Commun. Korean Math. Soc. **26** (2011), No. 1, pp. 151–161
DOI 10.4134/CKMS.2011.26.1.151

MAXIMAL INEQUALITIES AND STRONG LAW OF LARGE NUMBERS FOR AANA SEQUENCES

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Abstract. Let $\{X_n, n \geq 1\}$ be a sequence of asymptotically almost negatively associated random variables and $S_n = \sum_{i=1}^{n} X_i$. In the paper, we get the precise results of Hájek-Rényi type inequalities for the partial sums of asymptotically almost negatively associated sequence, which generalize and improve the results of Theorem 2.4–Theorem 2.6 in Ko et al. ([4]). In addition, the large deviation of S_n for sequence of asymptotically almost negatively associated random variables is studied. At last, the Marcinkiewicz type strong law of large numbers is given.

1. Introduction

Definition 1.1. A finite collection of random variables $X_1, X_2, \ldots, X_n$ is said to be negatively associated (NA, in short) if for every pair of disjoint subsets A_1, A_2 of {1, 2, ..., n},

$$Cov\{f(X_i : i \in A_1), g(X_j : j \in A_2)\} \leq 0, \tag{1.1}$$

whenever f and g are coordinatewise nondecreasing such that this covariance exists. An infinite sequence $\{X_n, n \geq 1\}$ is NA if every finite subcollection is NA.

Definition 1.2. A sequence $\{X_n, n \geq 1\}$ of random variables is called asymptotically almost negatively associated (AANA, in short) if there exists a nonnegative sequence $q(n) \to 0$ as $n \to \infty$ such that

$$\begin{aligned} &Cov(f(X_n), g(X_{n+1}, X_{n+2}, \ldots, X_{n+k})) \\ &\leq q(n) \left[Var(f(X_n))Var(g(X_{n+1}, X_{n+2}, \ldots, X_{n+k}))\right]^{1/2} \end{aligned}$$

Received September 27, 2009.
2010 *Mathematics Subject Classification.* 60E10, 60F15, 60F10.
Key words and phrases. Hájek-Rényi inequality, asymptotically almost negatively associated sequence, strong law of large numbers, large deviation.
Supported by the NNSF of China (10871001, 60803059), Provincial Natural Science Research Project of Anhui Colleges (KJ2010A005), Talents Youth Fund of Anhui Province Universities (2010SQRL016ZD), Youth Science Research Fund of Anhui University (2009QN011A).

for all $n, k \geq 1$ and for all coordinatewise nondecreasing continuous functions f and g whenever the variances exist.

The family of AANA sequence contains NA and independent sequences as special cases. An AANA sequence of random variables means roughly that asymptotically the future is almost negatively associated with the present. An example of an AANA sequence which is not NA was constructed by Chandra and Ghosal ([1]).

Since the concept of AANA sequence was introduced by Chandra and Ghosal ([1]), many applications have been found. See for example, Chandra and Ghosal ([1]) derived the Kolmogorov type inequality and the strong law of large numbers, Chandra and Ghosal ([2]) obtained the almost sure convergence of weighted averages, Ko et al. ([4]) studied the Hájek-Rényi type inequality, and Wang et al. ([5]) established the law of the iterated logarithm for product sums. Recently, Yuan and An ([6]) established some Rosenthal type inequalities for maximum partial sums of AANA sequences.

The main purpose of the paper is to further study the Hájek-Rényi type inequalities, which generalize and improve the results of Theorem 2.4–Theorem 2.6 in Ko et al. ([4]). In addition, the large deviation and Marcinkiewicz type strong law of large numbers for AANA sequence are studied.

Throughout the paper, let $\{X_n, n \geq 1\}$ be a sequence of AANA random variables defined on a fixed probability space $(\Omega, \mathcal{F}, P)$. Denote $S_n \doteq \sum_{i=1}^{n} X_i$ and $I(A)$ be the indicator function of the set A. For $p > 1$, let $q \doteq p/(p-1)$ be the dual number of p. C denotes a positive constant which may be different in various places.

