

ON THE $2k$ -TH POWER MEAN VALUE OF THE GENERALIZED QUADRATIC GAUSS SUMS

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ABSTRACT. The main purpose of this paper is using the elementary and analytic methods to study the properties of the $2k$ -th power mean value of the generalized quadratic Gauss sums, and give two exact mean value formulae for $k = 3$ and 4 .

1. Introduction

Let $q \geq 2$ be an integer, χ denotes a Dirichlet character modulo q . For any integer n , we define the generalized quadratic Gauss sums $G(n, \chi; q)$ as follows:

$$G(n, \chi; q) = \sum_{a=1}^q \chi(a) e\left(\frac{na^2}{q}\right),$$

where $e(y) = e^{2\pi iy}$. This sum is important, because it is a generalization of the classical quadratic Gauss sums $G(n, q)$, which is defined by

$$G(n; q) = \sum_{a=1}^q e\left(\frac{na^2}{q}\right).$$

About the properties of $G(n, \chi; q)$, some authors had studied it, and obtained many interesting results. For example, for any integer n with $(n, q) = 1$, from the general result of Cochrane and Zheng [2] we can deduce that

$$|G(n, \chi; q)| \leq 2^{\omega(q)} q^{\frac{1}{2}},$$

where $\omega(q)$ denotes the number of all distinct prime divisors of q . The case where q is a prime is due to Weil [4]. Zhang [5] proved that for any odd prime p and integer n with $(n, p) = 1$, we have

$$\sum_{\chi \bmod p} |G(n, \chi; p)|^4 = \begin{cases} (p-1)[3p^2 - 6p - 1 + 4\left(\frac{n}{p}\right)\sqrt{p}], & \text{if } p \equiv 1 \pmod{4}; \\ (p-1)(3p^2 - 6p - 1), & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

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and

$$\sum_{\chi \bmod p} |G(n, \chi; p)|^6 = (p-1)(10p^3 - 25p^2 - 4p - 1), \text{ if } p \equiv 3 \pmod{4},$$

where $\left(\frac{n}{p}\right)$ is the Legendre symbol.

Besides, W. Zhang and H. Liu [6] also proved the following conclusion:

Let $q \geq 3$ be a square-full number. Then for any integers n, k with $(nk, q) = 1$ and $k \geq 1$, we have the identity

$$\sum_{\chi \bmod q} |G(n, \chi; q)|^4 = q \cdot \phi^2(q) \prod_{p|q} (k, p-1)^2 \prod_{\substack{p|q \\ (k, p-1)=1}} \frac{\phi(p-1)}{p-1},$$

where $\prod_{p|q}$ denotes the product over all prime divisors of q , $\phi(q)$ is the Euler function.

In this paper, we use the elementary and analytic methods to study the calculating problem of the $2k$ -th power mean value of the generalized quadratic Gauss sums, and give two exact calculating formulae for $k = 3$ and 4. That is, we shall prove the following:

Theorem 1. *Let odd number $q > 1$ be a square-full number (i.e., for any prime p , $p | q$ if and only if $p^2 | q$). Then for any integer n with $(n, q) = 1$, we have the identity*

$$\sum_{\chi \bmod q} |G(n, \chi; q)|^6 = 16^{\omega(q)} \cdot q^2 \cdot \phi^2(q),$$

where $\omega(q)$ denotes the number of all distinct prime divisors of q .

Theorem 2. *Let odd number $q > 1$ be a square-full number. Then for any integer n with $(n, q) = 1$, we have*

$$\sum_{\chi \bmod q} |G(n, \chi; q)|^8 = 64^{\omega(q)} \cdot q^3 \cdot \phi^2(q).$$

From our theorems we know that the estimates in reference [2] is the best one. In fact from Theorem 2 we know that there exists at least one Dirichlet character modulo q such that the inequality:

$$|G(n, \chi; q)| \geq 2^{\frac{3}{4}\omega(q)} q^{\frac{3}{8}} \phi^{\frac{1}{8}}(q).$$

For general integer $k \geq 5$, we believe that the following conclusion is correct:

Conjecture. *Let odd number $q > 1$ be a square-full number, $k \geq 2$ be an integer. Then for any integer n with $(n, q) = 1$, we have the identity*

$$\sum_{\chi \bmod q} |G(n, \chi; q)|^{2k} = 4^{(k-1)\omega(q)} \cdot q^{k-1} \cdot \phi^2(q).$$

The proposed method is supposed to be capable of proving this formula. However, the calculation will be so complex when $k \geq 5$ that such a general

conclusion cannot be obtained. For general positive integer $q > 3$, it is an open problem whether there is a formula to calculate the $2k$ -th power mean value of the generalized quadratic Gauss sums.

