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$P^t_{(\omega_r,s)}$-CLOSED SPACES AND PRE-(ω_r,s)t-θ_f-CLUSTER SETS

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Abstract. Using (r,s)-preopen sets [14] and pre-ω_t-closures [6], a new kind of covering property $P^t_{(\omega_r,s)}$-closedness is introduced in a bitopological space and several characterizations via filter bases, nets and grills [30] along with various properties of such concept are investigated. Two new types of cluster sets, namely pre-$(\omega_r,s)t$-θ_f-cluster sets and $(r,s)t$-θ_f-precluster sets of functions and multifunctions between two bitopological spaces are introduced. Several properties of pre-$(\omega_r,s)t$-θ_f-cluster sets are investigated and using the degeneracy of such cluster sets, some new characterizations of some separation axioms in topological spaces or in bitopological spaces are obtained. A sufficient condition for $P^t_{(\omega_r,s)}$-closedness has also been established in terms of pre-$(\omega_r,s)t$-θ_f-cluster sets.

1. Introduction

Ever since the introduction of bitopological spaces by J. C. Kelly [17] in the year 1963, many topologists have introduced and investigated various concepts in bitopological spaces and also generalized certain existing topological properties. Because of having various applications, now a days, it has become a mature field of mathematical activity. Topologists are often keen in investigating properties closely related to compactness using certain new accessories which have been developed recently. The notion of ω-open sets introduced by H. Z. Hdeib [11] has been studied by a good number of researchers in recent times. Contributions of Noiri, Omari and Noorani [24, 25], Omari and Noorani [26, 27], Zoubi and Nashef [3] and C. K. Basu and B. M. Uzzal Afsan [6] in the arena of ω-open sets and associated concepts are worth to be mentioned. In this paper, in Section 3, we introduce pre-(ω_r,s)-open sets and allied concepts in bitopological spaces which have been exploited effectively in investigating certain concepts which have been developed in the subsequent sections. In

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Section 4, a new type of covering property $P^t_{(\omega_r,s)}$-closedness is introduced in a bitopological space (X, τ_1, τ_2) using (r, s)-preopen sets [14] and pre-ω_t-closures (=pre-ω-closures with respect to the topology τ_t) [6] . Such concept is identical to the P_ω-closedness [6] if the topologies on X are taken the same. Several characterizations via filter bases, nets and grills [30] along with various properties of such concept are investigated. Further, in the last section, two new types of cluster sets, namely pre-$(\omega_r, s)t$-θ_f-cluster sets (resp. pre-$(\omega_r, s)t$-θ_f-cluster sets) of functions and multifunctions have been introduced between two bitopological spaces in terms of pre-ω-closure [6] (resp. preclosure [21]), θ-closure [31] and (r, s)-preopen sets [14]. Finally, a sufficient condition for $P^t_{(\omega_r,s)}$-closedness in terms of pre-$(\omega_r, s)t$-θ_f-cluster sets has been given.

2. Prerequisites

Let (X, τ) be a topological space and $A \subset X$. Then a point $x \in X$ is called a condensation point of A if for each open set U containing x, $A \cap U$ is uncountable. A set A is called ω-closed [11] if it contains all of its condensation points and the complement of an ω-closed set is called an ω-open set [11] or equivalently, $A \subset X$ is ω-open if and only if for each $x \in A$ there exists an open set U containing x such that $U - A$ is countable. Throughout this paper, spaces (X, τ_1, τ_2) and $(Y, \varrho_1, \varrho_2)$ (or simply X and Y) represent non-empty bitopological spaces and r, s, t and f are indices varying over the set $\{1, 2\}$. The set of all ω-open sets of space X when the topology τ_r is considered is denoted by τ_{ω_r}. It is to be noted that τ_{ω_r} is a topology on X finer than τ_r [11]. A subset A of a topological space X is called semi-open [18] (resp. regular open, α-open [23], preopen [21], β-open [2], semi-ω-open [24], α-ω-open [24], pre-ω-open [24], β-ω-open [24]) if $A \subset cl(int(A))$ (resp. $A = int(cl(A))$, $A \subset int(cl(int(A)))$, $A \subset int(cl(A))$, $A \subset cl(int(cl(A)))$, $A \subset cl(int_\omega(A))$, $A \subset int_\omega(cl(int_\omega(A)))$, $A \subset int_\omega(cl(A))$ and $A \subset cl(int_\omega(cl(A)))$). The θ-closure of a subset A of a topological space (X, τ) is the set $\theta\text{-}cl(A) = \{x \in X : cl(U) \cap A \neq \emptyset, \forall U \in \tau\}$ [31]. The pre-ω-closure of a subset A of a topological space (X, τ) is the intersection of all the pre-ω-closed subsets of X containing A [6]. The (r, s)-θ-closure of a subset A of a bitopological space (X, τ_1, τ_2) is the set $r\text{-}cl_{\theta_s}(A) = \{x \in X : cl_s(U) \cap A \neq \emptyset, \forall U \in \tau_r\}$ [16]. The closure (resp. interior, θ-closure, ω-interior [11], ω-closure [11], pre-ω-interior [6], pre-ω-closure [6]) of a subset A of a space X with respect to the topology τ_r (where $r = 1, 2$) (read as r-closure (resp. r-interior, θ_r-closure, ω_r-interior, ω_r-closure, pre-ω_r-interior, pre-ω_r-closure)) are denoted by $cl_r(A)$ (resp. $int_r(A)$, $cl_{\theta_r}(A)$, $int_{\omega_r}(A)$, $cl_{\omega_r}(A)$, $pint_{\omega_r}(A)$, $pcl_{\omega_r}(A)$). A subset A of a bitopological space X is called (r, s)-regular open [9] (resp. (r, s)-preopen [14]) if $A = int_r(cl_s(A))$ (resp. $A \subset int_r(cl_s(A))$). The family of all (r, s)-preopen (resp. (r, s)-preclopen i.e., (r, s)-preclosed as well as (r, s)-preopen and (r, s)-regular open) subsets of X is denoted by (r, s)-$PO(X)$ (resp. (r, s)-$PCO(X)$ and (r, s)-$RO(X)$). The family of all (r, s)-preopen (resp. (r, s)-preclopen i.e., (r, s)-preclosed as well as (r, s)-preopen and (r, s)-regular open) subsets of X containing $x \in X$ is denoted by $(r, s)PO(X, x)$ (resp. $(r, s)PCO(X, x)$ and $(r, s)RO(X, x)$) and the family of all (r, s)-preopen subsets of X containing a subset $K \subset X$ is denoted by $(r, s)PO(X, K)$. The family of all preopen sets in the topological space (X, τ_r) is denoted by $(r)PO(X)$. $(r, s)pcl(A)$ is the intersection of all (r, s)-preclosed subsets of X containing A [14]. A topological space (X, τ) is called almost regular [28] if for every regular closed set A in X and for each $x \in A$, there exist disjoint open sets M and N containing x and V respectively. It is well known that in an almost regular space X, θ-$cl(A)$ is θ-closed for each $A \subset X$. A bitopological space X is said to satisfy f-T (where $f = 1, 2$) property if (X, τ_f) satisfy the property T. A bitopological space X is called pairwise (r, f)-Urysohn [19] if each pair of distinct points $x, y \in X$, there exist $U \in \tau_r(x)$ and $V \in \tau_f(y)$ such that $cl_r(U) \cap cl_f(V) = \emptyset$.

Thron [30] has defined a grill as a non-empty family $\mathcal{G}$ of non-empty subsets of X satisfying (a) $A \in \mathcal{G}$ and $A \subset B \Rightarrow B \in \mathcal{G}$ and (b) $A \cup B \in \mathcal{G} \Rightarrow$ either $A \in \mathcal{G}$ or $B \in \mathcal{G}$. Thron [30] also has shown that $\mathcal{F}(\mathcal{G}) = \{A \subset X : A \cap F \neq \emptyset, \forall F \in \mathcal{G}\}$ is a filter on X and there exists an ultrafilter $\mathcal{F}$ such that $\mathcal{F}(\mathcal{G}) \subset \mathcal{F} \subset \mathcal{G}$.