Lemma 1.1 (cf. Yuan and An, [6, Lemma 2.1])**.** *Let $\{X_n, n \geq 1\}$ be a sequence of AANA random variables with mixing coefficients $\{q(n), n \geq 1\}$, $f_1, f_2, \ldots$ be all nondecreasing (or nonincreasing) functions, then $\{f_n(X_n), n \geq 1\}$ is still a sequence of AANA random variables with mixing coefficients $\{q(n), n \geq 1\}$.*

Lemma 1.2. *Let $1 < p \leq 2$ and $\{X_n, n \geq 1\}$ be a sequence of AANA random variables with mixing coefficients $\{q(n), n \geq 1\}$ and $EX_n = 0$ for each $n \geq 1$. If $\sum_{n=1}^{\infty} q^2(n) < \infty$, then there exists a positive constant C_p depending only on p such that*

$$E\left(\max_{1 \leq i \leq n} |S_i|^p\right) \leq C_p \sum_{i=1}^{n} E|X_i|^p \tag{1.2}$$

for all $n \geq 1$, where $C_p = 2^p \left[2^{2-p}p + (6p)^p \left(\sum_{n=1}^{\infty} q^2(n)\right)^{p/q}\right]$.

We point out that Lemma 1.2 has been studied by Yuan and An ([6]). But here we give the accurate coefficient C_p. And Lemma 1.2 generalizes and improves the result of Lemma 2.2 in Ko et al. ([4]). The following Khintchine-Kolmogorov type convergence theorem is the immediate byproduct of Lemma 1.1 and Lemma 1.2.

Corollary 1.1 (Khintchine-Kolmogorov type convergence theorem). *Let $\{X_n, n \geq 1\}$ be a sequence of AANA random variables with mixing coefficients $\{q(n), n \geq 1\}$ and $\sum_{n=1}^{\infty} q^2(n) < \infty$. Assume that*

$$\sum_{n=1}^{\infty} Var(X_n) < \infty, \tag{1.3}$$

then $\sum_{n=1}^{\infty}(X_n - EX_n)$ converges a.s..

Lemma 1.3 (cf. Fazekas and Klesov, [3, Theorem 1.1]). *Let $\beta_1, \beta_2, \ldots, \beta_n$ be a nondecreasing sequence of positive numbers and $\alpha_1, \alpha_2, \ldots, \alpha_n$ be nonnegative numbers. Let r be a fixed positive number. Assume that for each m with $1 \leq m \leq n$,*

$$E\left(\max_{1\leq l\leq m}\left|\sum_{j=1}^{l} X_j\right|\right)^r \leq \sum_{l=1}^{m} \alpha_l, \tag{1.4}$$

then

$$E\left(\max_{1\leq l\leq n}\left|\frac{\sum_{j=1}^{l} X_j}{\beta_l}\right|\right)^r \leq 4\sum_{l=1}^{n} \frac{\alpha_l}{\beta_l^r}. \tag{1.5}$$

Lemma 1.4 (cf. Yuan and An, [6, Theorem 2.1]). *Let $\{X_n, n \geq 1\}$ be a sequence of AANA random variables with $EX_i = 0$ for all $i \geq 1$ and $p \in (3 \cdot 2^{k-1}, 4 \cdot 2^{k-1}]$, where integer number $k \geq 1$. If $\sum_{n=1}^{\infty} q^{q/p}(n) < \infty$, then there exists a positive constant D_p depending only on p such that for all $n \geq 1$*

$$E\left(\max_{1\leq i\leq n} |S_i|^p\right) \leq D_p \left\{\sum_{i=1}^{n} E|X_i|^p + \left(\sum_{i=1}^{n} EX_i^2\right)^{p/2}\right\}. \tag{1.6}$$

