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ON RANK ONE PERTURBATIONS OF THE UNILATERAL SHIFT

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Abstract. In this paper we study some properties of rank one perturbations of the unilateral shift operators $T = S+u\otimes v$. In particular, we give some criteria for eigenvalues of T. Also we characterize some conditions for T to be hyponormal.

1. Introduction

Let $\mathcal{H}$ be a separable complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. If $T \in \mathcal{L}(\mathcal{H})$, we write $\sigma(T)$, and $\sigma_p(T)$ for the spectrum, and the point spectrum of T, respectively.

An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *quasinormal* if T and T^*T commute. Also, if $T = U|T|$ is the polar decomposition of T, then T is quasinormal if and only if U and $|T|$ commute. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *hyponormal* if $T^*T \geq TT^*$ where T^* is the adjoint of T. It is known that the class of hyponormal operators is a larger class containing normal and quasinormal operators. A *spectral* operator is an operator with a countably additive resolution of the identity defined on the Borel sets of the plane (see [6]) and an operator $T \in \mathcal{L}(\mathcal{H})$ is *hypercyclic* if there is a vector $x \in \mathcal{H}$ with dense orbit $\{x, Tx, T^2x, \ldots\}$ (see [10]).

If u and v are nonzero vectors in $\mathcal{H}$, we write $u \otimes v$ for the operator of the *rank one* defined by

$$(u \otimes v)x = \langle x,\ v\rangle u,\ x \in \mathcal{H},$$

where $\langle , \rangle$ denotes the inner product of the Hilbert space $\mathcal{H}$.

Let $\{e_n\}_{n=0}^{\infty}$ denote an orthonormal basis for $\mathcal{H}$ which will remain fixed throughout this paper and let $S \in \mathcal{L}(\mathcal{H})$ be the unilateral shift of multiplicity one defined by $Se_n = e_{n+1}$ for $n = 0, 1, \ldots$. Throughout the paper we suppose

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that u and v are nonzero vectors in $\mathcal{H}$ and their expansions with respect to the orthonormal basis $\{e_n\}_{n=0}^{\infty}$ are

$$u = \sum_{n=0}^{\infty} a_n e_n \quad \text{and} \quad v = \sum_{n=0}^{\infty} b_n e_n,$$

where a_n and b_n are nonzero coefficients for all nonnegative integer n.

We say that an operator $T \in \mathcal{L}(\mathcal{H})$ is *a rank one perturbation of an operator* if there exist the vectors u and v (defined above) in the Hilbert space $\mathcal{H}$ such that $T = S + u \otimes v$. E. Ionascu has studied the several properties of rank one perturbations of diagonal operators (see [8]). It is natural to consider the rank one perturbations of subnormal operators. Also, it is unknown whether these operators have nontrivial invariant subspaces. As the special case of these operators, we study some properties of rank one perturbations of the unilateral shift $T = S + u \otimes v$. In particular, we give some criteria for eigenvalues of T. Also we characterize some conditions for T to be hyponormal.

2. Main results

First, we give some criteria for eigenvalues of $T = S + u \otimes v$.

Theorem 2.1. *Let $T = S + u \otimes v$. Then $0 \notin \sigma_p(T)$. Furthermore, a nonzero point $\mu \in \mathbb{C}$ is an eigenvalue of T if and only if*

(i) $\sum_{n=0}^{\infty} \sum_{j=0}^{n} \frac{a_{n-j}\bar{b}_n}{\mu^{j+1}} = 1$ *and*

(ii) $\sum_{n=0}^{\infty} |\sum_{j=0}^{n} \frac{a_{n-j}}{\mu^{j+1}}|^2 < \infty$.

Proof. If $Tx = 0$, then $(S + u \otimes v)x = Sx + \langle x, v\rangle u = 0$. Hence we obtain the following equation

$$\sum_{n=0}^{\infty} x_n e_{n+1} + \langle x, v\rangle \sum_{n=0}^{\infty} a_n e_n = 0,$$

where $x = \sum_{n=0}^{\infty} x_n e_n$. If we solve this equation, then we get $\langle x, v\rangle = 0$. Therefore $\sum_{n=0}^{\infty} x_n e_{n+1} = 0$. Thus $x = 0$. Hence $0 \notin \sigma_p(T)$.

Assume that $\mu \neq 0$. If $\mu \in \sigma_p(T)$, then there exists a nonzero vector y such that $Ty = \mu y$. Thus

$$\langle y, v\rangle u = (\mu - S)y.$$

If $\langle y, v\rangle = 0$, then $Sy = \mu y$. It is a contradiction since $\sigma_p(S) = \emptyset$. Thus $\langle y, v\rangle \neq 0$. Set $x = -\frac{1}{\langle y,v\rangle} y$. Then $x \neq 0$ and

$$(S - \mu)x = -\frac{1}{\langle y, v\rangle}(S - \mu)y = u.$$

Hence $u \in \operatorname{ran}(S - \mu)$ and $\langle x, v\rangle + 1 = 0$. Thus

$$0 = \langle x, v\rangle + 1 = \langle \sum_{n=0}^{\infty} x_n e_n, \sum_{m=0}^{\infty} b_m e_m\rangle + 1 = \sum_{n=0}^{\infty} x_n \bar{b}_n + 1. \tag{1}$$

Since $u = (S-\mu)x$, the equation $\sum_{n=0}^{\infty} a_n e_n = (S-\mu)\sum_{n=0}^{\infty} x_n e_n$ holds. Hence

$$\begin{cases} a_0 = -\mu x_0 \\ a_n = x_{n-1} - \mu x_n, \quad n = 1, 2, \dots . \end{cases}$$

