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# APPROXIMATE CONTROLLABILITY FOR DIFFERENTIAL EQUATIONS WITH QUASI-AUTONOMOUS OPERATORS

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Abstract. The approximate controllability for the nonlinear control system with nonlinear monotone hemicontinuous and coercive operator is studied. The existence, uniqueness and a variation of solutions of the system are also given.

## 1. Introduction

Let $H$ and $V$ be two real separable Hilbert spaces such that $V$ is a dense subspace of $H$. We are interested in the approximate controllability for the following nonlinear functional control system on $H$:

$$\text{(E)} \qquad \begin{cases} \dfrac{dx(t)}{dt} + Ax(t) \ni (Bu)(t), \quad 0 < t \leq T, \\ x(0) = x_0. \end{cases}$$

Assume that $A$ is a monotone hemicontinuous operator from $V$ to $V^*$ and satisfies the coercive condition. Here $V^*$ stands for the dual space of $V$. Let $U$ be a Banach space and the controller operator $B$ be a bounded linear operator from the Banach space $L^2(0,T;U)$ to $L^2(0,T;H)$. If $Bu \in L^2(0,T;V^*)$, it is well known as the quasi-autonomous differential equation (see Theorem 2.6 of Chapter III in Barbu [5]). In [5], the existence and the norm estimate of a solution of the above equation on $L^2(0,T;V) \cap W^{1,2}(0,T;V^*)$ was given, and results similar to this case were obtained by many authors (see bibliographical notes of [5, 6, 7, 10, 11]), which is also applicable to optimal control problem.

The optimal control problems for a class of systems governed by a class of nonlinear evolution equations with nonlinear operator $A$ have been studied in references by Ahmed, Teo and Xiang [1, 2, 3]. The condition equivalent to the approximate controllability for semilinear control system have been obtained in by Naito [9] and Zhou [11]. As for the semilinear control system with the linear operator $A$ generated $C_0$-semigroup, Naito [9] proved the approximate

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controllability under the range conditions of the controller $B$. The papers treating the controllability for systems with nonlinear principal operator $A$ are not many.

In the present article, we will prove the approximately controllable for (E) under a rather applicable assumption on the range of the control operator $B$, namely that $\{y : y(t) = Bu(t),\ u \in L^2(0,T;U)\}$ is dense subspace of $L^2(0,T,H)$, which is reasonable and widely used in case of the nonlinear system (refer to [11, 9, 8]).

## 2. Quasi-autonomous differential equations

If $H$ is identified with its dual space we may write $V \subset H \subset V^*$ densely and the corresponding injections are continuous. The norm on $V$, $H$ and $V^*$ will be denoted by $||\cdot||$, $|\cdot|$ and $||\cdot||_*$, respectively. Thus, in terms of the intermediate theory we may assume that

$$(V, V^*)_{\frac{1}{2},2} = H,$$

where $(V, V^*)_{\frac{1}{2},2}$ denotes the real interpolation space between $V$ and $V^*$. The duality pairing between the element $v_1$ of $V^*$ and the element $v_2$ of $V$ is denoted by $(v_1, v_2)$, which is the ordinary inner product in $H$ if $v_1, v_2 \in H$. For the sake of simplicity, we may consider

$$||u||_* \le |u| \le ||u||, \quad u \in V.$$

We note that a nonlinear operator $A$ is said to be *hemicontinuous* on $V$ if

$$w\text{-}\lim_{t\to 0} A(x+ty) = Ax$$

for every $x,\ y \in V$ where "$w$-lim" indicates the weak convergence on $V$.

Let $A : V \longrightarrow V^*$ be given as a monotone operator and hemicontinuous from $V$ to $V^*$ such that

$$\text{(F)} \qquad \begin{cases} A(0) = 0, \\ (Au - Av, u - v) \ge \omega_1 ||u - v||^2 - \omega_2 |u - v|^2, \\ ||Au||_* \le \omega_3(||u|| + 1) \end{cases}$$

for every $u, v \in V$, where $\omega_2$ is a real number and $\omega_1$, $\omega_3$ are some positive constants.