2. Hájek-Rényi type inequalities for AANA sequence

Theorem 2.1. *Let $\{X_n, n \geq 1\}$ be a sequence of AANA random variables with mixing coefficients $\{q(n), n \geq 1\}$ and $\{b_n, n \geq 1\}$ be a nondecreasing sequence of positive numbers. Assume that $EX_n = 0$ for each $n \geq 1$ and $\sum_{n=1}^{\infty} q^2(n) < \infty$. Then for any $\varepsilon > 0$ and any integer $n \geq 1$,*

$$P\left\{\max_{1\leq k\leq n}\left|\frac{1}{b_k}\sum_{j=1}^{k} X_j\right| \geq \varepsilon\right\} \leq \frac{2^p C_p}{\varepsilon^p}\sum_{j=1}^{n} \frac{E|X_j|^p}{b_j^p} \tag{2.1}$$

for all $1 < p \leq 2$, where $C_p = 2^p\left[2^{2-p}p + (6p)^p \left(\sum_{n=1}^{\infty} q^2(n)\right)^{p/q}\right]$.

Proof. Without loss of generality, we assume that $b_0 = 0$ and $\sum_{j=1}^{i-1} \frac{X_j}{b_j} = 0$ when $i = 1$. It is easy to check that

$$S_k = \sum_{j=1}^{k} X_j = \sum_{j=1}^{k}\sum_{i=1}^{j}(b_i - b_{i-1})\frac{X_j}{b_j} = \sum_{i=1}^{k}(b_i - b_{i-1})\sum_{j=i}^{k}\frac{X_j}{b_j}. \tag{2.2}$$

(2.2) and $\frac{1}{b_k}\sum_{i=1}^{k}(b_i - b_{i-1}) = 1$ imply that

$$\left(\left|\frac{S_k}{b_k}\right| \geq \varepsilon\right) \subset \left(\max_{1\leq i\leq k}\left|\sum_{j=i}^{k}\frac{X_j}{b_j}\right| \geq \varepsilon\right). \tag{2.3}$$

Therefore

$$\begin{aligned}\left(\max_{1\leq k\leq n}\left|\frac{S_k}{b_k}\right| \geq \varepsilon\right) &\subset \left(\max_{1\leq k\leq n}\max_{1\leq i\leq k}\left|\sum_{j=i}^{k}\frac{X_j}{b_j}\right| \geq \varepsilon\right)\\ &= \left(\max_{1\leq i\leq k\leq n}\left|\sum_{j=1}^{k}\frac{X_j}{b_j} - \sum_{j=1}^{i-1}\frac{X_j}{b_j}\right| \geq \varepsilon\right)\\ &\subset \left(\max_{1\leq i\leq n}\left|\sum_{j=1}^{i}\frac{X_j}{b_j}\right| \geq \frac{\varepsilon}{2}\right),\end{aligned}$$

which implies that

$$P\left(\max_{1\leq k\leq n}\left|\frac{S_k}{b_k}\right| \geq \varepsilon\right) \leq P\left(\max_{1\leq i\leq n}\left|\sum_{j=1}^{i}\frac{X_j}{b_j}\right| \geq \frac{\varepsilon}{2}\right). \tag{2.4}$$

By Lemma 1.1, we can see that $\{X_n/b_n, n \geq 1\}$ is still a sequence of AANA random variables with mixing coefficients $\{q(n), n \geq 1\}$. Thus, by (2.4), Markov's inequality and Lemma 1.2, we can obtain

$$P\left\{\max_{1\leq k\leq n}\left|\frac{1}{b_k}\sum_{j=1}^{k}X_j\right| \geq \varepsilon\right\} \leq \frac{2^p}{\varepsilon^p}E\left(\max_{1\leq i\leq n}\left|\sum_{j=1}^{i}\frac{X_j}{b_j}\right|^p\right) \leq \frac{2^pC_p}{\varepsilon^p}\sum_{j=1}^{n}\frac{E|X_j|^p}{b_j^p}.$$

The proof of the theorem is complete. □

Theorem 2.2. *Let $\{X_n, n \geq 1\}$ be a sequence of AANA random variables with mixing coefficients $\{q(n), n \geq 1\}$ and $\{b_n, n \geq 1\}$ be a nondecreasing sequence of positive numbers. Assume that $EX_n = 0$ for each $n \geq 1$ and $\sum_{n=1}^{\infty} q^2(n) < \infty$. Then for any $\varepsilon > 0$ and any positive integers $m < n$,*

$$P\left(\max_{m\leq k\leq n}\left|\frac{1}{b_k}\sum_{j=1}^{k}X_j\right| \geq \varepsilon\right) \leq \frac{2^pC_p}{\varepsilon^p}\left(\sum_{j=1}^{m}\frac{E|X_j|^p}{b_m^p} + 2^p\sum_{j=m+1}^{n}\frac{E|X_j|^p}{b_j^p}\right) \tag{2.5}$$

for all $1 < p \leq 2$, where $C_p = 2^p\left[2^{2-p}p + (6p)^p\left(\sum_{n=1}^{\infty} q^2(n)\right)^{p/q}\right]$.

