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DUALITY OF CO-POISSON HOPF ALGEBRAS

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Abstract. Let A be a co-Poisson Hopf algebra with Poisson co-bracket δ. Here it is shown that the Hopf dual A° is a Poisson Hopf algebra with Poisson bracket $\{f,g\}(x) = \langle\delta(x), f\otimes g\rangle$ for any $f,g\in A^\circ$ and $x\in A$ if A is an almost normalizing extension over the ground field. Moreover we get, as a corollary, the fact that the Hopf dual of the universal enveloping algebra $U(\mathfrak{g})$ for a finite dimensional Lie bialgebra $\mathfrak{g}$ is a Poisson Hopf algebra.

Let G be a Lie group with Lie algebra $\mathfrak{g}$. Then its coordinate ring $\mathcal{O}(G)$ is a Hopf algebra and can be replaced by the Hopf dual $U(\mathfrak{g})^\circ$ of the universal enveloping algebra $U(\mathfrak{g})$. In fact, it is well-known that $U(\mathfrak{g})^\circ$ is equal to $\mathcal{O}(G)$ if G is connected and simply connected. Moreover it is convenient to work on $U(\mathfrak{g})^\circ$ instead of $\mathcal{O}(G)$ since $U(\mathfrak{g})^\circ$ has a natural grading. For instance, see [3, Chapter 2] and [2].

Recall that a Lie group G is said to be a Poisson Lie group if its coordinate ring $\mathcal{O}(G)$ is a Poisson Hopf algebra. If G is a Poisson Lie group, then its Lie algebra $\mathfrak{g}$ becomes a finite dimensional Lie bialgebra with a co-bracket δ and the universal enveloping algebra $U(\mathfrak{g})$ is a co-Poisson Hopf algebra with Poisson co-bracket extended naturally from δ (See [1, §6.2]). Thus its Hopf dual $U(\mathfrak{g})^\circ$ would be a Poisson Hopf algebra with Poisson bracket induced by δ. At this moment, we would show the fact $\{f,g\}\in U(\mathfrak{g})^\circ$ for all $f,g\in U(\mathfrak{g})^\circ$.

Let A be a co-Poisson Hopf algebra. Since the concept of a co-Poisson Hopf algebra is a dual concept of Poisson Hopf algebra, the Hopf dual A° of A is anticipated a Poisson Hopf algebra. Here we give a complete proof that the Hopf dual A° is a Poisson Hopf algebra in the case that A is an almost normalizing extension over the ground field and we get, as a corollary, the fact that $U(\mathfrak{g})^\circ$ is a Poisson Hopf algebra if $\mathfrak{g}$ is a finite dimensional Lie bialgebra.

Assume throughout that $\mathbf{k}$ denotes a field of characteristic zero and all vector spaces are over $\mathbf{k}$.

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Recall the definition of co-Poisson Hopf algebra. Let $A = (A, \iota, \mu, \epsilon, \Delta, S)$ be a Hopf algebra over $\mathbf{k}$. Let τ be the flip on $A \otimes A$, that is, τ is a $\mathbf{k}$-linear map defined by

$$\tau : A \otimes A \longrightarrow A \otimes A, \quad x \otimes y \mapsto y \otimes x,$$

and set

$$\tau_{12} = \tau \otimes 1, \quad \tau_{23} = 1 \otimes \tau.$$

A Hopf algebra A is said to be a co-Poisson Hopf algebra if there exists a skew-symmetric $\mathbf{k}$-linear map $\delta : A \longrightarrow A \otimes A$, called a Poisson co-bracket, satisfying the following conditions:

(i) (co-Jacobi identity)

$$(\delta \otimes 1) \circ \delta + \tau_{12} \circ \tau_{23} \circ (\delta \otimes 1) \circ \delta + \tau_{23} \circ \tau_{12} \circ (\delta \otimes 1) \circ \delta = 0.$$

(ii) (co-Leibniz rule)

$$(\Delta \otimes \mathrm{id}) \circ \delta = (\mathrm{id} \otimes \delta) \circ \Delta + \tau_{23} \circ (\delta \otimes \mathrm{id}) \circ \Delta.$$

(iii) (Δ-derivation)

$$\delta(ab) = \delta(a)\Delta(b) + \Delta(a)\delta(b)$$

for all $a, b \in A$.

Definition 1 ([4, 1.6.10])**.** An algebra R over $\mathbf{k}$ is said to be an almost normalizing extension over $\mathbf{k}$ if R is a finitely generated $\mathbf{k}$-algebra with generators $x_1, \dots, x_n$ satisfying the condition

$$x_i x_j - x_j x_i \in \sum_{\ell=1}^{n} \mathbf{k} x_\ell + \mathbf{k}$$

for all i, j.

Lemma 2. *Let R be an almost normalizing extension of $\mathbf{k}$ with generators $x_1, \dots, x_n$. Then R is spanned by all standard monomials*

$$x_1^{r_1} x_2^{r_2} \cdots x_n^{r_n}, \quad r_i = 0, 1, \dots$$

together with the unity 1.