Here, we note that if $0 \ne A(0)$ we need the following assumption

$$(Au, u) \ge \omega_1 ||u||^2 - \omega_2 |u|^2$$

for every $u \in V$. It is also known that $A$ is maximal monotone and $R(A) = V^*$, where $R(A)$ denotes the range of $A$.

Let $h \in L^2(0,T;V^*)$ and $x$ be the solution of the following quasi-autonomous differential equation with $B = I$:

$$\text{(1)} \qquad \begin{cases} \frac{dx(t)}{dt} + Ax(t) \ni h(t), \quad 0 < t \le T, \\ x(0) = x_0, \end{cases}$$

where $A$ is given satisfying the hypotheses mentioned above. The following result is from Theorem 2.6 of Chapter III in [5].

**Proposition 2.1.** *Let* $x_0 \in H$ *and* $h \in L^2(0,T;V^*)$*. Then there exists a unique solution* $x$ *of* (2.1) *belonging to*

$$C([0,T];H) \cap L^2(0,T;H) \cap W^{1,2}(0,T;V^*)$$

*and satisfying*

$$|x(t)|^2 + \int_0^t ||x(s)||^2 ds \le C_1(|x_0|^2 + \int_0^t ||h(s)||_*^2 ds + 1), \tag{2}$$

$$\int_0^t ||\frac{dx(s)}{ds}||_*^2 dt \le C_1(|x_0|^2 + \int_0^t ||h(s)||_*^2 ds + 1), \tag{3}$$

*where* $C_1$ *is a constant.*

**Lemma 2.2.** *Let* $x_h$ *and* $x_k$ *be the solutions of* (1) *corresponding to* $h$ *and* $k$ *in* $L^2(0,T;V^*)$*. Then we have that*

$$\begin{aligned} &\frac{1}{2}|x_h(t) - x_k(t)|^2 + \omega_1 \int_0^t ||x_h(s) - x_k(s)||^2 ds \\ &\le \int_0^t e^{2\omega_2(t-s)} ||x_h(s) - x_k(s)|| \, ||h(s) - k(s)||_* ds, \end{aligned} \tag{4}$$

*and*

$$\begin{aligned} &\frac{1}{2}|x_h(t)|^2 + \omega_1 \int_0^t ||x_h(s)||^2 ds \\ &\le \frac{e^{2\omega_2 t}}{2}|x_0|^2 + \int_0^t e^{2\omega_2(t-s)} ||x_h(s)|| \, ||h(s)||_* ds. \end{aligned} \tag{5}$$

*Proof.* In order to prove (5), taking scalar product on both sides of (1) by $x(t)$,

$$\frac{1}{2}\frac{d}{dt}|x_h(t)|^2 + \omega_1 ||x_h(t)||^2 \le \omega_2 |x_h(t)|^2 + ||x_h(t)|| \, ||h(t)||_*.$$

Integrating on $[0,t]$, we get

$$\begin{aligned} &\frac{1}{2}|x_h(t)|^2 + \omega_1 \int_0^t ||x_h(s)||^2 ds \\ &\le \frac{1}{2}|x_0|^2 + \omega_2 \int_0^t |x_h(s)|^2 ds + \int_0^t ||x_h(s)|| \, ||h(s)||_* ds. \end{aligned} \tag{6}$$

From (6) it follows that

$$\begin{aligned} \frac{d}{dt}\{e^{-2\omega_2 t} \int_0^t |x_h(s)|^2 ds\} &= 2e^{-2\omega_2 t}\{\frac{1}{2}|x_h(t)|^2 - \omega_2 \int_0^t |x_h(s)|^2 ds\} \\ &\le 2e^{-2\omega_2 t}\{\frac{1}{2}|x_0|^2 + \int_0^t ||x_h(s)|| \, ||h(s)||_* ds\}. \end{aligned} \tag{7}$$