Proof. Observe that

$$\max_{m\leq k\leq n}\left|\frac{1}{b_k}\sum_{j=1}^{k}X_j\right| \leq \left|\frac{1}{b_m}\sum_{j=1}^{m}X_j\right| + \max_{m+1\leq k\leq n}\left|\frac{1}{b_k}\sum_{j=m+1}^{k}X_j\right|,$$

thus

$$
\begin{aligned}
&P\left(\max_{m\le k\le n}\left|\frac{1}{b_k}\sum_{j=1}^{k}X_j\right|\ge\varepsilon\right)\\
&\le P\left(\left|\frac{1}{b_m}\sum_{j=1}^{m}X_j\right|\ge\frac{\varepsilon}{2}\right)+P\left(\max_{m+1\le k\le n}\left|\frac{1}{b_k}\sum_{j=m+1}^{k}X_j\right|\ge\frac{\varepsilon}{2}\right)\\
&\doteq I+II.
\end{aligned}\tag{2.6}
$$

For I, by Markov's inequality and Lemma 1.2, we have

$$
I\le\frac{2^p}{\varepsilon^p b_m^p}E\left|\sum_{j=1}^{m}X_j\right|^p\le\frac{2^pC_p}{\varepsilon^p}\sum_{j=1}^{m}\frac{E|X_j|^p}{b_m^p}.\tag{2.7}
$$

For II, we will apply Theorem 2.1 to $\{X_{m+i},1\le i\le n-m\}$ and $\{b_{m+i},1\le i\le n-m\}$. Noting that

$$
\max_{m+1\le k\le n}\left|\frac{1}{b_k}\sum_{j=m+1}^{k}X_j\right|=\max_{1\le k\le n-m}\left|\frac{1}{b_{m+k}}\sum_{j=1}^{k}X_{m+j}\right|,
$$

thus, by Theorem 2.1, we get

$$
II\le\frac{2^pC_p}{(\varepsilon/2)^p}\sum_{j=1}^{n-m}\frac{E|X_{m+j}|^p}{b_{m+j}^p}=\frac{2^{2p}C_p}{\varepsilon^p}\sum_{j=m+1}^{n}\frac{E|X_j|^p}{b_j^p}.\tag{2.8}
$$

Therefore, the desired result (2.5) follows from (2.6)–(2.8) immediately. □

Theorem 2.3. *Let $\{X_n,n\ge1\}$ be a sequence of AANA random variables with mixing coefficients $\{q(n),n\ge1\}$ and $\sum_{n=1}^{\infty}q^2(n)<\infty$, $\{b_n,n\ge1\}$ be a nondecreasing sequence of positive numbers. Denote $T_n=\sum_{i=1}^{n}(X_i-EX_i)$ for $n\ge1$. Assume that*

$$
\sum_{j=1}^{\infty}\frac{Var(X_j)}{b_j^2}<\infty.\tag{2.9}
$$

Then for any $r\in(0,2)$,

$$
E\left(\sup_{n\ge1}\left|\frac{T_n}{b_n}\right|^r\right)\le1+\frac{4rC_2}{2-r}\sum_{j=1}^{\infty}\frac{Var(X_j)}{b_j^2}<\infty\tag{2.10}
$$

and

$$
E\left(\sup_{n\ge1}\left|\frac{T_n}{b_n}\right|^2\right)\le4C_2\sum_{j=1}^{\infty}\frac{Var(X_j)}{b_j^2}<\infty.\tag{2.11}
$$

