution $\sigma : [0, t] \to X$ such that (σ_{n_i}) converges uniformly to σ on $[0, t]$. Since $\sigma(s) = \lim_{i\to\infty} \sigma_{n_i}(s) \in N$, we have $\sigma : [0, t] \to N$. Since

$$\sigma(0) = \lim_{i\to\infty} \sigma_{n_i}(0) = x \text{ and } \sigma(t) = \lim_{i\to\infty} \sigma_{n_i}(t) = \lim_{i\to\infty} y_{n_i} = y,$$

we have $y \in f_N(x, t)$. Thus $f(x, t)$ is compact.

Assume that f_N is not usc at (x, t). There exists $\varepsilon > 0$ such that for any $\delta > 0$ there exist $y \in N, s \in \mathbb{R}$ such that

$$d(x, y) < \delta,\ |t - s| < \delta,\ f_N(y, s) \nsubseteq B(f_N(x, t), \varepsilon)\,.$$

For each positive integer n, there exist $x_n \in N, t_n \in \mathbb{R}$ such that

$$d(x, x_n) < \frac{1}{n},\ |t - s| < \frac{1}{n},\ f_N(x_n, t_n) \nsubseteq B(f_N(x, t), \varepsilon)\,.$$

Let $y_n \in f_N(x_n, t_n) - B(f_N(x, t), \varepsilon)$. Since N is compact, (y_n) has a convergent subsequence. Let $y_n \to y \in N$. Since $x_n \to x, t_n \to t, y_n \to y$, and $conn_N(t_n, x_n, y_n) \neq \emptyset$ for all n, by Proposition 2.12, we have $conn_N(t, x, y) \neq \emptyset$. Thus $y \in f_N(x, t)$. This is a contradiction. Hence f_N is usc.

The verification of properties (1) to (3) of Definition 2.6 is straightforward. □

Given a subset $N \subset X$, we introduce the following notation

$$\begin{aligned} inv^+N &= \{x \in N \mid sol_N(\mathbb{R}^+, x) \neq \emptyset\}, \\ inv^-N &= \{x \in N \mid sol_N(\mathbb{R}^-, x) \neq \emptyset\}, \\ invN &= \{x \in N \mid sol_N(\mathbb{R}, x) \neq \emptyset\}. \end{aligned}$$

By Proposition 2.8, we have $invN = inv^+N \cap inv^-N$.

Let $diam_N f = \sup\{diam f(x,t) \mid x \in N,\ 0 \leq t \leq 1\}$ and $dist(A,B) = \inf\{d(x,y) \mid x \in A,\ y \in B\}$, $A, B \subset X$. If $A \subset X$, we denote the boundary of A by ∂A, its interior by $intA$, and let $B(A,\varepsilon) = \{x \in X \mid d(x,A) < \varepsilon\}$ for $\varepsilon > 0$.

Definition 3.2. A compact set $N \subset X$ is called an *isolating neighborhood* for a dispersive dynamical system f if

$$B(invN, diam_N f) \subset intN,$$

or equivalently

$$dist(invN, \partial N) > diam_N f.$$

Definition 3.3. Let N be an isolating neighborhood for a dispersive dynamical system f. A pair $P = (P_1, P_2)$ of compact subsets $P_2 \subset P_1 \subset N$ is called an *index pair* if the following conditions are satisfied

(i) $f(P_i, t) \cap N \subset P_i,\ i = 1, 2,\ 0 \leq t \leq 1$;
(ii) $f(P_1 - P_2, t) \subset N,\ 0 \leq t \leq 1$;
(iii) $invN \subset int(P_1 - P_2)$.

Our first aim is to prove the following result.

Theorem 3.4. *Let N be an isolating neighborhood for a dispersive dynamical system f and W a neighborhood of $invN$. Then there exists an index pair $P = (P_1, P_2)$ for N with $P_1 - P_2 \subset W$.*

The proof is based on several lemmas. First, given $N \subset X, s \in N, \tau \in \mathbb{R}^+$, the following notation will be used

$$\begin{aligned} f_{N,\tau}(x) &= \{y \in N \mid conn_N(\tau, x, y) \neq \emptyset\}, \\ f_{N,-\tau}(x) &= \{y \in N \mid conn_N(-\tau, x, y) \neq \emptyset\}, \\ f_N^+(x) &= \bigcup_{r \in \mathbb{R}^+} f_{N,\tau}(x),\ f_N^-(x) = \bigcup_{r \in \mathbb{R}^+} f_{N,-\tau}(x). \end{aligned}$$

Lemma 3.5. *If $N \subset X$ is compact, then the map $f_{N,\tau} : N \to 2^N$ is usc for any $\tau \in \mathbb{R}$.*