Integrating (7) over $(0, t)$ we have

$$\begin{aligned}
&e^{-2\omega_2 t} \int_0^t |x_h(s)|^2 ds \\
&\le 2 \int_0^t e^{-2\omega_2 \tau} \int_0^\tau ||x_h(s)||\, ||h(s)||_* ds d\tau + \frac{1 - e^{-2\omega_2 t}}{2\omega_2} |x_0|^2 \\
&= 2 \int_0^t \int_s^t e^{-2\omega_2 \tau} d\tau ||x_h(s)||\, ||h(s)||_* ds + \frac{1 - e^{-2\omega_2 t}}{2\omega_2} |x_0|^2 \\
&= 2 \int_0^t \frac{e^{-2\omega_2 s} - e^{-2\omega_2 t}}{2\omega_2} ||x_h(s)||\, ||h(s)||_* ds + \frac{1 - e^{-2\omega_2 t}}{2\omega_2} |x_0|^2 \\
&= \frac{1}{\omega_2} \int_0^t (e^{-2\omega_2 s} - e^{-2\omega_2 t}) ||x_h(s)||\, ||h(s)||_* ds + \frac{1 - e^{-2\omega_2 t}}{2\omega_2} |x_0|^2,
\end{aligned}$$

and hence,

$$(8) \quad \omega_2 \int_0^t |x_h(s)|^2 ds \le \int_0^t (e^{2\omega_2 (t-s)} - 1) ||x_h(s)||\, ||h(s)||_* ds + \frac{e^{2\omega_2 t} - 1}{2} |x_0|^2.$$

Combining (6) with (8) it follows that

$$\frac{1}{2} |x_h(t)|^2 + \omega_1 \int_0^t ||x_h(s)||^2 ds \le \frac{e^{2\omega_2 t}}{2} |x_0|^2 + \int_0^t e^{2\omega_2 (t-s)} ||x_h(s)||\, ||h(s)||_* ds.$$

We also obtain (4) by the similar argument in the proof of (5). □

**Theorem 2.3.** *If* $(x_0, h) \in H \times L^2(0, T; V^*)$*, then* $x \in L^2(0, T; V) \cap C([0, T]; H)$ *and the mapping*

$$H \times L^2(0, T; V^*) \ni (x_0, h) \mapsto x \in L^2(0, T; V) \cap C([0, T]; H)$$

*is continuous.*

*Proof.* By virtue of Proposition 2.1 for any $(x_0, h) \in H \times L^2(0, T; V^*)$, the solution $x$ of (1) belongs to $L^2(0, T; V) \cap C([0, T]; H)$. Let $(x_{0i}, h_i) \in H \times L^2(0, T; V^*)$ and $x_i$ be the solution of (1) with $(x_{0i}, h_i)$ instead of $(x_0, h)$ for $i = 1,\ 2$. Multiplying on (1) by $x_1(t) - x_2(t)$, we have

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} |x_1(t) - x_2(t)|^2 + \omega_1 ||x_1(t) - x_2(t)||^2 \\
&\le \omega_2 |x_1(t) - x_2(t)|^2 + ||x_1(t) - x_2(t)||\, ||h_1(t) - h_2(t)||_*.
\end{aligned}$$

By the similar process of the proof of (5) it holds

$$\begin{aligned}
&\frac{1}{2} |x_1(t) - x_2(t)|^2 + \omega_1 \int_0^t ||x_1(s) - x_2(s)||^2 ds \\
&\le \frac{e^{2\omega_2 t}}{2} |x_{01} - x_{02}|^2 + \int_0^t e^{2\omega_2 (t-s)} ||x_1(s) - x_2(s)||\, ||h_1(s) - h_2(s)||_* ds.
\end{aligned}$$

We can choose a constant $c > 0$ such that

$$\omega_1 - e^{2\omega_2 T} \frac{c}{2} > 0$$

and, hence

$$\int_0^T e^{2\omega_2(t-s)}||x_1(s)-x_2(s)||\,||h_1(s)-h_2(s)||_* ds$$
$$\le e^{2\omega_2 T}\int_0^T \{\frac{c}{2}||x_1(s)-x_2(s)||^2+\frac{1}{2c}||h_1(s)-h_2(s)||_*^2\}ds.$$

Thus, there exists a constant $C>0$ such that

$$||x_1-x_2||_{L^2(0,T,V)\cap C([0,T];H)} \le C(|x_{01}-x_{02}|+||h_1-h_2||_{L^2(0,T;V^*)}). \tag{9}$$