Furthermore, if $\lim_{n\to\infty} b_n = +\infty$, *then* $\lim_{n\to\infty} \frac{1}{b_n}\sum_{j=1}^{n}(X_j - EX_j) = 0$ *a.s., where*

$$C_2 = 2^2 \left[2^{2-2} \times 2 + (6\times 2)^2 \left(\sum_{n=1}^{\infty} q^2(n) \right)^{2/2} \right] = 8 + 576 \sum_{n=1}^{\infty} q^2(n) < \infty.$$

Proof. By the continuity of probability and Theorem 2.1, we can see that

$$\begin{aligned} E\left(\sup_{n\ge 1} \left| \frac{T_n}{b_n} \right|^r \right) &= \int_0^1 P\left(\sup_{n\ge 1} \left| \frac{T_n}{b_n} \right|^r > t \right) dt + \int_1^{\infty} P\left(\sup_{n\ge 1} \left| \frac{T_n}{b_n} \right|^r > t \right) dt \\ &\le 1 + \int_1^{\infty} \lim_{N\to\infty} P\left(\max_{1\le n\le N} \left| \frac{T_n}{b_n} \right| > t^{1/r} \right) dt \\ &\le 1 + 4C_2 \sum_{j=1}^{\infty} \frac{Var(X_j)}{b_j^2} \int_1^{\infty} t^{-2/r} dt \\ &= 1 + \frac{4rC_2}{2-r} \sum_{j=1}^{\infty} \frac{Var(X_j)}{b_j^2} < \infty. \end{aligned}$$

So (2.10) is proved. By Lemma 1.2, we have

$$E\left(\max_{1\le i\le n} T_i^2 \right) \le C_2 \sum_{i=1}^{n} E|X_i - EX_i|^2 = C_2 \sum_{i=1}^{n} Var(X_i) \doteq \sum_{j=1}^{n} \alpha_j, \tag{2.12}$$

where $\alpha_j = C_2 Var(X_j) \ge 0,\ j = 1, 2, \ldots, n$. By (2.12) and Lemma 1.3,

$$E\left(\max_{1\le i\le n} \left| \frac{T_i}{b_i} \right|^2 \right) \le 4 \sum_{j=1}^{n} \frac{\alpha_j}{b_j^2} = 4C_2 \sum_{j=1}^{n} \frac{Var(X_j)}{b_j^2}. \tag{2.13}$$

Thus, by monotone convergence theorem and (2.13),

$$\begin{aligned} E\left(\sup_{n\ge 1} \left| \frac{T_n}{b_n} \right|^2 \right) &= E\left[\lim_{n\to\infty} \left(\max_{1\le i\le n} \left| \frac{T_i}{b_i} \right|^2 \right) \right] = \lim_{n\to\infty} E\left(\max_{1\le i\le n} \left| \frac{T_i}{b_i} \right|^2 \right) \\ &\le 4C_2 \sum_{j=1}^{\infty} \frac{Var(X_j)}{b_j^2} < \infty. \end{aligned}$$

This completes the proof of (2.11). Observe that

$$\begin{aligned} P\left(\bigcup_{n=m}^{\infty} \left(\left| \frac{T_n}{b_n} \right| > \varepsilon \right) \right) &= P\left(\bigcup_{N=m}^{\infty} \left(\max_{m\le n\le N} \left| \frac{T_n}{b_n} \right| > \varepsilon \right) \right) \\ &= \lim_{N\to\infty} P\left(\max_{m\le n\le N} \left| \frac{T_n}{b_n} \right| > \varepsilon \right). \end{aligned}$$

By Theorem 2.2 (for $p = 2$) we can obtain that

$$P\left(\max_{m\le n\le N} \left| \frac{T_n}{b_n} \right| > \varepsilon \right) \le \frac{4C_2}{\varepsilon^2} \left(\sum_{j=1}^{m} \frac{Var(X_j)}{b_m^2} + 4 \sum_{j=m+1}^{N} \frac{Var(X_j)}{b_j^2} \right).$$