The results of the following theorem are used frequently in this paper.

Theorem 2.1 ([6]). *For subsets A, B of a topological space X, following properties hold*:

(a) $pcl_\omega(A) \subset pcl(A)$ *and* $pcl_\omega(A) \subset cl_\omega(A)$.
(b) $A \subset B$ *implies* $pcl_\omega(A) \subset pcl_\omega(B)$ *and* $pint_\omega(A) \subset pint_\omega(B)$.
(c) $pcl_\omega(pcl_\omega(A)) = pcl_\omega(A)$ *and* $pint_\omega(pint_\omega(A)) = pint_\omega(A)$.
(d) A *is pre-ω-closed if and only if* $pcl_\omega(A) = A$.
(e) A *is pre-ω-open if and only if* $pint_\omega(A) = A$.
(f) $pcl_\omega(X - A) = X - pint_\omega(A)$.
(g) $pint_\omega(X - A) = X - pcl_\omega(A)$.

3. Pre-(ω_r, s)-open sets

Definition 3.1. A subset A of a bitopological space (X, τ_1, τ_2) is called pre-(ω_r, s)-open if $A \subset int_{\omega_r}(cl_s(A))$.

The complement of a pre-(ω_r, s)-open set is called a pre-(ω_r, s)-closed set.

The family of all pre-(ω_r, s)-open sets of X is denoted by $P(\omega_r, s)O(X)$ and the family of all pre-(ω_r, s)-open sets of X containing a point $x \in X$ is denoted by $P(\omega_r, s)O(X, x)$.

Remark 3.2. (i) If $r = s$, then every pre-(ω_r, s)-open set is pre-ω-open in (X, τ_r).

(ii) Every (r, s)-preopen set is pre-(ω_r, s)-open, but the converse need not be true that is established with the following example.

Example 3.3. Consider the real line $X = \mathbb{R}$ with the co-countable topology τ_1 and the topology $\tau_2 = \{S \subset \mathbb{R} : \mathbb{Q} \subset S, \mathbb{Q}$ is the set of all rational numbers$\}$. Then the set $\mathbb{N}$ of all positive integers is pre-$(\omega_2, 1)$-open but is not $(2,1)$-preopen in the bitopological space (X, τ_1, τ_2). In fact $int_{\omega_2}(\mathbb{N}) = \mathbb{N}$ and $int_2(cl_1(\mathbb{N})) = int_2(\mathbb{N}) = \emptyset$.

Lemma 3.4. *An arbitrary union of pre-(ω_r, s)-open sets in a bitopological space X is a pre-(ω_r, s)-open set.*

Proof. The proof is obvious and is thus omitted. □

Definition 3.5. Let A be a subset of a bitopological space X. Then pre-(ω_r, s)-interior (resp. pre-(ω_r, s)-closure) is denoted by $pint_{(\omega_r,s)}(A)$ (resp. $pcl_{(\omega_r,s)}(A)$) and is defined as the set $pint_{(\omega_r,s)}(A) = \cup\{G \subset A : G \in P(\omega_r, s)O(X)\}$ (resp. $pcl_{(\omega_r,s)}(A) = \cap\{G \supset A : X - G \in P(\omega_r, s)O(X)\}$).

Theorem 3.6. *For subsets A, B of a topological space X, following properties hold*:

(a) $pcl_{(\omega_r,s)}(A) \subset (r, s)pcl(A)$.

(b) $A \subset B$ *implies* $pcl_{(\omega_r,s)}(A) \subset pcl_{(\omega_r,s)}(B)$ *and* $pint_{(\omega_r,s)}(A) \subset pint_{(\omega_r,s)}(B)$.

(c) $pcl_{(\omega_r,s)}(pcl_{(\omega_r,s)}(A)) = pcl_{(\omega_r,s)}(A)$ *and* $pcl_{(\omega_r,s)}(pcl_{(\omega_r,s)}(A)) = pcl_{(\omega_r,s)}(A)$.

(d) *A is pre-(ω_r, s)-closed if and only if* $pcl_{(\omega_r,s)}(A) = A$.

(e) *A is pre-(ω_r, s)-open if and only if* $pint_{(\omega_r,s)}(A) = A$.

(f) $pcl_{(\omega_r,s)}(X - A) = X - pint_{(\omega_r,s)}(A)$.

(g) $pint_{(\omega_r,s)}(X - A) = X - pcl_{(\omega_r,s)}(A)$.

The reverse inclusion of the above Theorem 3.6(a) is not true in general. This fact is reflected in the following example.

Example 3.7. Consider the bitopological space $X = \mathbb{Q} \cup \{\sqrt{2}\}$ with the co-countable topology τ_1 and the topology τ_2 generated by the base $\{\{x, \sqrt{2}\} : x \in X\}$. Then $\tau_{\omega_1} = \tau_{\omega_2} = P(\omega_r, s)O(X) = P(X)$ and $(2,1)PO(X) = \tau_2$. Then $pcl_{(\omega_2,1)}(\{\sqrt{2}\}) = \{\sqrt{2}\}$ and $pcl_{(2,1)}(\{\sqrt{2}\}) = X$.

Definition 3.8. A point $x \in X$ is said to be a pre-(ω_r, s)-θ_t-accumulation (resp. (r, s)-θ_t-pre-accumulation) point of a subset A of a bitopological space X if $pcl_{\omega_t}(U) \cap A \neq \emptyset$ (resp. $pcl_t(U) \cap A \neq \emptyset$) for every $U \in (r, s)PO(X, x)$. The set of all pre-(ω_r, s)-θ_t-accumulation (resp. (r, s)-θ_t-pre-accumulation) points of A is called the pre-(ω_r, s)-θ_t-closure (resp. (r, s)-θ_t-preclosure) of A and is denoted by $p_{(\omega_r,s)}cl_{\theta_t}(A)$ (resp. $(r, s)pcl_{\theta_t}(A)$). A subset A of a bitopological space X is said to be pre-(ω_r, s)-θ_t-closed (resp. $(r, s)\theta_t$-preclosed set) if $p_{(\omega_r,s)}cl_{\theta_t}(A) = A$ (resp. $(r, s)pcl_{\theta_t}(A) = A$). The complement of a pre-(ω_r, s)-θ_t-closed set (resp. $(r, s)\theta_t$-preclosed set) is called pre-(ω_r, s)-θ_t-open set (resp. $(r, s)\theta_t$-preopen set).

Lemma 3.9. *A subset A of space X is a pre-(ω_r,s)-θ_t-open if and only if for each $x \in A$, there exists $V \in (r,s)PO(X,x)$ such that $pcl_{\omega_t}(V) \subset A$.*

Proof. Let A be a pre-(ω_r,s)-θ_t-open set and $x \in A$. Then $X - A$ is pre-(ω_r,s)-θ_t-closed and so for each $x \in A$, there exists a $V \in (r,s)PO(X,x)$ such that $pcl_{\omega_t}(V) \cap (X - A) = \emptyset$ and thus $pcl_{\omega_t}(V) \subset A$. □

Conversely, suppose the condition does not hold. Then there exists an $x \in A$ such that $pcl_{\omega_t}(V) \not\subset A$ for all $V \in (r,s)PO(X,x)$. Thus $pcl_{\omega_t}(V) \cap (X-A) \neq \emptyset$ for all $V \in (r,s)PO(X,x)$ and so x is a pre-(ω_r,s)-θ_t-accumulation point of $X - A$. Hence $X - A$ is not pre-(ω_r,s)-θ_t-closed.