Proof. It is enough to prove that the assertion for $\tau \in \mathbb{R}^+$ since the case of a negative τ is analogous. Suppose that $f_{N,\tau}$ is not usc at $x \in N$. There exists $\varepsilon > 0$ such that for each $\delta > 0$ there exists $y \in N$ such that $d(x,y) < \delta$ and $f_{N,\tau}(y) \nsubseteq B(f_{N,\tau}(x), \varepsilon)$. For each positive integer n, there exists $x_n \in N$

such that $d(x,x_n) < \frac{1}{n}$ and $f_{N,\tau}(x_n) \nsubseteq B(f_{N,\tau}(x),\varepsilon)$. Let $y_n \in f_{N,\tau}(x_n) - B(f_{N,\tau}(x),\varepsilon)$. There exists a solution $\sigma_n : [0,\tau] \to N$ such that $\sigma_n(0) = x_n$ and $\sigma_n(\tau) = y_n$. Since $\sigma_n(0) = x_n \to x$, by Barbashin's Theorem, there exist a subsequence (σ_{n_i}) of (σ_n) and a solution $\gamma : [0,\tau] \to X$ such that (σ_{n_i}) converges uniformly to γ on $[0,\tau]$. Clearly $\gamma : [0,\tau] \to N$. Since N is compact, (y_{n_i}) has a convergent subsequence. Let $y_{n_i} \to y \in N$. We have

$$\gamma(0) = \lim_{i\to\infty} \sigma_{n_i}(0) = \lim_{i\to\infty} x_{n_i} = \lim_{n\to\infty} x_n = x,$$
$$\gamma(\tau) = \lim_{i\to\infty} \sigma_{n_i}(\tau) = \lim_{i\to\infty} y_{n_i} = y.$$

Thus $y \in f_{N,\tau}(x)$. Since $d(y_n, f_{N,\tau}(x)) \geq \varepsilon$ for all i, we have $d(y, f_{N,\tau}(x)) \geq \varepsilon$. This is a contradiction. Hence $f_{N,\tau}$ is usc. □

Lemma 3.6. *Suppose that $D(f_{N,\tau}) \equiv \{x \in N \mid f_{N,\tau}(x) \neq \emptyset\}$ are nonempty for all $\tau \in \mathbb{R}$. Then invN is nonempty. Moreover, $inv^+N = \bigcap_{\tau\in\mathbb{R}^+} D(f_{N,\tau})$ and $inv^-N = \bigcap_{\tau\in\mathbb{R}^-} D(f_{N,\tau})$.*

Proof. Since $\{D(f_{N,\tau}) \mid \tau \in \mathbb{R}^+\}$ is a family of nonempty closed sets with finite intersection property, $K = \cap_{\tau\in\mathbb{R}^+} D(f_{N,\tau})$ is nonempty. We prove that $inv^+N = K$. The proof for inv^-N is analogous and the conclusion for $invN$ follows from Proposition 2.8. The inclusion $inv^+N \subset K$ is obvious. Let $x \in K$. For each positive integer n, there exists a solution $\sigma_n : [0,n] \to N$ such that $\sigma_n(0) = x$. By Proposition 2.13, there exists a solution $\sigma : \mathbb{R}^+ \to N$ such that $\sigma(0) = x$. Thus $x \in inv^+N$. □

Lemma 3.7. *If $N \subset X$ is compact, then*

(a) *the sets inv^+N, inv^-N and $invN$ are compact*;

(b) *if A is compact with $inv^-N \subset A \subset N$, then $f_N^+(A)$ is compact.*

Proof. (a) Obvious.

(b) It is sufficient to show that $f_N^+(A)$ is closed. Let $y \in \overline{f_N^+(A)}$. There exists a sequence (y_n) in $f_N^+(A)$ such that $y_n \to y$. For each positive integer n, there exist $t_n \in \mathbb{R}^+$ and a solution $\sigma_n : [0,t_n] \to N$ such that $\sigma_n(t_n) = y_n$ and $\sigma_n(0) \in A$.

Case 1 : (t_n) is bounded. (t_n) has a convergent subsequence. Let $t_n \to t \in \mathbb{R}^+$. By Proposition 2.12, we have $y \in f_{N,t}(A) \subset f_N^+(A)$.

Case 2 : (t_n) is unbounded. We may assume that $t_n \to \infty$. For each positive integer n, define $\gamma_n : [-t_n, 0] \to N$ by $\gamma_n(t) = \sigma_n(t + t_n)$. Since γ_n is a solution and $\gamma_n(0) = \sigma_n(t_n) = y_n$, we have $f_{N,-t_n}(y_n) \neq \emptyset$, that is $y_n \in D(f_{N,-t_n})$. Thus we have

$$y \in \bigcap_{n=1}^{\infty} D(f_{N,-t_n}) = inv^-N \subset A \subset f_N^+(A).$$

Hence $f_N^+(A)$ is closed. □

Lemma 3.8. *If $K \subset N$ is a compact subset of X with $K \cap inv^+N = \emptyset$ (respectively, $K \cap inv^-N = \emptyset$), then*

(a) *there exists $T \in \mathbb{R}^+$ (respectively, $\mathbb{R}^-$) such that $f_{N,\tau}(K) = \emptyset$ for all $\tau \geq T$ (respectively, $\tau \leq T$);*