Hence, by (2.9) and Kronecker's lemma, it follows that

$$\lim_{m\to\infty} P\left(\bigcup_{n=m}^{\infty}\left(\left|\frac{T_n}{b_n}\right| > \varepsilon\right)\right) = 0,\ \forall\, \varepsilon > 0,$$

which is equivalent to $\lim\limits_{n\to\infty} \frac{1}{b_n}\sum_{j=1}^{n}(X_j - EX_j) = 0$ a.s.. The desired results are proved. □

Remark 2.1. Hájek-Rényi type inequalities for AANA sequence have been studied by Ko et al. ([4]). But their results are based on the following conditions

$$\left(\sum_{k=1}^{n}\sigma_k^{M/(M-1)}\right)^{1-1/M} \le D\left(\sum_{k=1}^{n}\sigma_k^2\right)^{1/2} \text{ for some } M > 1,\ D > 0, \tag{2.14}$$

and $EX_k^2 < \infty$, where $\sigma_k^2 = EX_k^2$. Here (Theorem 2.1–Theorem 2.3) we remove the conditions above, and generalize $p = 2$ to the case of $1 < p \le 2$. In addition, we give the accurate coefficient C_p. So our Theorem 2.1–Theorem 2.3 generalize and improve the results of Theorem 2.4–Theorem 2.6 in Ko et al. ([4]), respectively.

3. Large deviations for AANA sequence

In this section, we will study the asymptotic behavior of the probabilities

$$P(S_n > nx),\ x > 0,\ n \to \infty. \tag{3.1}$$

In the following, we let $\|X\|_p = (E|X|^p)^{1/p}$ for some $p > 0$.

Theorem 3.1. *Let $1 < p \le 2$ and $\{X_n, n \ge 1\}$ be a sequence of AANA random variables with $\sum_{n=1}^{\infty} q^2(n) < \infty$ and $EX_i = 0$ for all $i \ge 1$. If there exists a positive constant $M < \infty$ such that $\|X_i\|_p \le M$ for all $i \ge 1$, then for every $x > 0$,*

$$P\left(\max_{1\le i\le n}|S_i| > nx\right) \le \frac{C_pM^p}{x^p}n^{1-p}, \tag{3.2}$$

where C_p is defined in Lemma 1.2.

Proof. By Markov's inequality and Lemma 1.2, we can see that

$$\begin{aligned} P\left(\max_{1\le i\le n}|S_i| > nx\right) &\le \frac{1}{n^px^p}E\left(\max_{1\le i\le n}|S_i|^p\right) \\ &\le \frac{C_p}{n^px^p}\sum_{i=1}^{n}E|X_i|^p \le \frac{C_pM^p}{x^p}n^{1-p}, \end{aligned} \tag{3.3}$$

which implies (3.2). □

Theorem 3.2. *Let $\{X_n, n \geq 1\}$ be a sequence of AANA random variables with $EX_i = 0$ for all $i \geq 1$. If there exists a positive constant $M < \infty$ such that $\|X_i\|_p \leq M$ for all $i \geq 1$ and some $p \in (3 \cdot 2^{k-1}, 4 \cdot 2^{k-1}]$, where integer number $k \geq 1$, then for every $x > 0$,*

$$P\left(\max_{1\leq i\leq n} |S_i| > nx\right) \leq \frac{2D_p M^p}{x^p} n^{-p/2},$$

where D_p is defined in Lemma 1.4.

Proof. By $0 < 2/p < 1$ and C_r's inequality,

$$\left(\sum_{i=1}^{n} |X_i|^p\right)^{2/p} \leq \sum_{i=1}^{n} X_i^2,$$

which implies that

$$\sum_{i=1}^{n} E|X_i|^p \leq E\left(\sum_{i=1}^{n} X_i^2\right)^{p/2}.$$

By Jensen's inequality, we have

$$\left(\sum_{i=1}^{n} EX_i^2\right)^{p/2} \leq E\left(\sum_{i=1}^{n} X_i^2\right)^{p/2}.$$

Therefore, the statements above and C_r's inequality imply that

$$\sum_{i=1}^{n} E|X_i|^p + \left(\sum_{i=1}^{n} EX_i^2\right)^{p/2} \leq 2n^{p/2-1} \sum_{i=1}^{n} E|X_i|^p \leq 2M^p n^{p/2}. \tag{3.5}$$