Now we state following theorem:

Theorem 3.10. *Let A and B be any subsets of a space X. Then the following properties hold*:

(a) *$(r,s)\theta_t$-preclosed sets are pre-(ω_r,s)-θ_t-closed sets,*

(b) *$p_{(\omega_r,s)}cl_{\theta_t}(A) \subset (r,s)pcl_{\theta_t}(A)$,*

(c) *if $A \subset B$, then $p_{(\omega_r,s)}cl_{\theta_t}(A) \subset p_{(\omega_r,s)}cl_{\theta_t}(B)$,*

(d) *intersection of an arbitrary family of pre-(ω_r,s)-θ_t-closed sets is pre-(ω_r,s)-θ_t-closed in X.*

Proof. Proof of (a), (b), (c) are straight forward. So we prove (d) only.

(d) Let $\{A_\alpha : \alpha \in \Delta\}$ be a family of pre-(ω_r,s)-θ_t-closed sets. Let $x \in p_{(\omega_r,s)}cl_{\theta_t}(\cap_{\alpha\in\Delta}(A_\alpha))$. Then for all $U \in (r,s)PO(X,x)$, $\emptyset \neq (\cap_{\alpha\in\Delta}(A_\alpha)) \cap pcl_{\omega_t}(U) = \cap_{\alpha\in\Delta}(A_\alpha \cap pcl_{\omega_t}(U))$. So for each $\alpha \in \Delta$, $A_\alpha \cap pcl_{\omega_t}(U) \neq \emptyset$. Thus $x \in p_{(\omega_r,s)}cl_{\theta_t}(A_\alpha) = A_\alpha$ for each $\alpha \in \Delta$ and hence $x \in \cap_{\alpha\in\Delta}(A_\alpha)$. Thus $\cap_{\alpha\in\Delta}(A_\alpha)$ is pre-(ω_r,s)-θ_t-closed in X. □

The following example shows that the converse of Theorem 3.9(a) is not true and so $p_{(\omega_r,s)}cl_{\theta_t}(A) \neq (r,s)pcl_{\theta_t}(A)$.

Example 3.11. Consider the same bitopological space X as in Example 3.7. Then $\tau_{\omega_2} = P\omega_2O(X) = P(X)$ and $(2,1)PO(X) = (2)PO(X) = \tau_2$. Then $pcl_{\omega_2}(\{\sqrt{2}\}) = \{\sqrt{2}\}$ and $pcl_2(\{\sqrt{2}\}) = X$ and thus $p_{(\omega_2,1)}cl_{\theta_2}(\mathbb{Q}) = \mathbb{Q}$ and $(2,1)pcl_{\theta_2}(\mathbb{Q}) = X$. So $\mathbb{Q}$ is a pre-$(\omega_2,1)$-θ_2-closed set but not $(2,1)\theta_2$-preclosed set in the bitopological space X.

4. $P^t_{(\omega_r,s)}$-closed spaces

Definition 4.1. A subset G of a bitopological space X is called $P^t_{(\omega_r,s)}$-closed (resp. $P^t_{(r,s)}$-closed) relative to X if every (r,s)-preopen cover of X has a finite subfamily whose pre-ω_t-closures (resp. t-preclosures) cover G. If $G = X$, then the $P^t_{(\omega_r,s)}$-closed (resp. $P^t_{(r,s)}$-closed) set G relative to X is called $P^t_{(\omega_r,s)}$-closed (resp. $P^t_{(r,s)}$-closed) bitopological space.

It is obvious to note that $P^t_{(\omega_r,s)}$-closedness is identical to the P_ω-closedness if the topologies on X are taken the same. Also we note that every $P^t_{(\omega_r,s)}$-closed space is $P^t_{(r,s)}$-closed. But a $P^t_{(r,s)}$-closed space need not be $P^t_{(\omega_r,s)}$-closed. For example, the bitopological space X as in Example 3.7, is $P^2_{(2,1)}$-closed but not $P^2_{(\omega_2,1)}$-closed. In fact the $(2,1)$-preopen cover $\{\{x, \sqrt{2}\}, x \in \mathbb{Q}\}$ has no finite subfamily whose pre-ω_2-closures cover X.

Definition 4.2. A filter base $\mathcal{F}$ (resp. a grill $\mathcal{G}$) on a topological space (X, τ_1, τ_2) is said to pre-(ω_r, s)-θ_t-converge to a point $x \in X$ if for each $V \in (r,s)PO(X,x)$, there exists $F \in \mathcal{F}$ (resp. $F \in \mathcal{G}$) such that $F \subset pcl_{\omega_t}(V)$. A filter base $\mathcal{F}$ is said to pre-(ω_r, s)-θ_t-accumulate (or pre-(ω_r, s)-θ_t-adhere) at $x \in X$ if $pcl_{\omega_t}(V) \cap F \neq \emptyset$ for every $V \in (r,s)PO(X,x)$ and every $F \in \mathcal{F}$. The collection of all the points of X at which the filter base $\mathcal{F}$ pre-(ω_r, s)-θ_t-adheres is denoted by $p_{(\omega_r,s)}$-θ_t-$ad\mathcal{F}$.

Theorem 4.3. *An ultrafilter base $\mathcal{F}$ pre-(ω_r, s)-θ_t-converges to a point $x \in X$ if and only if it pre-(ω_r, s)-θ_t-accumulates to the point x.*

Proof. Here only to prove is that if $\mathcal{F}$ pre-(ω_r, s)-θ_t-accumulates to the point x, then $\mathcal{F}$ pre-(ω_r, s)-θ_t-converges to the point x. If $\mathcal{F}$ does not pre-(ω_r, s)-θ_t-converge to the point x, there exists a $V \in (r,s)PO(X,x)$ such that $F \not\subset pcl_{\omega_t}(V)$ and so $(X - pcl_{\omega_t}(V)) \cap F \neq \emptyset$ for every $F \in \mathcal{F}$. Since $\mathcal{F}$ is an ultrafilter base on X, $pcl_{\omega_t}(V) \in \mathcal{F}$. Again since $\mathcal{F}$ pre-(ω_r, s)-θ_t-accumulates to the point x, $pcl_{\omega_t}(V) \cap F \neq \emptyset$ for every $F \in \mathcal{F}$ and so $pcl_{\omega_t}(V) \in \mathcal{F}$. □

Theorem 4.4. *For a topological space (X, τ_1, τ_2) the following conditions are equivalent*:

(a) *X is $P^t_{(\omega_r,s)}$-closed,*

(b) *for every family $\{V_\alpha : \alpha \in \Delta\}$ of (r,s)-preclosed subsets such that $\cap\{V_\alpha : \alpha \in \Delta\} = \emptyset$, there exist $\alpha_1, \alpha_2, \ldots, \alpha_n \in \Delta$ such that $\cap_{i=1}^n pint_{\omega_t}(V_{\alpha_i}) = \emptyset$,*

(c) *every ultrafilter base pre-(ω_r, s)-θ_t-converges to some point of X,*

(d) *every filter base pre-(ω_r, s)-θ_t-adheres at some point of X,*

(e) *every grill on X pre-ω-θ_t-converges to some point of X.*

Proof. (a)⇔(b). Let $\Sigma = \{V_\alpha : \alpha \in \Delta\}$ be a cover of X by (r,s)-preclosed sets such that $\cap\{V_\alpha : \alpha \in \Delta\} = \emptyset$. Then there exists $\alpha_1, \alpha_2, \ldots, \alpha_k \in \Delta$ such that $\cup_{i=1}^n pcl_{\omega_t}(X - V_{\alpha_i}) = X$. Hence $X - \cup_{i=1}^n pcl_{\omega_t}(X - V_{\alpha_i}) = \emptyset$ and so by Theorem 2.1, $\cap_{i=1}^n pint_{\omega_t}(V_{\alpha_i}) = \emptyset$.