Suppose $(x_{0n},h_n)\to(x_0,h)$ in $H\times L^2(0,T;V^*)$, and let $x_n$ and $x$ be the solutions (E) with $(x_{0n},h_n)$ and $(x_0,h)$, respectively. Then, by virtue of (9), we see that $x_n\to x$ in $L^2(0,T,V)\cap C([0,T];H)$. □

## 3. Approximate controllability

In what follows we assume that the embedding $V\subset H$ is compact. Let $x_h$ be the solution of (1) corresponding to $h$ in $L^2(0,T;V^*)$. We define the solution mapping $S$ from $L^2(0,T;V^*)$ to $L^2(0,T;V)$ by

$$(Sh)(t)=x_h(t),\quad h\in L^2(0,T;V^*).$$

Let $\mathcal{A}$ be the Nemitsky operator corresponding to the map $A$, which is defined by $\mathcal{A}(x)(\cdot)=Ax(\cdot)$. Then

$$x_h(t)=\int_0^t((I-\mathcal{A}S)h)(s)ds,$$

and with the aid of Proposition 2.1

$$||Sh||_{L^2(0,T;V)\cap W^{1,2}(0,T;V^*)}=||x_h||_{L^2(0,T;V)\cap W^{1,2}(0,T;V^*)} \le C_1(|x_0|+||h||_{L^2(0,T;V^*)}+1). \tag{10}$$

Hence if $h$ is bounded in $L^2(0,T;V^*)$, then so is $x_h$ in $L^2(0,T;V)\cap W^{1,2}(0,T;V^*)$. Since $V$ is compactly embedded in $H$ by assumption, the embedding $L^2(0,T;V)\cap W^{1,2}(0,T;V^*)\subset L^2(0,T;H)$ is compact in view of Theorem 2 of Aubin [4]. Hence, since the embedding $L^2(0,T;H)\subset L^2(0,T;V^*)$ is continuous, the mapping $h\mapsto Sh=x_h$ is compact from $L^2(0,T;V^*)$ to itself.

The solution of (E) is denoted by $x(T;u)$ associated with the control $u$ at time $T$. The system (E) is said to be *approximately controllable* at time $T$ if $Cl\{x(T;u):u\in L^2(0,T;U)\}=H$, where $Cl$ denotes the closure in $H$.

We assume

**(B)** $Cl\{y:y(t)=(Bu)(t),\quad \text{a.e.}\quad u\in L^2(0,T;U)\}=L^2(0,T;H)$,

where $Cl$ denotes also the closure in $L^2(0,T;H)$.

The main results of this paper is the following:

**Theorem 3.1.** *Let the assumption* (B) *be satisfied. If our constants condition in* (F) *contains the following inequality*: $\omega_3 < \omega_1$, *then*

$$Cl\{(I-\mathcal{A}S)h : h \in L^2(0,T;V^*)\} = L^2(0,T;V^*). \tag{11}$$

*Therefore, the nonlinear differential control system* (E) *is approximately controllable at time* $T$.

*Proof.* Let us fix $T_0 > 0$ so that

$$N = \omega_1^{-1}\omega_3 e^{\omega_2 T_0} < 1. \tag{12}$$

Let $z \in L^2(0,T_0;V^*)$ and $r$ be a constant such that

$$z \in U_r = \{x \in L^2(0,T_0;V^*) : ||x||_{L^2(0,T_0;V^*)} < r\}.$$

Take a constant $d > 0$ such that

$$(r + \omega_3 + \omega_3\omega_1^{-1/2} e^{\omega_2 T_0}|x_0|)(1-N)^{-1} < d. \tag{13}$$

(5) in Lemma 2.2 implies

$$\omega_1||x_h||^2_{L^2(0,T_0;V)} \le \frac{e^{2\omega_2 T_0}}{2}|x_0|^2 + \frac{\omega_1}{2}||x_h||^2_{L^2(0,T_0;V)} + \frac{e^{2\omega_2 T_0}}{2\omega_1}||h||^2_{L^2(0,T_0;V^*)},$$

that is,

$$\begin{aligned} ||Sh||_{L^2(0,T_0;V)} &= ||x_h||_{L^2(0,T_0;V)} \\ &\le e^{\omega_2 T_0}(\omega_1^{-1/2}|x_0| + \omega_1^{-1}||h||_{L^2(0,T_0;V^*)}). \end{aligned} \tag{14}$$