Combining Lemma 1.4 and (3.5),

$$E\left(\max_{1\leq i\leq n} |S_i|^p\right) \leq 2D_p M^p n^{p/2}. \tag{3.6}$$

It follows from Markov's inequality and (3.6) that

$$P\left(\max_{1\leq i\leq n} |S_i| > nx\right) \leq \frac{1}{n^p x^p} E\left(\max_{1\leq i\leq n} |S_i|^p\right) \leq \frac{2D_p M^p}{x^p} n^{-p/2}, \tag{3.7}$$

this completes the proof of the theorem. □

4. Marcinkiewicz type strong law of large numbers for AANA sequence

Theorem 4.1. *Let $\{X_n, n \geq 1\}$ be a sequence of identically distributed AANA random variables with $\sum_{n=1}^{\infty} q^2(n) < \infty$ and $E|X_1|^p < \infty$ for $0 < p < 2$. Assume that $EX_1 = 0$ if $1 \leq p < 2$. Then*

$$\frac{1}{n^{1/p}} \sum_{k=1}^{n} X_k \to 0 \ a.s., \ n \to \infty. \tag{4.1}$$

Proof. Denote

$$Y_n = -n^{1/p} I(X_n \le -n^{1/p}) + X_n I(|X_n| < n^{1/p}) + n^{1/p} I(X_n \ge n^{1/p}),$$

then

$$\sum_{n=1}^{\infty} P(X_n \ne Y_n) = \sum_{n=1}^{\infty} P(|X_n| \ge n^{1/p}) \le CE|X_1|^p < \infty,$$

which implies that $P(X_n \ne Y_n,\ i.o.) = 0$ by the Borel-Cantelli lemma. Thus $\frac{1}{n^{1/p}} \sum_{k=1}^{n} X_k \to 0$ a.s. if and only if $\frac{1}{n^{1/p}} \sum_{k=1}^{n} Y_k \to 0$ a.s.. So we only need to show that

$$\frac{1}{n^{1/p}} \sum_{k=1}^{n} (Y_k - EY_k) \to 0 \text{ a.s.},\ n \to \infty, \tag{4.2}$$

and

$$\frac{1}{n^{1/p}} \sum_{k=1}^{n} EY_k \to 0,\ n \to \infty. \tag{4.3}$$

By Corollary 1.1 and Kronecker's lemma, to prove (4.2), it suffices to show that

$$\sum_{n=1}^{\infty} Var\left(\frac{Y_n}{n^{1/p}}\right) < \infty. \tag{4.4}$$

In fact,

$$\begin{aligned}
\sum_{n=1}^{\infty} Var\left(\frac{Y_n}{n^{1/p}}\right) &\le C \sum_{n=1}^{\infty} P(|X_n| \ge n^{1/p}) + C \sum_{n=1}^{\infty} \frac{EX_1^2 I(|X_1| < n^{1/p})}{n^{2/p}} \\
&\le C + C \sum_{n=1}^{\infty} \frac{1}{n^{2/p}} \sum_{k=1}^{n} EX_1^2 I(k-1 \le |X_1|^p < k) \\
&= C + C \sum_{k=1}^{\infty} \sum_{n=k}^{\infty} \frac{1}{n^{2/p}} E|X_1|^p |X_1|^{2-p} I(k-1 \le |X_1|^p < k) \\
&\le C + C \sum_{k=1}^{\infty} k^{1-2/p} E|X_1|^p k^{(2-p)/p} I(k-1 \le |X_1|^p < k) \\
&< \infty.
\end{aligned}$$

Hence (4.2) holds. Next, we will prove (4.3). It will be divided into two cases:

(i) If $p = 1$, by $E|X_1|^p < \infty$ and Lebesgue dominated convergence theorem, we have

$$\lim_{n\to\infty} n^{1/p} P(|X_n| \ge n^{1/p}) = 0, \tag{4.5}$$

$$\begin{aligned}
\lim_{n\to\infty} EX_n I(|X_n| < n^{1/p}) &= \lim_{n\to\infty} \int_{\Omega} X_1(\omega) I(|X_1(\omega)| < n^{1/p}) P(d\omega) \\
&= EX_1 = 0.
\end{aligned} \tag{4.6}$$

Thus,

$$|EY_n| \leq n^{1/p}P(|X_n| \geq n^{1/p}) + |EX_nI(|X_n| < n^{1/p})| \to 0 \text{ as } n \to \infty.$$

By the Toeplitz lemma, we obtain $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} EY_k = 0$.