Conversely, let $\{U_\alpha : \alpha \in \Delta\}$ be a family of (r,s)-preopen subsets of X covering X. Then $\{X - U_\alpha : \alpha \in \Delta\}$ is a family of (r,s)-preclosed subsets of X having empty intersection. Thus by (b), there exist $\alpha_1, \alpha_2, \ldots, \alpha_n \in \Delta$ such that $\cap_{i=1}^n pint_{\omega_t}(X - U_{\alpha_i}) = \emptyset$, i.e., by Theorem 2.1, $\cup_{i=1}^n pcl_{\omega_t}(U_{\alpha_i}) = X$. So X is $P^t_{(\omega_r,s)}$-closed.

(b)⇒(c). Let $\mathcal{F}$ be an ultrafilter base on X which does not pre-(ω_r, s)-θ_t-converge to any point of X. Then by Theorem 4.3, $\mathcal{F}$ can not pre-ω-θ_t-accumulate at any point of X. Thus for each $x \in X$, there are an $F_x \in \mathcal{F}$ and a $V_x \in (r,s)PO(X,x)$ such that $pcl_{\omega_t}(V_x) \cap F_x = \emptyset$ and so $F_x \subset X - pcl_{\omega_t}(V_x) =$

$pint_{\omega_t}(V_x)$ (by Theorem 2.1). Again $\{X - V_x : x \in X\}$ is a family of (r,s)-preclosed subsets of X having empty intersection. But then by (b), there exists a finite subset $X_0 \subset X$ such that $\cap_{x\in X_0} pint_{\omega_t}(V_x) = \emptyset$. Since $\mathcal{F}$ is a filter base on X, there exists $F_0 \in \mathcal{F}$ such that $F_0 \subset \cap_{x\in X_0}(F_x)$ and thus $F_0 = \emptyset$, which is a contradiction.

(c)$\Rightarrow$(d). Let $\mathcal{F}$ be any filter base on X and $\mathcal{F}_0$ be an ultrafilter base on X such that $\mathcal{F} \subset \mathcal{F}_0$. Then (c) ensures that $\mathcal{F}_0$ pre-(ω_r,s)-θ_t-converges to some point $x \in X$. Therefore for each $V \in (r,s)PO(X,x)$, there exists $F_0 \in \mathcal{F}_0$ such that $F_0 \subset pcl_{\omega_t}(V)$. Now since for each $F \in \mathcal{F}$, $\emptyset \neq F_0 \cap F \subset pcl_{\omega_t}(V) \cap F$, $pcl_{\omega_t}(V) \cap F \neq \emptyset$ for every $V \in (r,s)PO(X,x)$ and every $F \in \mathcal{F}$. So the filter base $\mathcal{F}$ pre-(ω_r,s)-θ_t-accumulates at $x \in X$.

(c)$\Rightarrow$(e). Let $\mathcal{G}$ be a grill on X. Then Thron [30] has shown that $\mathcal{F}(\mathcal{G}) = \{F \subset X : F \cap E \neq \emptyset, \forall E \in \mathcal{G}\}$ is a filter on X and there exists an ultrafilter $\mathcal{F}$ such that $\mathcal{F}(\mathcal{G}) \subset \mathcal{F} \subset \mathcal{G}$. Let $x \in X$ at which filter base $\mathcal{F}$ pre-(ω_r,s)-θ_t-converges. If possible, let $\mathcal{G}$ does not pre-(ω_r,s)-θ_t-converge to x. Then there exists $V \in (r,s)PO(X,x)$ such that $E \not\subset pcl_{\omega_t}(V)$ and so $E\cap(X - pcl_{\omega_t}(V)) \neq \emptyset$ for all $E \in \mathcal{G}$. So $X - pcl_{\omega_t}(V) \in \mathcal{F}(\mathcal{G}) \subset \mathcal{G}$. Again by Theorem 4.3, $pcl_{\omega_t}(V) \cap F \neq \emptyset$ for every $F \in \mathcal{F}$. Therefore $pcl_{\omega_t}(V) \in \mathcal{F}$. Hence $pcl_{\omega_t}(V) \in \mathcal{G}$, which is a contradiction.

(e)$\Rightarrow$(c). Since every ultrafilter base is a grill, (c) immediately follows.

(d)$\Rightarrow$(b). Let $\{V_\alpha : \alpha \in \Delta\}$ be a family of (r,s)-preclosed subsets of X such that $\cap\{V_\alpha : \alpha \in \Delta\} = \emptyset$. If possible, let $\cap_{\lambda\in\Gamma} pint_{\omega_t}(V_\lambda) \neq \emptyset$ for each finite subset Γ of Δ. Then the family $\mathcal{F} = \{\cap\{pint_{\omega_t}(V_\gamma), \gamma \in \Gamma\}, \Gamma \subset \Delta, \text{card}(\Gamma) < \mathcal{N}_0\}$, where $\mathcal{N}_0$ is the cardinality of the set of all natural numbers is a filter base on X. Then by (d), $\mathcal{F}$ pre-(ω_r,s)-θ_t-accumulates at some point x of X. Since $\{X - V_\alpha : \alpha \in \Delta\}$ is a (r,s)-preopen cover of X, $x \in X - V_{\alpha_0}$ for some $\alpha_0 \in \Delta$. Let $G = X - V_{\alpha_0}$. Then $G \in (r,s)PO(X,x)$ and $pint_{\omega_t}(V_{\alpha_0}) \in \mathcal{F}$ such that $pcl_{\omega_t}(G) \cap pint_{\omega_t}(V_{\alpha_0}) = \emptyset$, which is a contradiction. □

Definition 4.5. A bitopological space X is called pre-$(\omega_r,s)t$-regular if for each $x \in X$ and $U \in (r,s)PO(X,x)$, there exists a $V \in (r,s)PO(X,x)$ such that $cl_{\omega_t}(V) \subset U$.

Theorem 4.6. *For a pre-$(\omega_r,s)t$-regular bitopological space X, the following two conditions are equivalent*:

(i) *X is $P^t_{(\omega_r,s)}$-closed,*

(ii) *every cover of X by pre-(ω_r,s)-θ_t-open sets of X has a finite subcover.*

Proof. (i)$\Rightarrow$(ii). Let X be $P^t_{(\omega_r,s)}$-closed and Σ be any cover of X by pre-(ω_r,s)-θ_t-open sets of X. Then for each $x \in X$, there exists an $U_x \in \Sigma$ and so Theorem 3.9 ensures the existence of a $V_x \in (r,s)PO(X,x)$ such that $pcl_{\omega_t}(V_x) \subset cl_{\omega_t}(V_x) \subset U_x$. Since $\{V_x : x \in X\}$ is a cover of X by the (r,s)-preopen sets of X, there exist finite number of points $x_1, x_2, \ldots, x_k \in X$ such that $X = \cup_{i=1}^k pcl_{\omega_t}(V_{x_i})$ and so $X = \cup_{i=1}^k U_{x_i}$. Hence $\{U_{x_i} : x \in X, i = 1,2,\ldots,k\}$ is the required finite subcover of Σ.

(ii)⇒(i). Let X be a $(\omega_r, s)t$-regular bitopological space. Consider Ω be a cover of X by (r, s)-preopen sets of X. Let $x \in X$ and $x \in U_x$ for some $U_x \in \Omega$. Since X is pre-$(\omega_r, s)t$-regular, there exists $V_x \in (r, s)PO(X, x)$ such that $pcl_{\omega_t}(V_x) \subset cl_{\omega_t}(V_x) \subset U_x$. Then by Lemma 3.9 for each $x \in X$, U_x is pre-(ω_r, s)-θ_t-open set of X. Thus by (ii), there exist $x_1, x_2, \ldots, x_k \in X$ such that $X = \cup_{i=1}^{k} U_{x_i} \subset \cup_{i=1}^{k} pcl_{\omega_t}(U_{x_i})$. □

Theorem 4.7. *For a pre-$(\omega_r, s)t$-regular bitopological space X, the following two conditions are equivalent*:

(i) *X is $P^t_{(\omega_r,s)}$-closed,*

(ii) *every family of pre-(ω_r, s)-θ_t-closed subsets of X with finite intersection property has a nonempty intersection.*

Proof. (i)⇒(ii). Let X be $P^t_{(\omega_r,s)}$-closed and $\{V_\alpha : \alpha \in \Delta\}$ be a family of pre-(ω_r, s)-θ_t-closed subsets of X with finite intersection property having empty intersection. Then $\{X - V_\alpha : \alpha \in \Delta\}$ is a cover of X by a family of pre-(ω_r, s)-θ_t-open subsets of X. Then Theorem 4.6 ensures the existence of a finite subset Δ_0 of Δ such that $\{X - V_\alpha : \alpha \in \Delta_0\}$ covers X and so $\cap\{V_\alpha : \alpha \in \Delta_0\} = \emptyset$, which is a contradiction.