2. Several lemmas

To complete the proof of our theorems, we need the following several lemmas.

Lemma 1. *For any integer $q \geq 1$, we have the identity*

$$G(1; q) = \frac{1}{2}\sqrt{q}(1+i) \left(1 + e^{\frac{-\pi iq}{2}}\right) = \begin{cases} \sqrt{q}, & \text{if } q \equiv 1 \pmod{4}; \\ 0, & \text{if } q \equiv 2 \pmod{4}; \\ i\sqrt{q}, & \text{if } q \equiv 3 \pmod{4}; \\ (1+i)\sqrt{q}, & \text{if } q \equiv 0 \pmod{4}. \end{cases}$$

Proof. This is a remarkable formula of Gauss. See Theorem 9.16 of [1]. \square

Lemma 2. *Let p be an odd prime and $\alpha \geq 2$ be an integer. Then for any integer n with $(p, n) = 1$, we have the identity*

$$\sum_{b=1}^{p^\alpha} e\left(\frac{nb^2}{p^\alpha}\right) = 0.$$

Proof. First we know that for any positive integers $q \geq 2$ and integer n with $(n, q) = 1$, we have the identity

$$\sum_{u=0}^{q-1} e\left(\frac{un}{q}\right) = 0.$$

From this identity and the properties of reduce residue system we have

$$\begin{aligned} \sum_{b=1}^{p^\alpha} e\left(\frac{nb^2}{p^\alpha}\right) &= \sum_{u=0}^{p-1} \sum_{v=1}^{p^{\alpha-1}} e\left(\frac{n(up^{\alpha-1} + v)^2}{p^\alpha}\right) = \sum_{u=0}^{p-1} \sum_{v=1}^{p^{\alpha-1}} e\left(\frac{2nuvp^{\alpha-1} + v^2}{p^\alpha}\right) \\ &= \sum_{v=1}^{p^{\alpha-1}} e\left(\frac{v^2}{p^\alpha}\right) \sum_{u=0}^{p-1} e\left(\frac{2nuv}{p}\right) = 0. \end{aligned}$$

This proves Lemma 2. \square

Lemma 3. *Let $m, n \geq 2$ and u be three integers with $(m, n) = 1$ and $(u, mn) = 1$. Then for any character $\chi = \chi_1\chi_2$ with χ_1 mod m and χ_2 mod n , we have the identity*

$$G(u, \chi; mn) = \chi_1(n)\chi_2(m)G(un, \chi_1; m)G(um, \chi_2; n).$$

Proof. See Lemma 6 of [6]. \square

Lemma 4. *Let p be an odd prime, $\alpha \geq 2$ and n be two integers with $(n, p) = 1$. Then we have*

$$\sum_{\chi \bmod p^\alpha} |G(n, \chi; p^\alpha)|^6 = 16\phi^2(p^\alpha)p^{2\alpha}.$$

Proof. From the definition of $G(n, \chi; p^\alpha)$ we have

$$\begin{aligned} |G(n, \chi; p^\alpha)|^2 &= \sum_{a=1}^{p^\alpha} \sum_{b=1}^{p^\alpha} \chi(a) \overline{\chi}(b) e\left(\frac{n(a^2 - b^2)}{p^\alpha}\right) \\ &= \sum_{a=1}^{p^\alpha} \chi(a) \sum_{b=1}^{p^\alpha} e\left(\frac{nb^2(a^2 - 1)}{p^\alpha}\right). \end{aligned}$$