(b) *the map f_N^+ (respectively f_N^-) is usc on K;*

(c) *$f_N^+(K) \cap inv^+N = \emptyset$ (respectively, $f_N^- \cap inv^-N = \emptyset$).*

Proof. (a) Assume that $K \cap inv^+N = \emptyset$. For each $x \in K$, since $x \notin inv^+N$, by Lemma 3.6, there exists $t_x \in \mathbb{R}^+$ such that $f_{N,t_x}(x) = \emptyset$. Since f_{N,t_x} is usc, there exists a neighborhood V_x of x such that $f_{N,t_x}(y) = \emptyset$ for all $y \in V_x$. Then the family $\{V_x \mid x \in K\}$ is an open cover of K. Since K is compact, there are finitely many $x_1, \ldots, x_n \in K$ such that $K \subset \cup_{i=1}^n V_{x_i}$. Let $T = \max t_{x_i}$. If $\tau \geq T$, then

$$f_{N,\tau}(K) \subset f_{N,\tau}(\bigcup_{i=1}^n V_{x_i}) = \bigcup_{i=1}^n f_{N,\tau}(V_{x_i}) \subset \bigcup_{i=1}^n f_{N,t_{x_i}}(V_{x_i}) = \emptyset.$$

Thus $f_{N,\tau}(K) = \emptyset$.

(b) For any open subset U of X, $(f_N^+)^{-1}(U) = \cup_{\tau \in \mathbb{R}^+}(f_{N,\tau})^{-1}(U)$ is open. Thus f_N^+ is usc.

(c) Let $x \in f_N^+(K) \cap inv^+N$. Since $x \in f_N^+(K)$, there exist $\tau \in \mathbb{R}^+$ and a solution $\sigma_1 : [0, \tau] \to N$ such that $\sigma_1(\tau) = x$ and $\sigma_1(0) \in K$. Since $x \in inv^+N$, there exists a solution $\sigma_2 : \mathbb{R}^+ \to N$ such that $\sigma_2(0) = x$. Define $\sigma : \mathbb{R}^+ \to N$ by

$$\sigma(t) = \begin{cases} \sigma_1(t), & 0 \leq t \leq \tau \\ \sigma_2(t - \tau), & t \geq \tau. \end{cases}$$

Since σ is a solution and $\sigma(0) = \sigma_1(0) \in K$, we have $\sigma(T) \in f_{N,T}(K) \cap inv^+N$. This is a contradiction. Thus $f_N^+(K) \cap inv^+N = \emptyset$. □

Lemma 3.9. *If N is a compact subset of X, then for any neighborhood V of inv^-N there exists a compact neighborhood A of inv^-N such that $f_N^+(A) \subset V$.*

Proof. Since $N - V$ is compact and $(N - V) \cap inv^-N = \emptyset$, by Lemma 3.8, there exists $T \in \mathbb{R}^+$ such that $f_{N,-T}(N - V) = \emptyset$. For each $x \in inv^-N$, since $f_{N,T}$ is usc, there exists a compact neighborhood V_x of x such that $f_{N,T}(V_x) \subset V$. Since inv^-N is compact, there are finitely many points $x_1, \ldots, x_n \in inv^-N$ such that $K \subset \cup_{i=1}^n V_{x_i}$. Let $A = \cup_{i=1}^n V_{x_i}$. Then A is a compact neighborhood of inv^-N and

$$f_{N,T}(A) \subset f_{N,T}(\bigcup_{i=1}^n V_{x_i}) = \bigcup_{i=1}^n f_{N,T}(V_{x_i}) \subset V.$$

Let $y \in f_N^+(A)$. Then there exist $\tau \in \mathbb{R}^+$ and $x \in A$ such that $y \in f_{N,\tau}(x)$. If $\tau \leq T$, then

$$y \in f_{N,\tau}(x) \subset f_{N,\tau}(A) \subset f_{N,T}(A) \subset V.$$

If $\tau > T$, then we have $x \in f_{N,-\tau}(y) \subset f_{N,-T}(y)$. Since $f_{N,-T}(N - V) = \emptyset$, we have $y \in V$. Thus $f_N^+(A) \subset V$. □

Proof of Theorem 3.4. Since N is an isolating neighborhood, we have

$$invN \subset B(invN, diam_N f) \subset invN.$$

Thus we may assume that $W \subset intN$. Choose $0 < \varepsilon < dist(invN, \partial N) - diam_n f$ and let $\gamma = \varepsilon + diam_N f$. Since $\gamma < dist(invN, \partial N)$, we have $B(invN, \gamma) \subset intN$. Since the set $\{x \in X \mid f(x, [0,1]) \subset B(invN, \gamma)\}$ is an open neighborhood of $invN$, we may assume that $f(W, t) \subset intN$ for all $0 \le t \le 1$. Let U and V be open neighborhoods of inv^+N and inv^-N, respectively, such that $U \cap V \subset W$. By Lemma 3.9, there exists a compact neighborhood A of inv^-N such that $f_N^+(A) \subset V$. We define