(ii)⇒(i). Let X be a pre-$(\omega_r, s)t$-regular bitopological space. If X be not $P^t_{(\omega_r,s)}$-closed, Theorem 4.6 ensures the existence of a cover $\{U_\alpha : \alpha \in \Delta\}$ of X by a family of pre-(ω_r, s)-θ_t-open subsets of X without any finite subcover. Then $\{X - U_\alpha : \alpha \in \Delta\}$ is a family of pre-(ω_r, s)-θ_t-closed subsets of X with finite intersection property. Therefore by the hypothesis (ii), $\cap\{X - U_\alpha : \alpha \in \Delta\} \neq \emptyset$ and so $\cup\{U_\alpha : \alpha \in \Delta\} \neq X$, which is a contradiction. □

As an application of Theorem 4.7, we prove the following "fixed set theorem" for multifunction:

Theorem 4.8. *Let X be a $P^t_{(\omega_r,s)}$-closed bitopological space and $\Omega : X \to X$ be any multifunction preserving pre-(ω_r, s)-θ_t-closed subsets to pre-(ω_r, s)-θ_t-closed subsets. Then there exists a subset $K \subset X$ such that $\Omega(K) = K$.*

Proof. Clearly, $\Gamma = \{G \subset X : \Omega(G) \subset G, G \neq \emptyset, G$ is pre-(ω_r, s)-θ_t-closed$\}$ is a totally order set under the set inclusion. Theorem 3.10(d) implies that every subfamily of Γ has a lower bound and hance Zorns Lemma implies that there is a minimal element K of Γ. Since the multifunction Ω preserves pre-(ω_r, s)-θ_t-closed subsets to pre-(ω_r, s)-θ_t-closed subsets and K is minimal, $K \subset \Omega(K) \subset K$ and hance $\Omega(K) = K$. □

Definition 4.9. A point $x \in X$ is called pre-(ω_r, s)-θ_t-complete adherent point of a subset A of X if for each pre-(ω_r, s)-θ_t-open set U containing x, card$(A \cap U) =$ card(A).

Definition 4.10. A net $(x_\lambda)_{\lambda \in \Upsilon}$ (where Υ is a directed set) on a bitopological space X is called pre-(ω_r, s)-θ_t-adheres at a point $x \in X$ if for every $U \in$

$(r,s)PO(X,x)$ and for every $\lambda \in \Upsilon$, there exists $\mu(\succeq \lambda) \in \Upsilon$ such that $x_\mu \in pcl_{\omega_t}(U)$.

Theorem 4.11. *For a pre-$(\omega_r,s)t$-regular bitopological space X, the following three statements are equivalent*:

(i) *X is $P^t_{(\omega_r,s)}$-closed,*

(ii) *every net $(x_\lambda)_{\lambda\in\Upsilon}$, where Υ is a well-ordered index set pre-(ω_r,s)-θ_t-adheres at a point in X,*

(iii) *every infinite subset A of X has a pre-(ω_r,s)-θ_t-complete adherent point in X.*

Proof. (i)⇒(ii). Let X be $P^t_{(\omega_r,s)}$-closed and $(x_\lambda)_{\lambda\in\Upsilon}$, where Υ is a well-ordered index set be a net on X. If possible, let $(x_\lambda)_{\lambda\in\Upsilon}$ does not pre-(ω_r,s)-θ_t-adhere at any point of X. So for each $x \in X$, there exists an $U_x \in (r,s)PO(X,x)$ and a $\lambda(x) \in \Upsilon$ such that $x_\mu \notin pcl_{\omega_t}(U_x)$ for all $\mu(\succeq \lambda(x)) \in \Upsilon$. Since $\{V_x : x \in X\}$ forms a cover of X by (r,s)-preopen sets of X, there exist finite number of points $x_1, x_2, \ldots, x_k \in X$ such that $X = \cup_{i=1}^k pcl_{\omega_t}(U_{x_i})$. Consider an $\eta \in \Upsilon$ such that $\eta \succeq \lambda(x_i)$ for all $i = 1,2,\ldots,n$. Then for each $i = 1,2,\ldots,n$, $x_\mu \notin pcl_{\omega_t}(U_{x_i})$ for all $\mu \succeq \eta$, which is a contradiction.

(ii)⇒(iii). Let A be an infinite subset of X. Then A can be well-ordered by some minimal well-ordering $\preceq$. Thus A may be thought as a net with a well-ordered index set as domain. Then by (ii), the net A pre-(ω_r,s)-θ_t-adheres at a point $x \in X$. Now consider an $U \in (r,s)PO(X,x)$. Since X is pre-$(\omega_r,s)t$-regular, there exists $V \in (r,s)PO(X,x)$ such that $pcl_{\omega_t}(V) \subset cl_{\omega_t}(V) \subset U$. Now since net A pre-(ω_r,s)-θ_t-adheres at a point $x \in X$, so for any $\lambda \in A$, there exists $\mu(\succeq \lambda) \in A$ such that $x_\mu \in pcl_{\omega_t}(V) \cap A$ and hence card$(A) =$ card$(A \cap pcl_{\omega_t}(V))$ and for similar cause card$(A \cap U) =$ card$(A \cap pcl_{\omega_t}(V))$. Therefore card$(A\cap U) =$ card(A). Hence x is a pre-(ω_r,s)-θ_t-complete adherent point of A.

(iii)⇒(i). Let X be not $P^t_{(\omega_r,s)}$-closed. Then Theorem 4.6 implies that X has a cover Σ by pre-(ω_r,s)-θ_t-open sets of X without any finite subcover. Let ρ be the minimum of the cardinal numbers of the subcover Σ_0 of Σ. Then clearly $\rho \geq \aleph_0$. Let Σ_0 be well-ordered by minimal well-ordering $\prec$. Then for each $U \in \Sigma_0$, card$(\{W \in \Sigma : W \prec U\}) < \rho$ and so $\{W \in \Sigma : W \prec U\}$ can not be a cover of X. Then for each $U \in \Sigma_0$, there exists $x_U \in X - \cup\{W \in \Sigma : W \prec U\}$. Now consider $P = \{x_U : U \in \Sigma_0\}$. Since $U, V \in \Sigma_0, U \neq V$ implies $x_U \neq x_V$, card$(P) = \rho$. So P is an infinite set. Now consider any $x \in X$. Then $x \in U_0$ for some $U_0 \in \Sigma_0$. Since $x_U \in U_0$ implies $U \prec U_0$. Therefore $\{U \in \Sigma_0 : x_U \in U_0\} \subset \{U \in \Sigma_0 : U \prec U_0\}$ and so by the minimality of $\prec$, we get card$(\{U \in \Sigma_0 : x_U \in U_0\}) < \rho$. Thus card$(A \cap U_0) < \rho =$ card(A). So x can not be pre-(ω_r,s)-θ_t-complete adherent point of A. □

5. Pre-$(\omega_r,s)t$-θ_f-cluster sets

Definition 5.1. Let $\varphi : (X,\tau_1,\tau_2) \to (Y,\varrho_1,\varrho_2)$ be a function. Then the pre-

$(\omega_r, s)t$-θ_f-cluster set (resp. $(r,s)t$-θ_f-precluster set) of φ at the point $x \in X$ is the set $P^{\theta_f}_{(\omega_r,s)t}(\varphi, x) = \cap\{cl_{\theta_f}(\varphi(pcl_{\omega_t}(U))) : U \in (r,s)PO(X,x)\}$ (resp. $P^{\theta_f}_{(r,s)t}(\varphi, x) = \cap\{cl_{\theta_f}(\varphi(pcl_t(U))) : U \in (r,s)PO(X,x)\}$).