Let us consider the equation

$$z = (I - \lambda\mathcal{A}S)h, \quad 0 \le \lambda \le 1. \tag{15}$$

Let $h$ be the solution of (14). Then, for the element $z \in U_r$, from (13) and (14), it follows that

$$\begin{aligned} ||h||_{L^2(0,T_0;V^*)} &\le ||z|| + ||\mathcal{A}Sh|| \le r + \omega_3(||Sh|| + 1) \\ &\le r + \omega_3\{e^{\omega_2 T_0}(\omega_1^{-1/2}|x_0| + \omega_1^{-1}||h||_{L^2(0,T_0;V^*)}) + 1\}, \end{aligned}$$

and hence

$$\begin{aligned} ||h|| &\le (r + \omega_3 + \omega_1^{-1/2}\omega_3 e^{\omega_2 T_0}|x_0|)(1-N)^{-1} \\ &< d \end{aligned}$$

it follows that $h \notin \partial U_d$, where $\partial U_d$ stands for the boundary of $U_d$. Thus the homotopy property of topological degree theory there exists $h \in U_d$ such that the equation

$$z = (I - \mathcal{A}S)h$$

holds. Since the assumption (B), there exists a sequence $\{u_n\} \in L^2(0,T_0;U)$ such that $Bu_n \mapsto h$ in $L^2(0,T_0;V^*)$. Then by Theorem 2.3 we have that $x(\cdot;u_n) \mapsto x_h$ in $L^2(0,T_0;V) \cap C([0,T_0];H)$. Let $y \in H$. We can choose $g \in W^{1,2}(0,T_0;V^*)$ such that $g(0) = x_0$ and $g(T_0) = y$ and from the equation

(15) there is $h \in L^2(0, T_0; V^*)$ such that $g' = (I - \mathcal{A}S)h$. By the assumption (B) there exists $u \in L^2(0, T_0; U)$ such that

$$||h - Bu||_{L^2(0,T_0;V^*)} \leq \frac{\sqrt{2}\omega_1}{e^{\omega_2 T_0}}\epsilon$$

for every $\epsilon > 0$. From (4)

$$\begin{aligned}
&\frac{1}{2}|x_h(t) - x_{Bu}(t)|^2 + \omega_1 \int_0^t ||x_h(s) - x_{Bu}(s)||^2 ds \\
&\leq \int_0^t e^{2\omega_2(t-s)}||x_h(s) - x_{Bu}(s)|| \, ||h(s) - (Bu)(s)||_* ds \\
&\leq \omega_1 \int_0^t ||x_h(s) - x_{Bu}(s)||^2 ds + \frac{e^{2\omega_2 t}}{4\omega_1} \int_0^t ||h(s) - (Bu)(s)||^2 ds,
\end{aligned}$$

it holds

$$||x_h - x_{Bu}||_{C([0,T_0];H)} \leq \frac{e^{\omega_2 T_0}}{\sqrt{2\omega_1}}||h - Bu||_{L^2(0,T_0;V^*)},$$

thus, we have

$$\begin{aligned}
|y - x_h(T)| &= |\int_0^{T_0} ((I - \mathcal{A}S)h)(s)ds - \int_0^{T_0} ((I - \mathcal{A}S)Bu)(s)ds| \\
&\leq \frac{e^{\omega_2 T_0}}{\sqrt{2\omega_1}}||h - Bu||_{L^2(0,T_0;V^*)} \leq \epsilon.
\end{aligned}$$

Therefore, the system (E) is approximately controllable at time $T_0$. Since the condition (12) is independent of initial values, we can solve the equation in $[T_0, 2T_0]$ with the initial value $x(T_0)$. By repeating this process, the approximate controllability for (E) can be extended the interval $[0, nT_0]$ for natural number $n$, i.e., for the initial $x(nT_0)$ in the interval $[nT_0, (n+1)T_0]$. □

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