Then by this formula and the orthogonality relation for character sums modulo p^α we may get

$$\begin{aligned} &\sum_{\chi \bmod p^\alpha} |G(n, \chi; p^\alpha)|^6 \\ &= \phi(p^\alpha) \sum_{\substack{a=1 \\ abc \equiv 1 \pmod{p^\alpha}}}^{p^\alpha} \sum_{b=1}^{p^\alpha} \sum_{c=1}^{p^\alpha} \left(\sum_{u=1}^{p^\alpha} e\left(\frac{nu^2(a^2 - 1)}{p^\alpha}\right) \right) \\ (1) \quad &\times \left(\sum_{v=1}^{p^\alpha} e\left(\frac{nv^2(b^2 - 1)}{p^\alpha}\right) \right) \left(\sum_{w=1}^{p^\alpha} e\left(\frac{nw^2(c^2 - 1)}{p^\alpha}\right) \right). \end{aligned}$$

Let $(a^2 - 1, p^\alpha) = p^m$. If $m \leq \alpha - 2$, note that $(n(a^2 - 1)/p^m, p) = 1$, then from Lemma 2 we have

$$\sum_{u=1}^{p^\alpha} e\left(\frac{nu^2(a^2 - 1)}{p^\alpha}\right) = p^m \sum_{u=1}^{p^{\alpha-m}} e\left(\frac{nu^2(a^2 - 1)/p^m}{p^{\alpha-m}}\right) = 0.$$

If $m = \alpha$, then

$$\sum_{u=1}^{p^\alpha} e\left(\frac{nu^2(a^2 - 1)}{p^\alpha}\right) = \phi(p^\alpha).$$

If $m = \alpha - 1$, then $a = rp^{\alpha-1} \pm 1$, $1 \leq r \leq p - 1$. Note that for any prime p with $p \nmid n$, by Theorem 7.5.4 of [3] we have

$$(2) \quad G(n; p) = \left(\frac{n}{p}\right) G(1; p).$$

Then from (2) and Lemma 1 we may get

$$\begin{aligned} \sum_{u=1}^{p^\alpha} e\left(\frac{nu^2(a^2 - 1)}{p^\alpha}\right) &= p^{\alpha-1} \sum_{u=1}^p e\left(\frac{nu^2(a^2 - 1)/p^{\alpha-1}}{p}\right) \\ &= p^{\alpha-1} \left[\left(\frac{\pm 2rn}{p}\right) G(1; p) - 1 \right]. \end{aligned}$$

Note that the number of the solutions of the congruent equation $1 \leq a, b, c \leq p^\alpha - 1$ with $p^\alpha \mid a^2 - 1$, $p^\alpha \mid b^2 - 1$, $p^\alpha \mid c^2 - 1$ and $abc \equiv 1 \pmod{p^\alpha}$ are 4, the

number of the solutions of the congruent equation $1 \leq a, b, c \leq p^\alpha - 1$ with $p^\alpha \mid a^2 - 1, p^\alpha \mid b^2 - 1, p^{\alpha-1} \parallel c^2 - 1$ and $abc \equiv 1 \pmod{p^\alpha}$ are 0, and

$$\begin{aligned}
& \sum'_{\substack{a=1 \\ p^\alpha \mid a^2-1}}^{p^\alpha} \sum'_{\substack{b=1 \\ p^{\alpha-1} \parallel b^2-1}}^{p^\alpha} \sum'_{\substack{c=1 \\ p^{\alpha-1} \parallel c^2-1 \\ abc \equiv 1 \pmod{p^\alpha}}}^{p^\alpha} \left(\sum'_{u=1}^{p^\alpha} e\left(\frac{nu^2(a^2-1)}{p^\alpha}\right) \right) \\
& \times \left(\sum'_{v=1}^{p^\alpha} e\left(\frac{nv^2(b^2-1)}{p^\alpha}\right) \right) \left(\sum'_{w=1}^{p^\alpha} e\left(\frac{nw^2(c^2-1)}{p^\alpha}\right) \right) \\
& = \phi(p^\alpha) \sum'_{\substack{a=1 \\ p^\alpha \mid a^2-1}}^{p^\alpha} \sum'_{\substack{b=1 \\ p^{\alpha-1} \parallel b^2-1}}^{p^\alpha} \sum'_{\substack{c=1 \\ p^{\alpha-1} \parallel c^2-1 \\ abc \equiv 1 \pmod{p^\alpha}}}^{p^\alpha} \left(\sum'_{v=1}^{p^\alpha} e\left(\frac{nv^2(b^2-1)}{p^\alpha}\right) \right) \left(\sum'_{w=1}^{p^\alpha} e\left(\frac{nw^2(c^2-1)}{p^\alpha}\right) \right) \\
& = 4\phi(p^\alpha) p^{2(\alpha-1)} \sum_{r=1}^{p-1} \left[\left(\frac{-1}{p}\right) G^2(1; p) + 1 \right] \\
& \quad - 4\phi(p^\alpha) p^{2(\alpha-1)} \sum_{r=1}^{p-1} \left[\left(\frac{-2rn}{p}\right) + \left(\frac{2rn}{p}\right) \right] G(1; p) \\
& = 4\phi(p^\alpha) p^{2(\alpha-1)} (p^2 - 1).
\end{aligned}$$