It is clear that $P^{\theta_f}_{(\omega_r,s)t}(\varphi, x) \subset P^{\theta_f}_{(r,s)t}(\varphi, x)$. The following example establishes that $P^{\theta_f}_{(\omega_r,s)t}(\varphi, x) \neq P^{\theta_f}_{(r,s)t}(\varphi, x)$.

Example 5.2. Consider the same bitopological space X as in Example 3.7 and the identity function $\varphi : X \to X$. Then $P^{\theta_1}_{(\omega_2,1)2}(\varphi, \sqrt{2}) = \cap\{cl_{\theta_1}(\varphi(pcl_{\omega_2}(U))) : U \in (2,1)PO(X, \sqrt{2})\} = \cap\{cl_{\theta_1}(pcl_{\omega_2}(U)) : U \in \tau_2\} = \cap\{cl_{\theta_1}(U) : U \in \tau_2\} = \cap\{U : U \in \tau_2\} = \{\sqrt{2}\}$ and $P^{\theta_1}_{(2,1)2}(\varphi, \sqrt{2}) = X$.

Theorem 5.3. *For any function $\varphi : (X, \tau_1, \tau_2) \to (Y, \varrho_1, \varrho_2)$, following statements are equivalent*:

(i) $y \in P^{\theta_f}_{(\omega_r,s)t}(\varphi, x)$,

(ii) *The filter base $\mathcal{F} = \{\varphi^{-1}(cl_f(U)) : U \in \varrho_f(y)\}$ pre-(ω_r, s)-θ_t-adheres at x,*

(iii) *There exists a grill $\mathcal{G}$ on X pre-(ω_r, s)-θ_t-converging to x and $y \in \cap\{cl_{\theta_f}(\varphi(K)) : K \in \mathcal{G}\}$.*

Proof. (i)⇒(ii). Let $y \in P^{\theta_f}_{(\omega_r,s)t}(\varphi, x)$. It is obvious to note that $\mathcal{F}$ is a filter base on X. If possible, let the filter base $\mathcal{F}$ does not pre-(ω_r, s)-θ_t-adhere at x. Then there exist a $V \in (r,s)PO(X,x)$ and an $U \in \varrho_f(y)$ such that $pcl_{\omega_t}(V) \cap \varphi^{-1}(cl_f(U)) = \emptyset$. So $\varphi(pcl_{\omega_t}(V)) \cap cl_f(U) = \emptyset$. Hence $y \notin cl_{\theta_f}(\varphi(pcl_{\omega_t}(U)))$, which is a contradiction.

(ii)⇒(iii). Consider the family $\mathcal{G} = \{G \subset X : G \cap F \neq \emptyset$ for all $F \in \mathcal{F}\}$. We claim that $\mathcal{G}$ is a grill on X. It is clear that $\mathcal{G}$ is nonempty and does not contained empty set. Let $G \notin \mathcal{G}$ and $H \notin \mathcal{G}$. Then there exist $E, F \in \mathcal{F}$ such that $E \cap G = \emptyset$ and $F \cap H = \emptyset$. But since $\mathcal{F}$ is a filter base on X, there exists $K \in \mathcal{F}$ such that $K \subset E \cap F$ and hence $K \cap (G \cup H) = \emptyset$. So $G \cup H \notin \mathcal{G}$. Again $G \in \mathcal{G}$ and $H \supset G$ imply $H \in \mathcal{G}$ obviously. Hence $\mathcal{G}$ is a grill on X. Let now $V \in (r,s)PO(X,x)$. Then by (ii), $pcl_{\omega_t}(V) \cap F \neq \emptyset$ for all $F \in \mathcal{F}$. So $pcl_{\omega_t}(V) \in \mathcal{G}$. Hence the grill $\mathcal{G}$ pre-(ω_r, s)-θ_t-converges to x. Let $K \in \mathcal{G}$. The definition of $\mathcal{G}$ implies that $\varphi^{-1}(cl_f(U)) \cap K \neq \emptyset$ for all $U \in \varrho_f(y)$. Therefore $cl_f(U) \cap \varphi(K) \neq \emptyset$ for all $U \in \varrho_f(y)$. So $y \in cl_{\theta_f}(\varphi(K))$ for all $K \in \mathcal{G}$.

(iii)⇒(i). Let $U \in (r,s)PO(X,x)$. Then (iii) ensures the existence of a $K \in \mathcal{G}$ such that $K \subset pcl_{\omega_t}(U)$ and $y \in cl_{\theta_f}(\varphi(K))$. Therefore $y \in cl_{\theta_f}(\varphi(pcl_{\omega_t}(U)))$. □

Theorem 5.4. *Let $(Y, \varrho_1, \varrho_2)$ be a pairwise (r,f)-Urysohn space. Then for some bitopological space (X, τ_1, τ_2), and for some function $\varphi : (X, \tau_1, \tau_2) \to (Y, \varrho_1, \varrho_2)$, $P^{\theta_f}_{(\omega_r,s)r}(\varphi, x)$ at every point $x \in X$ is degenerate.*

Proof. Let $(Y, \varrho_1, \varrho_2)$ be a pairwise (r,f)-Urysohn space. If the theorem does not hold, the identity function $I : Y \to Y$, there exists $y \in Y$ such that

$P^{\theta_f}_{(\omega_r,s)r}(\varphi,y)$ is not degenerate. So there exists $y_0(\neq y)\in Y$ such that $y_0\in P^{\theta_f}_{(\omega_r,s)r}(\varphi,y)$. Then for each $U_y\in(r,s)PO(Y,y)$ and $U_{y_0}\in\varrho_f(y_0)$ such that $pcl_{\omega_r}(U_y)\cap(cl_f(U_{y_0}))\neq\emptyset$. Since $\varrho_r\subset P(\omega_r,s)O(X)$, $cl_r(U_y)\cap(cl_f(U_{y_0}))\neq\emptyset$ for each $U_y\in\varrho_r(y)$ and for each $U_{y_0}\in\varrho_f(y_0)$, which is a contradiction. □

Definition 5.5. A function $\varphi:(X,\tau_1,\tau_2)\to(Y,\varrho_1,\varrho_2)$ is called pre-$(\omega_r,s)t$-θ_f-irresolute if $cl_f(\varphi(V))\subset\varphi(pcl_{\omega_t}(V))$ for every $V\subset X$.

Definition 5.6. A function $\varphi:(X,\tau_1,\tau_2)\to(Y,\varrho_1,\varrho_2)$ is called $(r,s)f$-preopen if the image of every (r,s)-preopen set of X is f-open in Y.

Theorem 5.7. *Let $\varphi:(X,\tau_1,\tau_2)\to(Y,\varrho_1,\varrho_2)$ be any pre-$(\omega_r,s)t$-θ_f-irresolute, $(r,s)f$-preopen surjection whose pre-$(\omega_r,s)t$-θ_f-cluster set at every point $x\in X$ is degenerate. Then Y is f-Urysohn space.*

Proof. Let $y_1,y_2\in Y$ and $y_1\neq y_2$. Since φ is surjective, there exists $x_1,x_2\in X$ such that $\varphi(x_1)=y_1$ and $\varphi(x_2)=y_2$. The degeneracy of $P^{\theta_f}_{(\omega_r,s)t}(\varphi,x_1)$ implies that $\varphi(x_2)\notin P^{\theta_f}_{(\omega_r,s)t}(\varphi,x_1)$. So by Theorem 5.3(ii), there exist $V_{x_2}\in(r,s)PO(X,x_2)$ and $U_{y_1}\in\varrho_f(y_1)$ such that $pcl_{\omega_t}(V_{x_2})\cap\varphi^{-1}(cl_f(U_{y_1}))=\emptyset$ i.e., $\varphi(pcl_{\omega_t}(V_{x_2}))\cap cl_f(U_{y_1})=\emptyset$. Thus $cl_f(\varphi(V_{x_2}))\cap cl_f(U_{y_1})=\emptyset$. So the space Y is f-Urysohn. □

Definition 5.8. A function $\varphi:(X,\tau_1,\tau_2)\to(Y,\varrho_1,\varrho_2)$ is called strongly pre-$(\omega_r,s)t$-θ_f-continuous if for each $x\in X$ and for each $U\in\varrho_f(\varphi(x))$, there exists $V\in(r,s)PO(X,x)$ such that $\varphi(pcl_{\omega_t}(V))\subset U$.