Proof. This follows immediately from induction on the degree of monomials. □

Note that the Hopf dual A° of a Hopf algebra A consists of

$$A^\circ = \{f \in A^* \mid f(I) = 0 \text{ for some cofinite ideal } I \text{ of } A\},$$

where A^* is the dual vector space of A.

Theorem 3. *Let A be a co-Poisson Hopf algebra with Poisson co-bracket δ. If A is an almost normalizing extension over $\mathbf{k}$, then the Hopf dual A° is a Poisson Hopf algebra with Poisson bracket*

$$\{f,g\}(x) = \langle \delta(x), f\otimes g\rangle, \qquad x\in A \tag{1}$$

for any $f,g\in A^\circ$, where $\langle\cdot,\cdot\rangle$ is the natural pairing between the vector space $A\otimes A$ and its dual vector space.

Proof. Step 1. The Poisson bracket (1) *is well-defined. That is, $\{f,g\}\in A^\circ$ for every $f,g\in A^\circ$*: There exist cofinite ideals I,J of A such that $f(I)=0$ and $g(J)=0$. Since the canonical map $A/(I\cap J)\longrightarrow A/I\times A/J$ is a monomorphism, the ideal $I\cap J$ is also cofinite. Set

$$K=(I\cap J)\otimes A + A\otimes (I\cap J).$$

Note that $\langle K, f\otimes g\rangle = 0$. The canonical map from $[A/(I\cap J)]\otimes[A/(I\cap J)]$ into $(A\otimes A)/K$ is surjective and thus $(A\otimes A)/K$ is finite dimensional.

Note that A is spanned by the standard monomials

$$x_1^{r_1}x_2^{r_2}\cdots x_n^{r_n}, \quad r_i=0,1,\dots$$

together with the unity 1 by Lemma 2. For each $i=1,2,\dots,n$, the set of cosets

$$\{\delta(x_i^k)+K \mid k=1,2,\dots\}$$

is linearly dependent since $(A\otimes A)/K$ is finite dimensional and thus there exists a nonzero polynomial $h\in\mathbf{k}[x]$ such that $\delta(h(x_i))\in K$, where x is an indeterminate. Consider the set

$$S=\{0\neq h\in \mathbf{k}[x] \mid \delta(h(x_i))\in K\}.$$

Note that S is an infinite set since K is an ideal and S is not empty. For instance, if $h\in S$, then $h^k\in S$ for all positive integer k by the Δ-derivation of δ. Since S is an infinite set and $(A\otimes A)/K$ is finite dimensional, there exists a nonzero polynomial $h_i\in S$ such that $\Delta(h_i(x_i))\in K$. That is,

$$\delta(h_i(x_i))\in K, \quad \Delta(h_i(x_i))\in K.$$

Let $s_i=\deg(h_i)$ and let L be the ideal of A generated by

$$h_1(x_1), h_2(x_2),\dots,\ h_n(x_n).$$

For any standard monomial $X=x_1^{r_1}x_2^{r_2}\cdots x_n^{r_n}$, there exist polynomials $q_1,\dots$, q_n, $t_1,\dots$, t_n of $\mathbf{k}[x]$ such that

$$x_i^{r_i}=q_i(x_i)h_i(x_i)+t_i(x_i), \qquad \deg(t_i)<s_i \tag{2}$$

for $i=1,2,\dots,n$. Replacing each factor $x_i^{r_i}$ in X by the right hand of the equation (2), we have the fact that X is congruent to a $\mathbf{k}$-linear combination of finite standard monomials

$$x_1^{p_1}x_2^{p_2}\cdots x_n^{p_n}, \ \ p_i<s_i \text{ for } i=1,2,\dots,n$$

modulo L. Thus A/L is finite dimensional and hence L is a cofinite ideal.

Note that every element of L is a sum of elements of the form $ah_i(x_i)b$, where $a, b \in A$ and $i = 1, \ldots, n$. For every element $ah_i(x_i)b$, we have

$$\begin{aligned}\delta(ah_i(x_i)b) = \delta(a)\Delta(h_i(x_i))\Delta(b) + \Delta(a)\delta(h_i(x_i))\Delta(b)\\ + \Delta(a)\Delta(h_i(x_i))\delta(b) \in K.\end{aligned}$$

Hence $\{f, g\}(L) = \langle \delta(L), f \otimes g\rangle = 0$ and thus $\{f, g\} \in A^\circ$.

Step 2. For every $f, g \in A^\circ$, $\{f, g\} = -\{g, f\}$: Since $\tau \circ \delta = -\delta$, we have immediately that

$$\begin{aligned}\{f, g\}(x) &= \langle \delta(x), f \otimes g\rangle = \langle \tau \circ \delta(x), g \otimes f\rangle\\ &= -\langle \delta(x), g \otimes f\rangle = -\{g, f\}(x)\end{aligned}$$

for all $x \in A$. Thus we have $\{f, g\} = -\{g, f\}$.