So combining the above several cases and (1) we have

$$\begin{aligned}
& \sum_{\chi \pmod{p^\alpha}} |G(n, \chi; p^\alpha)|^6 \\
& = 4\phi^4(p^\alpha) + 12\phi^2(p^\alpha) p^{2(\alpha-1)} (p^2 - 1) \\
& \quad + \phi(p^\alpha) p^{3(\alpha-1)} \sum_{\substack{r=1 \\ (rp^{\alpha-1} \pm 1)(sp^{\alpha-1} \pm 1)(tp^{\alpha-1} \pm 1) \equiv 1 \pmod{p^\alpha}}}^{p-1} \sum_{s=1}^{p-1} \sum_{t=1}^{p-1} \left[\left(\frac{\pm 2rn}{p}\right) G(1; p) - 1 \right] \\
& \quad \times \left[\left(\frac{\pm 2sn}{p}\right) G(1; p) - 1 \right] \left[\left(\frac{\pm 2tn}{p}\right) G(1; p) - 1 \right] \\
& = 4\phi^4(p^\alpha) + 12\phi^2(p^\alpha) p^{2(\alpha-1)} (p^2 - 1) \\
& \quad + \phi(p^\alpha) p^{3(\alpha-1)} \sum_{\substack{r=1 \\ r+s+t \equiv 0 \pmod{p}}}^{p-1} \sum_{s=1}^{p-1} \sum_{t=1}^{p-1} \left[\left(\frac{2rn}{p}\right) G(1; p) - 1 \right] \\
& \quad \times \left[\left(\frac{2sn}{p}\right) G(1; p) - 1 \right] \left[\left(\frac{2tn}{p}\right) G(1; p) - 1 \right] \\
& \quad + 3\phi(p^\alpha) p^{3(\alpha-1)} \sum_{\substack{r=1 \\ s+t \equiv r \pmod{p}}}^{p-1} \sum_{s=1}^{p-1} \sum_{t=1}^{p-1} \left[\left(\frac{2rn}{p}\right) G(1; p) - 1 \right]
\end{aligned}$$

$$\begin{aligned}
& \times \left[\left(\frac{-2sn}{p} \right) G(1; p) - 1 \right] \left[\left(\frac{-2tn}{p} \right) G(1; p) - 1 \right] \\
& = 4\phi^4(p^\alpha) + 12\phi^2(p^\alpha)p^{2(\alpha-1)}(p^2 - 1) \\
& \quad + 4\phi(p^\alpha)p^{3(\alpha-1)} \sum_{\substack{r=1 \\ r+s+t \equiv 0 \pmod p}}^{p-1} \sum_{s=1}^{p-1} \sum_{t=1}^{p-1} \left[\left(\frac{2rn}{p} \right) G(1; p) - 1 \right] \\
(3) \quad & \times \left[\left(\frac{2sn}{p} \right) G(1; p) - 1 \right] \left[\left(\frac{2tn}{p} \right) G(1; p) - 1 \right].
\end{aligned}$$