Theorem 5.9. *A space (Y,ϱ_1,ϱ_2) is f-Hausdorff if and only if for each space (X,τ_1,τ_2) and for any pre-$(\omega_r,s)t$-θ_f-continuous surjection $\varphi:(X,\tau_1,\tau_2)\to(Y,\varrho_1,\varrho_2)$ pre-$(\omega_r,s)t$-θ_f-cluster set at every point $x\in X$ is degenerate.*

Proof. Let Y be f-Hausdorff and for any X, $\varphi:(X,\tau_1,\tau_2)\to(Y,\varrho_1,\varrho_2)$ be a strongly pre-$(\omega_r,s)t$-θ_f-continuous surjection. Let $x\in X$ and $y\in Y$ with $y\neq\varphi(x)$. Since φ is a strongly pre-$(\omega_r,s)t$-θ_f-continuous surjection, for each $U\in\varrho_f(\varphi(x))$, there exists $V\in(r,s)PO(X,x)$ such that $\varphi(pcl_{\omega_t}(V))\subset U$. Now $P^{\theta_f}_{(\omega_r,s)t}(\varphi,x)=\cap\{cl_{\theta_f}(\varphi(pcl_{\omega_t}(V))):V\in(r,s)PO(X,x)\}\subset\cap\{cl_{\theta_f}(U):U\in\varrho_f(\varphi(x))\}=\cap\{cl_f(U):U\in\varrho_f(\varphi(x))\}$. Again since Y is f-Hausdorff, there exist disjoint sets $Q_1\in\varrho_f(\varphi(x))$ and $Q_2\in\varrho_f(y)$ and so $cl_f(Q_1)\cap Q_2=\emptyset$. Hence $y\notin cl_f(Q_1)$ and so $y\notin P^{\theta_f}_{(\omega_r,s)t}(\varphi,x)$. □

Conversely, let for each space (X,τ_1,τ_2) and for any strongly pre-$(\omega_r,s)t$-θ_f-continuous surjection $\varphi:(X,\tau_1,\tau_2)\to(Y,\varrho_1,\varrho_2)$, $P^{\theta_f}_{(\omega_r,s)t}(\varphi,x)$ is degenerate for each $x\in X$. Let y_1,y_2 be any two distinct points of Y. Then there exist two points $x_1,x_2\in X$ such that $y_1=\varphi(x_1)$ and $y_2=\varphi(x_2)$. The degeneracy of $P^{\theta_f}_{(\omega_r,s)t}(\varphi,x_1)$ implies that $y_2\notin P^{\theta_f}_{(\omega_r,s)t}(\varphi,x_1)$. Hence there exist $V\in(r,s)PO(X,x_1)$ and $U\in\varrho_f(y_2)$ such that $\varphi(pcl_{\omega_t}(V))\cap cl_f(U)=\emptyset$ and so $\varphi(pcl_{\omega_t}(V)) \subset Y - cl_f(U)$. Therefore $Y - cl_f(U)$ and U are required disjoint f-open sets containing y_1 and y_2 respectively and making Y as a f-Hausdorff space.

Theorem 5.10. *Let the bitopological space (X, τ_1, τ_2) be f-almost regular and for a space (X, τ_1, τ_2), $\varphi : (X, \tau_f) \to (Y, \varrho_f)$ be a θ-closed map. If $\varphi^{-1}(y)$ is θ_f-closed in X for each $y \in Y$ and τ_r is finer than τ_f, then $P^{\theta_f}_{(\omega_r,s)f}(\varphi, x)$ is degenerate for each $x \in X$.*

Proof. Since every f-open set is a pre-ω_f-open set of X, $P^{\theta_f}_{(\omega_r,s)f}(\varphi, x) = \cap\{cl_{\theta_f}(\varphi(pcl_{\omega_f}(V))) : V \in (r, s)PO(X, x)\} \subset \cap\{cl_{\theta_f}(\varphi(cl_f(V))) : V \in (r, s)PO(X, x)\} \subset \cap\{cl_{\theta_f}(\varphi(cl_{\theta_f}(V))) : V \in (r, s)PO(X, x)\}$. Since X is f-almost regular and $\varphi : (X, \tau_f) \to (Y, \varrho_f)$ is a θ-closed map, $cl_{\theta_f}(\varphi(cl_{\theta_f}(V)) = \varphi(cl_{\theta_f}(V))$ for each $V \in (r, s)PO(X, x)$. So $P^{\theta_f}_{(\omega_r,s)f}(\varphi, x) \subset \cap\{\varphi(cl_{\theta_f}(V)) : V \in (r, s)PO(X, x)\}$. Let $x \in X$ and $y \in Y$ with $y \neq \varphi(x)$. Then θ_f-closedness of $\varphi^{-1}(y)$ ensures the existence of a $G \in \tau_f(x)$ such that $cl_f(G) \cap \{\varphi^{-1}(y)\} = \emptyset$ and then the f-openness of G gives $\varphi(cl_f(G)) \cap \{y\} = \varphi(cl_{\theta_f}(G)) \cap \{y\} = \emptyset$. Since $\tau_f \subset \tau_r \subset (r, s)PO(X, x)$, $y \notin P^{\theta_f}_{(\omega_r,s)f}(\varphi, x)$. □

Theorem 5.11. *Let space (X, τ_1, τ_2) be f-almost regular f-Hausdorff and for a space (X, τ_1, τ_2), $\varphi : (X, \tau_f) \to (Y, \varrho_f)$ be a θ-closed injection. If τ_r is finer than τ_f, then $P^{\theta_f}_{(\omega_r,s)f}(\varphi, x)$ is degenerate for each $x \in X$.*

Proof. The f-almost regularity of X and θ-closedness of φ (as in Theorem 5.10) imply that $P^{\theta_f}_{(\omega_r,s)f}(\varphi, x) \subset \cap\{\varphi(cl_{\theta_f}(V)) : V \in (r, s)PO(X, x)\}$. Let any $x' \in X$ with $x \neq x'$. Since X is f-Hausdorff, there exist $H \in \tau_f(x) \subset \tau_r(x) \subset (r, s)PO(X, x)$ such that $x' \notin cl_{\theta_f}(H)$ and so $\varphi(x') \notin \varphi(cl_{\theta_f}(H))$. Hence $\varphi(x') \notin P^{\theta_f}_{(\omega_r,s)f}(\varphi, x)$. □

Definition 5.12. Let $\Omega : (X, \tau_1, \tau_2) \to (Y, \varrho_1, \varrho_2)$ be a multifunction. Then the pre-$(\omega_r, s)t$-θ_f-cluster set of Ω at the point $x \in X$ is the set $P^{\theta_f}_{(\omega_r,s)t}(\Omega, x) = \cap\{cl_{\theta_f}(\Omega(pcl_{\omega_t}(U))) : U \in (r, s)PO(X, x)\}$. For any $M \subset X$, the symbol $P^{\theta_f}_{(\omega_r,s)t}(\Omega, M)$ is the set $\cup\{P^{\theta_f}_{(\omega_r,s)t}(\Omega, x) : x \in M\}$.