$$P_1 = f_N^+(A),\ P_2 = f_N^+(P_1 - U)\,.$$

Then $P_1 \subset V$ and $P_1 - U \subset P_2$ which implies that $P_1 - P_2 \subset U$. Thus $P_1 - P_2 \subset U \cap V \subset W$. We verify that (P_1, P_2) is an index pair. By Lemma 3.7, P_1 is compact. Since $P_1 - U$ is compact, by Lemma 3.7, P_2 is compact. We have

$$P_2 = f_N^+(P_1 - U) \subset f_N^+(P_1) = f_N^+(f_N^+(A)) = f_N^+(A) = P_1.$$

To verify (i), let $x \in P_i$ and $y \in f(x,t) \cap N$. Since $x \in P_i$, there exist $\tau \in \mathbb{R}^+$ and a solution $\sigma_1 : [0, \tau] \to N$ such that $\sigma_1(\tau) = x$ and $\sigma_1(0) \in A$ in the case of $i = 1$, $\sigma_1(0) \in P_1 - U$ in the case $i = 2$. There exists a solution $\sigma_2 : [0, t] \to N$ such that $\sigma_2(0) = x$ and $\sigma_2(t) = y$. Define $\gamma : [0, \tau + t] \to N$ by

$$\gamma(s) = \begin{cases} \sigma_1(s), & 0 \le s \le \tau \\ \sigma_2(s - \tau), & \tau \le s \le \tau + t\,. \end{cases}$$

Then γ is a solution and $\gamma(0) = \sigma_1(0), \gamma(\tau + t) = \sigma_2(t) = y$. Thus $y \in P_i$. Since $P_1 - P_2 \subset W$, we have $f(P_1 - P_2, t) \subset f(W, t) \subset N$ for all $0 \le t \le 1$. Thus (ii) is verified. In order to verify (iii), observe that P_1 is a neighborhood of inv^-N. Since $P_1 - U$ is compact and $(P_1 - U) \cap inv^+N = \emptyset$, by Lemma 3.8,

$$P_2 \cap inv^+N = f_N^+(P_1 - U) \cap inv^+N = \emptyset.$$

Thus $N - P_2$ is a neighborhood of inv^+N. Hence $P_1 - P_2 = P_1 \cap (N - P_2)$ is a neighborhood of $inv^-N \cap inv^+N = invN$. □

We shall now discuss several properties of index pairs which will be used in next section. In the remainder in this section, we denote $f(x, 1)$ by $F(x)$.

Proposition 3.10. (a) *If P is an index pair for N, then $(P_1 \cap F(P_2)) - (P_2 \cap F(P_2)) = P_1 - P_2$.*

(b) *If P and Q are index pairs for N, then so is $P \cap Q$.*

(c) *If $P \subset Q$ are index pairs for N, then so are $(P_1, P_1 \cap Q_2)$ and $(P_1 \cup Q_2, Q_2)$.*

Proof. (a) We have $(P_1 \cup F(P_2)) - (P_2 \cup F(P_2)) = P_1 - (P_2 \cup F(P_2)) \subset P_1 - P_2$. Let $x \in P_1 - P_2$. If $x \in F(P_2)$, then we have $x \in F(P_2) \cap N \subset P_2$. This is a contradiction. Thus $x \in (P_1 \cup F(P_2)) - (P_2 \cup F(P_2))$.
(b) Verification of (i) and (ii) is obvious. For (iii), let us note that

$$\begin{aligned} int(P_1 - P_2) \cap int(Q_1 - Q_2) &\subset int(P_1 \cap Q_1 - (P_2 \cap Q_2)) \\ &\subset int(P_1 \cap Q_1 - (P_2 \cup Q_2)). \end{aligned}$$

(c) Obvious. □

Proposition 3.11. *Let $P \subset Q$ be index pairs for N such that $P_1 = Q_1$ or $P_2 = Q_2$. Define a pair of sets $G(P,Q)$ by*

$$G_i(P,Q) = P_i \cup (F(Q_i) \cap N),\ i = 1,2.$$

Then (a) *if $P_i = Q_i$, then $G_i(P,Q) = P_i = Q_i$, $i = 1,2$;*
(b) $P \subset G(P,Q) \subset Q$;
(c) $G(P,Q)$ *is an index pair*;
(d) $F(Q_i) \cap N \subset G_i(P,Q),\ i = 1,2.$