Step 3. The equation (1) *satisfies the Leibniz rule*: Since

$$\{fg, h\}(x) = \langle (\Delta \otimes 1) \circ \delta(x), f \otimes g \otimes h\rangle$$

and

$$\begin{aligned}&(f\{g, h\} + \{f, h\}g)(x)\\ &= \langle (1 \otimes \delta) \circ \Delta(x), f \otimes g \otimes h\rangle + \langle \tau_{23} \circ (\delta \otimes 1) \circ \Delta(x), f \otimes g \otimes h\rangle\end{aligned}$$

for $x \in A$ and $f, g, h \in A^\circ$, it is enough to show that

$$(\Delta \otimes 1) \circ \delta = (1 \otimes \delta) \circ \Delta + \tau_{23} \circ (\delta \otimes 1) \circ \Delta. \tag{3}$$

But the equation (3) is just the co-Leibniz rule of δ.

Step 4. The equation (1) *satisfies the Jacobi identity*: Observe that

$$\begin{aligned}\{\{f, g\}, h\}(x) &= \langle (\delta \otimes 1) \circ \delta(x), f \otimes g \otimes h\rangle,\\ \{\{g, h\}, f\}(x) &= \langle \tau_{12} \circ \tau_{23} \circ (\delta \otimes 1) \circ \delta(x), f \otimes g \otimes h\rangle,\\ \{\{h, f\}, g\}(x) &= \langle \tau_{23} \circ \tau_{12} \circ (\delta \otimes 1) \circ \delta(x), f \otimes g \otimes h\rangle\end{aligned}$$

for $x \in A$ and $f, g, h \in A^\circ$. Hence (1) satisfies the Jacobi identity if and only if δ satisfies

$$(\delta \otimes 1) \circ \delta + \tau_{12} \circ \tau_{23} \circ (\delta \otimes 1) \circ \delta + \tau_{23} \circ \tau_{12} \circ (\delta \otimes 1) \circ \delta = 0. \tag{4}$$

But the equation (4) is the co-Jacobi identity of δ. Hence (1) satisfies the Jacobi identity.

Step 5. $\Delta(\{f, g\}) = \{\Delta(f), \Delta(g)\}$ *for all* $f, g, \in A^\circ$: For any $x, y \in A$,

$$\begin{aligned}\Delta(\{f, g\})(x \otimes y) &= \{f, g\}(xy) = \langle \delta(xy), f \otimes g\rangle\\ &= \langle \delta(x)\Delta(y), f \otimes g\rangle + \langle \Delta(x)\delta(y), f \otimes g\rangle\\ &= \sum \langle \delta(x), f' \otimes g'\rangle\langle \Delta(y), f'' \otimes g''\rangle\\ &\quad + \sum \langle \Delta(x), f' \otimes g'\rangle\langle \delta(y), f'' \otimes g''\rangle\\ &= \{f', g'\}(x)(f''g'')(y) + (f'g')(x)(\{f'', g''\})(y)\\ &= \{\Delta(f), \Delta(g)\}(x \otimes y),\end{aligned}$$

where $\Delta(f) = \sum f' \otimes f'', \Delta(g) = g' \otimes g''$. Thus we have

$$\Delta(\{f, g\}) = \{\Delta(f), \Delta(g)\}$$

for $f, g \in A^\circ$. This completes the proof of Theorem 3. □

Refer to [1, 1.3] for the definition of Lie bialgebra. Let $(\mathfrak{g}, \delta)$ be a Lie bialgebra, $U(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$ and Δ the comultiplication of $U(\mathfrak{g})$. The cobracket δ is extended uniquely to a Δ-derivation $\overline{\delta}$. That is,

$$\overline{\delta} : U(\mathfrak{g}) \longrightarrow U(\mathfrak{g}) \otimes U(\mathfrak{g})$$

is a $\mathbf{k}$-linear map such that $\overline{\delta}|_{\mathfrak{g}} = \delta$ and $\overline{\delta}(xy) = \overline{\delta}(x)\Delta(y) + \Delta(x)\overline{\delta}(y)$ for all $x, y \in U(\mathfrak{g})$. Then, by [1, Proposition 6.2.3], $U(\mathfrak{g})$ is a co-Poisson Hopf algebra with Poisson co-bracket $\overline{\delta}$.

Corollary 4. *Let $(\mathfrak{g}, \delta)$ be a finite dimensional Lie bialgebra. Then the Hopf dual $U(\mathfrak{g})^\circ$ of the universal enveloping algebra $U(\mathfrak{g})$ is a Poisson Hopf algebra with Poisson bracket*

$$\{f, g\}(x) = \langle \overline{\delta}(x), f \otimes g \rangle, \qquad x \in U(\mathfrak{g})$$

for $f, g \in U(\mathfrak{g})^\circ$.

Proof. Let $\{x_1, \ldots, x_n\}$ be a basis of $\mathfrak{g}$. Then $U(\mathfrak{g})$ is an almost normalizing extension over $\mathbf{k}$ with generators $x_1, \ldots, x_n$. Thus the result follows immediately from Theorem 3. □

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