From the properties of the Legendre symbol (see reference [1]) we know that

$$\begin{aligned}
& \sum_{\substack{r=1 \\ r+s \pm t \equiv 0 \pmod p}}^{p-1} \sum_{s=1}^{p-1} \sum_{t=1}^{p-1} \left(\frac{r}{p} \right) \left(\frac{s}{p} \right) \left(\frac{t}{p} \right) = \sum_{r=1}^{p-1} \sum_{s=1}^{p-1} \left(\frac{\mp rs(r+s)}{p} \right) \\
(4) \quad & = \sum_{r=1}^{p-1} \sum_{s=1}^{p-1} \left(\frac{\mp rs^2(rs+s)}{p} \right) = \sum_{r=1}^{p-1} \left(\frac{\mp r(r+1)}{p} \right) \sum_{s=1}^{p-1} \left(\frac{s}{p} \right) = 0,
\end{aligned}$$

$$\begin{aligned}
& \sum_{\substack{r=1 \\ r+s \pm t \equiv 0 \pmod p}}^{p-1} \sum_{s=1}^{p-1} \sum_{t=1}^{p-1} \left(\frac{r}{p} \right) \left(\frac{s}{p} \right) = \sum_{r=1}^{p-1} \sum_{s=1}^{p-1} \left(\frac{rs}{p} \right) - \sum_{\substack{r=1 \\ r+s \equiv 0 \pmod p}}^{p-1} \sum_{s=1}^{p-1} \left(\frac{rs}{p} \right) \\
(5) \quad & = - \sum_{r=1}^{p-1} \left(\frac{-r^2}{p} \right) = - \left(\frac{-1}{p} \right) (p-1),
\end{aligned}$$

$$\begin{aligned}
(6) \quad & \sum_{\substack{r=1 \\ r+s \pm t \equiv 0 \pmod p}}^{p-1} \sum_{s=1}^{p-1} \sum_{t=1}^{p-1} \left(\frac{r}{p} \right) = \sum_{r=1}^{p-1} \sum_{s=1}^{p-1} \left(\frac{r}{p} \right) - \sum_{\substack{r=1 \\ r+s \equiv 0 \pmod p}}^{p-1} \sum_{s=1}^{p-1} \left(\frac{r}{p} \right) = 0,
\end{aligned}$$

$$\begin{aligned}
(7) \quad & \sum_{\substack{r=1 \\ r+s \pm t \equiv 0 \pmod p}}^{p-1} \sum_{s=1}^{p-1} \sum_{t=1}^{p-1} 1 = \sum_{r=1}^{p-1} \sum_{s=1}^{p-1} 1 - \sum_{\substack{r=1 \\ r+s \equiv 0 \pmod p}}^{p-1} \sum_{s=1}^{p-1} 1 = (p-1)(p-2).
\end{aligned}$$

Note that $\left(\frac{-1}{p}\right)G^2(1; p) = p$, from (3), (4), (5), (6) and (7) we may get

$$\begin{aligned}
& \sum_{\chi \pmod{p^\alpha}} |G(n, \chi; p^\alpha)|^6 = 4\phi^4(p^\alpha) + 12\phi^2(p^\alpha)p^{2(\alpha-1)}(p^2 - 1) \\
& \quad + 12\phi^2(p^\alpha)p^{2\alpha-1} - 4\phi^2(p^\alpha)p^{2(\alpha-1)}(p-2) \\
& = 16\phi^2(p^\alpha)p^{2\alpha}.
\end{aligned}$$

This proves Lemma 4. □

3. Proof of theorems

In this section, we shall complete the proof of our theorems. We only prove Theorem 1. Similarly, we can deduce Theorem 2. In fact if q is an odd square-full number, let $q = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be the factorization of q into prime powers, then $\alpha_i \geq 2$, $i = 1, 2, \dots, k$. Then for any integer n with $(n, q) = 1$, from Lemma 3, Lemma 4 and the properties of Dirichlet character we have

$$\begin{aligned} \sum_{\chi \bmod q} |G(n, \chi; q)|^6 &= \prod_{p^\alpha \parallel q} \left[\sum_{\chi \bmod p^\alpha} |G(nq/p^\alpha, \chi; p^\alpha)|^6 \right] \\ &= \prod_{p^\alpha \parallel q} [8\phi^2(p^\alpha) p^{2\alpha}] \\ &= 8^{\omega(q)} \cdot q^2 \cdot \phi^2(q), \end{aligned}$$

where $\prod_{p^\alpha \parallel q}$ denotes that $p^\alpha \mid q$ and $p^{\alpha+1} \nmid q$.

This completes the proof of Theorem 1.

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