Proof. (a), (b) and (d) are clear. It remains to prove that (c). For (i), let $x \in G_i(P,Q), y \in F(x) \cap N$. If $x \in P_i$, then $y \in F(P_i) \cap N \subset P_i \subset G_i(P,Q)$. If $x \in F(Q_i) \cap N \subset Q_i$, then $y \in F(Q_i) \cap N \subset P_i \subset G_i(P,Q)$. For (ii), since $G_1(P,Q) = P_1 \cup (F(Q_1) \cap N) \subset P_1 \cup Q_1 = Q_1, P_2 \subset G_2(P,Q)$, we have

$$G_1(P,Q) - G_2(P,Q) \subset Q_1 - P_2.$$

If $P_1 = Q_1$, we have $F(G_1(P,Q) - G_2(P,Q)) \subset F(Q_1 - P_2) = F(P_1 - P_2) \subset N$. If $P_2 = Q_2$, we have $F(G_1(P,Q) - G_2(P,Q)) \subset F(Q_1 - P_2) = F(Q_1 - Q_2) \subset N$. For (iii), we show that $(P_1 - P_2) \cap (Q_1 - Q_2) \subset G_1(P,Q) - G_2(P,Q)$. Let $y \in (P_1 - P_2) \cap (Q_1 - Q_2)$. Since $P_1 \subset G_1(P,Q)$ and $G_2(P,Q) = P_2 \cup (F(Q_2) \cap N) \subset P_2 \cup Q_2 = Q_2$, we have $y \in G_1(P,Q)$ and

$$\begin{aligned} invN &\subset int(P_1 - P_2) \cap int(Q_1 - Q_2) \\ &\subset int((P_1 - P_2) \cap (Q_1 - Q_2)) \\ &\subset int(G_1(P,Q) - G_2(P,Q)). \end{aligned}$$ □

Proposition 3.12. *Let $P \subset Q$ be index pairs such that $P_1 = Q_1$ or $P_2 = Q_2$. Then there exists a sequence of pairs*

$$P = Q^n \subset Q^{n-1} \subset \cdots \subset Q^1 \subset Q^0 = Q$$

with the following properties:
(a) *if $P_i = Q_i$, then $Q_i^k = P_i = Q_i$ for all $k = 1,2,\ldots,n-1$, $i = 1,2$;*
(b) *Q^k is an index pair for all $k = 1,2,\ldots,n-1$;*
(c) $F(Q_i^k) \cap N \subset Q_i^{k+1}$, $i = 1,2$, $k = 0,1,\ldots,n-1$.

Proof. We denote $Q^0 = Q, Q^{k+1} = G(P,Q^k)$. By Proposition 3.11, (Q^k) is a decreasing sequence of index pairs containing P and satisfying (a), (b) and (c). It remains to show that $Q^n = P$ for some n. Suppose that the inclusion $P \subset Q^k$ is strict for all k. We may assume that $Q_2^k \neq P_2$ for all k. For any positive integer k, choose $\sigma(k) \in Q_2^k - P_2$. Since $\sigma(k) \in F(Q_2^{k-1}) \cap N$, there exists $\sigma(k-1) \in Q_2^{k-1}$ such that $\sigma(k) \in F(\sigma(k-1))$. If $\sigma(k-1) \in P_2$, then $\sigma(k) \in F(P_2) \cap N \subset P_2$. This is a contradiction. Thus $\sigma(k-1) \in Q_2^{k-1} - P_2$. Repeating this process, we have solution $\sigma_k : [0,k] \to Q_2 - intP_2$ such that $\sigma_k(i) = \sigma(i), 0 \leq i \leq k$. Since $D(f_{Q_2 - intP_2,k}) \neq \emptyset$ for all k, by Lemma 3.6, we have $\emptyset \neq inv(Q_2 - intP_2) \subset invQ_2$. On the other hand $invQ_2 \subset Q_2$ and $invQ_2 \subset int(Q_1 - Q_2) \subset Q_1 - Q_2$ implies that $invQ_2 = \emptyset$, a contradiction. $\square$

4. Existence of stable index pairs

Let (X,d) be a locally compact metric space and $\Omega(X)$ the set of all continuous dispersive dynamical systems on X. Define a metric $\rho : \Omega(X) \times \Omega(X) \to \mathbb{R}$ by

$$\rho(f,g) = \sup\Big\{ \max\{\frac{1}{|t|} D(f(x,t), g(x,t)) \mid x \in X\} \mid t \neq 0 \Big\}$$

for all $f, g \in \Omega(X)$, where D is the Hausdorff metric induced by d. Let $\Lambda \subset \mathbb{R}$ be a compact interval.

Definition 4.1. A map $\phi : \Lambda \to \Omega(X)$ will be called a *parametrized family* of dispersive dynamical systems if the map $(\lambda, x, t) \in \Lambda \times X \times \mathbb{R} \mapsto \phi(\lambda)(x,t) \in 2^X$ is usc.

Define a map $\psi : \Lambda \times X \times \mathbb{R} \to 2^{\Lambda \times X}$ by $\psi(\lambda, x, t) = \{\lambda\} \times \phi(\lambda)(x,t)$. It is easy to show that ψ is a dispersive dynamical system on $\Lambda \times X$.

Given a compact set $N \subset X$ and $\lambda \in \Lambda$, the sets $inv^{(\pm)}N$ with respect to $\phi(\lambda)$ are denoted by $inv^{(\pm)}(N,\lambda)$.

We need the following propositions to prove the existence of an stable index pair.

Proposition 4.2. *Let $N \subset X$ be compact. Then the mappings*

$$\begin{aligned} \lambda \in \Lambda &\mapsto inv^+(N,\lambda) \in 2^N, \\ \lambda \in \Lambda &\mapsto inv^-(N,\lambda) \in 2^N, \\ \lambda \in \Lambda &\mapsto inv(N,\lambda) \in 2^N \end{aligned}$$

are usc.