(ii) If $p \neq 1$, by the Kronecker's lemma, to prove (4.3), it suffices to show that

$$\sum_{n=1}^{\infty} \frac{|EY_n|}{n^{1/p}} < \infty. \tag{4.7}$$

For $0 < p < 1$,

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{|EY_n|}{n^{1/p}} &\leq \sum_{n=1}^{\infty} P(|X_1| \geq n^{1/p}) + \sum_{n=1}^{\infty} \frac{E|X_n|I(|X_n| < n^{1/p})}{n^{1/p}} \\
&\leq C + \sum_{n=1}^{\infty}\sum_{j=1}^{n} n^{-1/p}E|X_1|I(j-1 \leq |X_1|^p < j) \\
&= C + \sum_{j=1}^{\infty}\sum_{n=j}^{\infty} n^{-1/p}E|X_1|I(j-1 \leq |X_1|^p < j) \\
&\leq C + C\sum_{j=1}^{\infty} j^{1-1/p}E|X_1|^p j^{(1-p)/p}I(j-1 \leq |X_1|^p < j) < \infty.
\end{aligned}$$

For $1 \leq p < 2$, by $EX_n = 0$, we can see that

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{|EY_n|}{n^{1/p}} &\leq \sum_{n=1}^{\infty} P(|X_n| \geq n^{1/p}) + \sum_{n=1}^{\infty} \frac{|EX_nI(|X_n| < n^{1/p})|}{n^{1/p}} \\
&\leq C + \sum_{n=1}^{\infty} n^{-1/p}E|X_n|I(|X_n| \geq n^{1/p}) \\
&= C + \sum_{j=1}^{\infty}\sum_{n=1}^{j} n^{-1/p}E|X_1|I(j \leq |X_1|^p < j+1) \\
&\leq C + C\sum_{j=1}^{\infty} j^{1-1/p}E|X_1|^p j^{(1-p)/p}I(j \leq |X_1|^p < j+1) < \infty.
\end{aligned}$$

Thus (4.7) holds, which implies (4.3) by Kronecker's lemma. We get the desired result. □

Acknowledgements. The authors are most grateful to the editor and anonymous referees for careful reading of the manuscript and valuable suggestions which helped in improving an earlier version of this paper.

References

[1] T. K. Chandra and S. Ghosal, *Extensions of the strong law of large numbers of Marcinkiewicz and Zygmund for dependent variables*, Acta Math. Hungar. **71** (1996), no. 4, 327–336.

[2] ———, *The strong law of large numbers for weighted averages under dependence assumptions*, J. Theoret. Probab. **9** (1996), no. 3, 797–809.

[3] I. Fazekas and O. Klesov, *A general approach to the strong laws of large numbers*, Teor. Veroyatnost. i Primenen. **45** (2000), no. 3, 568–583; translation in Theory Probab. Appl. **45** (2002), no. 3, 436–449.

[4] M. H. Ko, T. S. Kim, and Z. Y. Lin, *The Hájeck-Rènyi inequality for the AANA random variables and its applications*, Taiwanese J. Math. **9** (2005), no. 1, 111–122.

[5] Y. B. Wang, J. G. Yan, and F. Y. Cheng, *The strong law of large numbers and the law of the iterated logarithm for product sums of NA and AANA random variables*, Southeast Asian Bull. Math. **27** (2003), no. 2, 369–384.

[6] D. M. Yuan and J. An, *Rosenthal type inequalities for asymptotically almost negatively associated random variables and applications*, Sci. China Ser. A **52** (2009), no. 9, 1887–1904.

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