Definition 5.13. A multifunction $\Omega : (X, \tau_1, \tau_2) \to (Y, \varrho_1, \varrho_2)$ is said to have pre-$(\omega_r, s)t$-θ_f-closed graph if for each $(x, y) \notin G(\Omega)$, there exist $U \in (r, s)PO(X, x)$ and $V \in \varrho_f(y)$ such that $(pcl_{\omega_t}(U) \times cl_f(V)) \cap G(\Omega) = \emptyset$.

Theorem 5.14. *For any multifunction $\Omega : (X, \tau_1, \tau_2) \to (Y, \varrho_1, \varrho_2)$, following two conditions are equivalent*:

(a) $P^{\theta_f}_{(\omega_r,s)t}(\Omega, x) = \Omega(x)$ *for each* $x \in X$,

(b) Ω *has pre-*$(\omega_r, s)t$-θ_f*-closed graph.*

Proof. (a)⇒(b). Let $(x,y)\notin G(\Omega)$. Then $y\notin\Omega(x)$. Hence (a) ensures the existence of $U\in(r,s)PO(X,x)$ and $V\in\varrho_f(y)$ such that $\Omega(pcl_{\omega_t}(U))\cap cl_f(V)=\emptyset$. Thus $(pcl_{\omega_t}(U)\times cl_f(V))\cap G(\Omega)=\emptyset$. So Ω has pre-$(\omega_r,s)t$-θ_f-closed graph.

(b)⇒(a). Let Ω have pre-$(\omega_r,s)t$-θ_f-closed graph. Let $y\notin\Omega(x)$. Then $(x,y)\notin G(\Omega)$. Since Ω has pre-$(\omega_r,s)t$-θ_f-closed graph, there exist $U\in(r,s)PO(X,x)$ and $V\in\varrho_f(y)$ such that $(pcl_{\omega_t}(U)\times cl_f(V))\cap G(\Omega)=\emptyset$ and hence $\Omega(pcl_{\omega_t}(U))\cap cl_f(V)=\emptyset$. Thus $y\notin P^{\theta_f}_{(\omega_r,s)t}(\Omega,x)$ and so $P^{\theta_f}_{(\omega_r,s)t}(\Omega,x)=\Omega(x)$. □

Theorem 5.15. *If a multifunction $\Omega:(X,\tau_1,\tau_2)\to(Y,\varrho_1,\varrho_2)$ has θ-closed graph in the product space $(X,\tau_f)\times(Y,\varrho_f)$, then $P^{\theta_f}_{(\omega_r,s)f}(\Omega,x)=\Omega(x)$ for each $x\in X$.*

Proof. Let $x\in X$ and $y\in P^{\theta_f}_{(\omega_r,s)f}(\Omega,x)$. Then for each $U\in(r,s)PO(X,x)$ and for each $V\in\varrho_f(y)$, $\Omega(pcl_{\omega_f}(U))\cap cl_f(V)\neq\emptyset$ and so $cl_f(U)\cap\Omega^-(cl_f(V))\supset pcl_{\omega_f}(U)\cap\Omega^-(cl_f(V))\neq\emptyset$. Hence in particular there exist open sets $M\in\tau_f(x)$ and $N\in\varrho_f(y)$ such that $cl_f(M)\cap\Omega^-(cl_f(N))\neq\emptyset$ and so $cl_f(M\times N)\cap G(\Omega)=(cl_f(M)\times cl_f(N))\cap G(\Omega)\neq\emptyset$. So $(x,y)\in cl_\theta(G(\Omega))=G(\Omega)$. Hence $y\in\Omega(x)$. Hence $P^{\theta_f}_{(\omega_r,s)f}(\Omega,x)=\Omega(x)$. □

Theorem 5.16. *If a bitopological space X satisfies any one of the following two conditions*:

(a) *For any bitopological space (Y,ϱ_1,ϱ_2) and any multifunction $\Omega:(X,\tau_1,\tau_2)\to(Y,\varrho_1,\varrho_2)$, $\cap\{r\text{-}cl_{\theta_t}(\Omega(U)):U\in(r,s)PO(X,Q)\}\subset P^{\theta_t}_{(\omega_r,s)t}(\Omega,Q)$ for each pre-(ω_r,s)-θ_t-closed subset Q of X,*

(b) *for any any bitopological space (Y,ϱ_1,ϱ_2) and any multifunction $\Omega:(X,\tau_1,\tau_2)\to(Y,\varrho_1,\varrho_2)$, $\cap\{p_{(\omega_r,s)}cl_{\theta_t}(\Omega(U)):U\in(r,s)PO(X,Q)\}\subset P^{\theta_t}_{(\omega_r,s)t}(\Omega,Q)$ for each pre-(ω_r,s)-θ_t-closed subset Q of X, then X is $P^t_{(\omega_r,s)}$-closed.*

Proof. Since for any subset $A\subset X$, $p_{(\omega_r,s)}cl_{\theta_t}(\Omega(A))\subset r\text{-}cl_{\theta_t}(\Omega(A))$, (a) implies (b). Now let (b) satisfies and $\mathcal{F}$ be a filter base. Consider a point $q\notin X$ and the set $Y=X\cup\{q\}$. Then the family $\varrho_1=\varrho_2=P(X)\cup\{U\subset Y:q\in U$ and $F\in U$ for some $F\in\mathcal{F}\}$ is a topology on Y [15]. Then for the identity function $\varphi:(X,\tau_1,\tau_2)\to(Y,\varrho_1,\varrho_1)$, we shall show that $P^{\theta_t}_{(\omega_r,s)t}(\varphi,X)=p_{(\omega_r,s)}cl_{\theta_t}(X)$. Here (b) implies that $P^{\theta_t}_{(\omega_r,s)t}(\varphi,X)\supset\cap\{p_{(\omega_r,s)}cl_{\theta_t}(\varphi(U)):U\in(r,s)PO(Y,X)\}=\cap\{p_{(\omega_r,s)}cl_{\theta_t}(U):U\in(r,s)PO(Y,X)\}=p_{(\omega_r,s)}cl_{\theta_t}(X)$(in (Y,ϱ_1,ϱ_1)). To show $P^{\theta_t}_{(\omega_r,s)t}(\varphi,X)\subset p_{(\omega_r,s)}cl_{\theta_t}(X)$, it is sufficient to prove that $q\in p_{(\omega_r,s)}cl_{\theta_t}(X)$ (in (Y,ϱ_1,ϱ_1)). It is clear that $\{q\}\notin\delta_Y$. Again we note that $int_r(cl_s(\{q\}))=\emptyset$ and so $\{q\}\notin(r,s)PO(Y,p)$. Thus $pcl_{\omega_t}(G)\cap X\neq\emptyset$ for every $G\in(r,s)PO(Y,q)$ and hence $q\in p_{(\omega_r,s)}cl_{\theta_t}(X)$. Therefore $q\in P^{\theta_t}_{(\omega_r,s)t}(\varphi,x)$ for some $x\in X$. Now consider $F\in\mathcal{F}$ and $V\in(r,s)PO(X,x)$. Since $Y-(F\cup\{q\}),F\cup\{q\}\in\varrho_t$, $cl_t(F\cup\{q\})=F\cup\{q\}$. Again since $q\in P^{\theta_t}_{(\omega_r,s)t}(\varphi,x)$, Theorem 5.3 ensures that $\varphi^{-1}(cl_t(F\cup\{q\}))\cap pcl_{\omega_t}(V)\neq\emptyset$

i.e., $cl_t(F \cup \{q\}) \cap \varphi(pcl_{\omega_t}(V)) \neq \emptyset$. Therefore $pcl_{\omega_t}(V) \cap F = \varphi(pcl_{\omega_t}(V)) \cap (F \cup \{q\}) = \varphi(pcl_{\omega_t}(V)) \cap cl_t(F \cup \{q\}) \neq \emptyset$. Hence by the Theorem 4.4, X is $P^t_{(\omega_r,s)}$-closed. □

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