Proof. We prove the assertion for the first map only, since the other two proofs are by extending the same argument to negative numbers. Assume that the map

$$\lambda \in \Lambda \mapsto inv^+(N,\lambda) \in 2^N$$

is not usc at $\lambda_0 \in \Lambda$. Then there exist a neighborhood U of $inv^+(N,\lambda_0)$ and a sequence (λ_n) in Λ such that $\lambda_n \to \lambda_0$ and $inv^+(N,\lambda_n) \not\subseteq$ for all n. Choose $x_n \in inv^+(N,\lambda_n) - U$ and

$$\sigma_n \in sol_N(\mathbb{R}^+, x_n, \lambda_n) \subset sol_{\Lambda \times N}(\mathbb{R}^+, (\lambda_n, x_n)).$$

Since N is compact, (x_n) has a convergent subsequence. Let $x_n \to x \in N - U$. By Proposition 2.12, there exists $\sigma \in sol_{\Lambda\times N}(\mathbb{R}^+,(\lambda_0,x))$. Since $\sigma(0) = (\lambda_0,x) \in \{\lambda_0\}\times N$, we have $\sigma \in sol_N(\mathbb{R}^+,x,\lambda_0)$. Thus $x \in inv^+(N,\lambda_0) \subset U$, which contradicts $x \notin U$. □

Proposition 4.3. *Let $\lambda_0 \in \Lambda$ and let N be an isolating neighborhood for $\phi(\lambda_0)$. Then N is an isolating neighborhood for $\phi(\lambda)$ for all λ sufficiently close to λ_0.*

Proof. Since $dist(inv(N,\lambda_0),\partial N) > diam_N\phi(\lambda_0)$, choose ε such that

$$0 < 3\varepsilon < dist(inv(N,\lambda_0),\partial N) - diam_N\phi(\lambda_0).$$

Then $B(inv(N,\lambda_0), diam_N\phi(\lambda_0)+3\varepsilon) \subset intN$. Since the map

$$(\lambda,x,t) \in \Lambda\times X\times\mathbb{R} \mapsto \phi(\lambda)(x,t) \in 2^X$$

is usc and N is compact, $\phi(\lambda)(x,t) \subset B(\phi(\lambda_0)(x,t),\varepsilon)$ for all λ close to λ_0, all $x \in N$ and all $0 \le t \le 1$. By compactness of N, $diam_N\phi(\lambda) < diam_N\phi(\lambda_0)+2\varepsilon$ for all λ close to λ_0. By Proposition 4.2, $inv(N,\lambda) \subset B(inv(N,\lambda_0),\varepsilon)$ for all λ close to λ_0 and we get

$$\begin{aligned} B(inv(N,\lambda), diam_N\phi(\lambda)) &\subset B\Big(B(inv(N,\lambda_0),\varepsilon), diam_N\phi(\lambda_0)+2\varepsilon\Big) \\ &= B(inv(N,\lambda_0), diam_N\phi(\lambda_0)+3\varepsilon) \\ &\subset intN. \end{aligned}$$

□

Proposition 4.4. *Let $f : X\times\mathbb{R} \to 2^X$ be a dispersive dynamical system, N an isolating neighborhood for f and P an index pair for N and f. If $g : X\times\mathbb{R} \to 2^X$ is a dispersive dynamical system such that $f(x,t) \subset f(x,t)$ for all $(x,t) \in X\times\mathbb{R}$, then N is an isolating neighborhood for g, $inv^{(\pm)}(N,g) \subset inv^{(\pm)}(N,f)$ and P is also an index pair for N and g.*

Proof. It is clear. □

The main result of this section is the following.

Theorem 4.5. *Let $f : X\times\mathbb{R} \to 2^X$ be a continuous dispersive dynamical system, N an isolating neighborhood for f and W an open neighborhood of $invN$. Then there exists an index pair $P = (P_1,P_2)$ for N with $P_1 - P_2 \subset W$ which is stable under small continuous perturbations of f, i.e., there exists $\varepsilon > 0$ such that if $g \in B_\rho(f,\varepsilon)$, then P also is an index pair for g.*

Proof. Define a family of dispersive dynamical systems $\phi : [0,1] \to \Omega(X)$ by

$$\begin{aligned} &\phi_\lambda(x,t) = \overline{B}(f(x,t),\lambda t) \text{ for } x \in X, t \in \mathbb{R}^+, \\ &\phi_\lambda(x,t) = \{y \in X \mid x \in \phi_\lambda(y,-t)\} \text{ for } x \in X, t \in \mathbb{R}^-. \end{aligned}$$

By Propositions 4.2 and 4.3, there exists $\tau > 0$ such that N is an isolating neighborhood for ϕ_λ and $inv(N,\lambda) \subset W$ provided $0 \le \lambda \le \tau$. Define

$$P_1 = inv^-(N,\tau),\ P_2 = P_1 - int(inv^+(N,\tau)).$$

Note that $\overline{P_1 - P_2} = inv(N, \tau) \subset W$. We shall verify below that $P = (P_1, P_2)$ is an index pair for all ϕ_λ with $0 \leq \lambda < \tau$. In particular, it is an index pair for $\phi(0) = f$. Moreover, if $g \in B_\rho(f, \varepsilon)$ for $\varepsilon < \tau$, then $g(x,t) \subset \phi_\tau(x,t)$ for all $(x,t) \in X \times \mathbb{R}$. Thus the conclusion follows from Proposition 4.4. We will show that P is an index pair for ϕ_λ provided $0 \leq \lambda < \tau$.

(i) $\phi_\lambda(P_i, t) \cap N \subset P_i,\ i = 1, 2,\ 0 \leq t \leq 1$.

Let $y \in \phi_\lambda(P_1, t) \cap N$. Then there exists $x \in P_1$ such that $y \in \phi_\lambda(x,t)$. Since $P_1 = inv^-(N, \tau)$, there exists a solution $\sigma_1 : \mathbb{R}^- \to N$ for ϕ_τ such that $\sigma_1(0) = x$. Also there exists a solution $\sigma_2 : [0,t] \to N$ for ϕ_λ such that $\sigma_2(0) = x, \sigma_2(t) = y$. Since

$$\sigma_2(s) \in \phi_\lambda(x,s) \subset \phi_\tau(x,s)$$

for all $0 \leq s \leq t$, σ_2 is a solution for ϕ_τ. Define $\sigma : \mathbb{R}^- \to N$ by

$$\sigma(s) = \begin{cases} \sigma_2(s+t), & -t \leq s \leq 0 \\ \sigma_1(s+t), & s \leq t\,. \end{cases}$$

Then σ is a solution for ϕ_τ and since $\sigma(0) = y$, we have $y \in inv^-(N, \tau) = P_1$. Let $y \in \phi_\lambda(P_2, t) \cap N$. Then there exists $x \in P_2$ such that $y \in \phi_\lambda(s,t)$. Since $x \in P_1$, we have $y \in P_1$. In order to show that $y \notin int(inv^+(N, \tau))$, suppose that $y \in int(inv^+(N, \tau))$. Then there exists $\varepsilon > 0$ such that $B(y, \varepsilon) \subset inv^+(N, \tau)$. Since $\phi_\tau \in \Omega(X)$, if $d(x,z) < \varepsilon$, then there exists $\delta > 0$ such that $\phi_\tau(x,t) \subset B(\phi_\tau(z,t), \varepsilon)$. Since $y \in \phi_\lambda(x,t) \subset \phi_\tau(x,t)$ for $z \in B(x, \delta)$, there exists $p \in \phi_\tau(z,t)$ such that $d(y,p) < \varepsilon$. Since $p \in B(y, \varepsilon) \subset inv^+(N, \tau)$, there exists a solution $\sigma_1 : \mathbb{R}^+ \to N$ for ϕ_τ such that $\phi_1(0) = p$. Since $p \in \phi_\tau(z,t)$, there exists a solution $\sigma_2 : [0,t] \to N$ for ϕ_τ such that $\sigma_2(0) = z$, $\sigma_2(t) = p$. Define $\sigma : \mathbb{R}^+ \to N$ by

$$\sigma(s) = \begin{cases} \sigma_2(s), & 0 \leq s \leq t \\ \sigma_1(s-t), & s \geq t\,. \end{cases}$$

Then σ is a solution for ϕ_τ. Since $\sigma(0) = z$, we have $z \in inv^+(N, \tau)$, that is, $B(x, \delta) \subset inv^+(N, \tau)$. Therefore we have $x \in int(inv^+(N, \tau))$. This contradicts $x \in P_2$.

(ii) $\phi_\lambda(P_1 - P_2, t) \subset N$.

Since $\phi_\lambda(P_1 - P_2, t) \subset \phi_\tau(P_1 - P_2, t) \subset N$, it holds.

(iii) $inv(N, \lambda) \subset int(P_1 - P_2)$.

Since $int(P_1 - P_2) = int(inv(N, \tau))$, we show only that the following holds;

$$0 \leq \lambda < \tau \ \text{ implies } \ inv^\pm(N, \lambda) \subset int(inv^\pm(N, \tau)).$$

Since $\tau - \lambda > 0$, if $d(x,z) < \delta$, then there exists $\delta > 0$ such that $f(x,1) \subset B(f(z,1), \tau - \lambda)$. We have

$$\phi_\lambda(x,1) = \overline{B}(f(x,1), \lambda) \subset \overline{B}\Big(B(f(z,1), \tau - \lambda), \lambda\Big) = \overline{B}(f(z,1), \tau) = \phi_\tau(z,1).$$

Let $x \in inv^+(N, \lambda)$. Then there exists a solution $\sigma_1 : \mathbb{R}^+ \to N$ for ϕ_λ such that $\sigma_1(0) = x$. For any $z \in B(x, \delta)$, let $y \in \phi_\lambda(x,1) \subset \phi_\tau(z,1)$. Then there exist a solution $\sigma_2 : [0,1] \to N$ for ϕ_λ and a solution $\sigma_3 : [0,1] \to N$ for ϕ_τ such that $\sigma_2(0) = x, \sigma_2(1) = y$ and $\sigma_3(0) = z, \sigma_3(1) = y$. Define $\sigma : \mathbb{R}^+ \to N$ by

$$\sigma(s) = \begin{cases} \sigma_3(s), & 0 \leq s \leq 1 \\ \sigma_2(2-s), & 1 \leq s \leq 2 \\ \sigma_1(s-2), & s \geq 2. \end{cases}$$

Then σ is a solution for ϕ_τ. Since $\sigma(0) = z$, we have $z \in inv^+(N,\tau)$. Thus we have

$$B(inv^+(N,\lambda),\delta) \subset inv^+(N,\tau)$$

and so it follows that $inv^+(N,\lambda) \subset int(inv^+(N,\tau))$.

Let $x \in inv^-(N,\lambda)$. Then there exists a solution $\sigma_1 : \mathbb{R}^- \to N$ for ϕ_λ such that $\sigma_1(0) = x$. Let $\sigma(-1) = y$. Then we have $x = \sigma(0) \in \phi_\lambda(\sigma(-1),1) = \phi_\lambda(y,1)$. For any $z \in B(x,\tau-\lambda)$, we have

$$z \in \overline{B}(\phi_\lambda(y,1),\tau-\lambda) = \overline{B}\Big(\overline{B}(f(y,1),\lambda),\tau-\lambda\Big) = \overline{B}(f(y,1),\tau) = \phi_\tau(y,1).$$

Therefore there exists a solution $\sigma_2 : [0,1] \to N$ for ϕ_τ such that $\sigma_2(0) = y, \sigma_2(1) = z$. Define $\sigma : \mathbb{R}^- \to N$ by

$$\sigma(s) = \begin{cases} \sigma_2(s+1), & -1 \leq s \leq 0 \\ \sigma_1(s+1), & s \leq -1\,. \end{cases}$$

Then σ is a solution for ϕ_τ. Since $\sigma(0) = z$, we have $z \in inv^-(N,\tau)$. Hence we have

$$B(inv^-(N,\lambda),\tau-\lambda) \subset inv^-(N,\tau).$$

It follows that $inv^-(N,\lambda) \subset int(inv^-(N,\tau))$. □

References

[1] R. Churchill, *Isolated invariant sets in compact metric spaces*, J. Differential Equations **12** (1972), 330–352.

[2] C. C. Conley, *Isolated Invariant Sets and the Morse Index*, CBMS Regional Conf. Ser. Math. No. 38, Amer. Math. Soc., Providence RI, 1978.

[3] E. N. Dancer, *Perturbation of zeros in the presence of symmetries*, J. Austral. Math. Soc. Ser. A **36** (1984), no. 1, 106–125.

[4] ———, *Degenerate critical points, homotopy indices and Morse inequalities*, J. Reine Angew. Math. **350** (1984), 1–22.

[5] A. Dold, *Fixed point index and fixed point theorem for Euclidean neighborhood retracts*, Topology **4** (1965), 1–8.

[6] T. Kaczynski and M. Morzec, *Conley index for discrete multi-valued dynamical systems*, Topology Appl. **65** (1995), no. 1, 83–96.

[7] ———, *Stable index pairs for discrete dynamical systems*, Canad. Math. Bull. **40** (1997), no. 4, 448–455.

[8] P. E. Kloeden, *Asymptotic invariance and limit sets of general control systems*, J. Differential Equations **19** (1975), no. 1, 91–105.

[9] H. L. Kurland, *The Morse index of an isolated invariant set is a connected simple system*, J. Differential Equations **42** (1981), no. 2, 234–259.

[10] D. Li, *On dynamical stability in general dynamical systems*, J. Math. Anal. Appl. **263** (2001), no. 2, 455–478.

[11] J. T. Montgomery, *Cohomology of isolated invariant sets under perturbation*, J. Differential Equations **13** (1973), 257–299.
[12] E. Roxin, *Stability in general control systems*, J. Differential Equations **1** (1965), 115–150.
[13] K. Rybakowski, *On the homotopy index for infinite dimensional semiflows*, Trans. Amer. Math. Soc. **269** (1982), no. 2, 351–382.
[14] K. S. Sibirsky, *Introduction to Topological Dynamics*, Noordhoff International Publishing, Leiden, 1975.
[15] H. Siegberg and G. Skordev, *Fixed point index and chain approximations*, Pacific J. Math. **102** (1982), no. 2, 455–486.

Yoon Hoe Goo
Department of Mathematics
Hanseo University
Chungnam 356-706, Korea
E-mail address: yhgoo@hanseo.ac.kr

Jong-Suh Park
Department of Mathematics
Chungnam National University
Daejeon 305-764, Korea
E-mail address: jspark@math.cnu.ac.kr