If we solve this system with respect to x_n, then we obtain the following equations:

$$x_n = -\sum_{j=0}^{n} \frac{a_{n-j}}{\mu^{j+1}}, \quad n = 0, 1, 2, \dots . \tag{2}$$

If we substitute (2) into (1), we get that

$$1 = \sum_{n=0}^{\infty}\sum_{j=0}^{n} \frac{a_{n-j}\bar{b}_n}{\mu^{j+1}}.$$

In order to get (ii), from $\|x\| < \infty$ and (2) we have

$$\infty > \|x\|^2 = \sum_{n=0}^{\infty} |x_n|^2 = \sum_{n=0}^{\infty} |\sum_{j=0}^{n} \frac{a_{n-j}}{\mu^{j+1}}|^2.$$

Conversely, if (i) and (ii) hold, set

$$x = -\sum_{n=0}^{\infty}\left(\sum_{j=0}^{n} \frac{a_{n-j}}{\mu^{j+1}}\right) e_n.$$

Then we obtain the following equations

$$\begin{aligned} \langle x, v\rangle + 1 &= \langle -\sum_{n=0}^{\infty}\left(\sum_{j=0}^{n} \frac{a_{n-j}}{\mu^{j+1}}\right) e_n, \sum_{m=0}^{\infty} b_m e_m\rangle + 1 \\ &= -\sum_{n=0}^{\infty}\left(\sum_{j=0}^{n} \frac{a_{n-j}}{\mu^{j+1}}\right) \bar{b}_n + 1 = 0. \end{aligned}$$

Hence $x \neq 0$ and

$$(S-\mu)x = (S-\mu)\left(-\sum_{n=0}^{\infty}\left(\sum_{j=0}^{n} \frac{a_{n-j}}{\mu^{j+1}}\right) e_n\right) = u.$$

Therefore

$$Tx = Sx + \langle x, v\rangle u = \mu x.$$

Thus $\mu \in \sigma_p(T)$. □

Corollary 2.2. *Let $T = S + u \otimes v$. If $\mu \in \sigma(T)$ and $|\mu| > 1$, then $\mu \in \sigma_p(T)$. Hence* (i) *and* (ii) *in Theorem 2.1 hold.*

Proof. Assume that $\mu \in \sigma(T)$ and $|\mu| > 1$. Since $\sigma(S) = \overline{\mathbb{D}}$ where $\mathbb{D}$ is the unit disc, it follows that $\mu \notin \sigma(S)$. Hence $S - \mu$ is invertible. Since $T - \mu = S - \mu + u \otimes v = (S - \mu)[1 + (S - \mu)^{-1} u \otimes v]$, $(S - \mu)^{-1} u \otimes v$ is a rank one operator, and $T - \mu$ is not invertible, it follows that $-1 \in \sigma_p((S - \mu)^{-1} u \otimes v)$. Thus there exists a nonzero eigenvector x such that $((S - \mu)^{-1} u \otimes v)x = -x$. So we get that $(u \otimes v)x = -(S - \mu)x$. Thus $(T - \mu)x = (S - \mu)x + (u \otimes v)x = 0$. Hence $\mu \in \sigma_p(T)$. Thus (i) and (ii) in Theorem 2.1 hold. □

Corollary 2.3. *Let $T = S + u \otimes v$. If*

$$F(\mu) = \sum_{n=0}^{\infty} \sum_{j=0}^{n} \frac{a_{n-j} \bar{b}_n}{\mu^{j+1}}$$

for all nonzero $\mu \in \sigma_p(T)$, then $F(\mu)$ converges absolutely.

Proof. Since

$$\sum_{n=0}^{\infty} \left| \sum_{j=0}^{n} \frac{a_{n-j} \bar{b}_n}{\mu^{j+1}} \right| \leq \left(\sum_{n=0}^{\infty} \left| \sum_{j=0}^{n} \frac{a_{n-j}}{\mu^{j+1}} \right|^2 \right)^{\frac{1}{2}} \left(\sum_{n=0}^{\infty} |b_n|^2 \right)^{\frac{1}{2}} < \infty$$

for all nonzero $\mu \in \sigma_p(T)$ by Theorem 2.1, $F(\mu)$ converges absolutely. □

As an application of Theorem 2.1, we obtain the following example.

Example 2.4. Let

$$T = S + \left(\sum_{n=0}^{\infty} \frac{1}{2^n} e_n \right) \otimes \left(\sum_{n=0}^{\infty} \frac{1}{n+1} e_n \right).$$

Then $\{2\} \subset \sigma_p(T)$.

Proof. We want to show that (i) and (ii) in Theorem 2.1 hold.

$$\sum_{n=0}^{\infty} \sum_{j=0}^{n} \frac{\frac{1}{2^{n-j}} \frac{1}{n+1}}{2^{j+1}} = \sum_{n=0}^{\infty} \sum_{j=0}^{n} \frac{1}{(n+1)2^{n+1}} = 1.$$

Moreover,

$$\sum_{n=0}^{\infty} \left| \sum_{j=0}^{n} \frac{\frac{1}{2^{n-j}}}{2^{j+1}} \right|^2 = \sum_{n=0}^{\infty} \left(\frac{n+1}{2^{n+1}} \right)^2 < \infty.$$

Hence $2 \in \sigma_p(T)$ by Theorem 2.1. □

Next, we show that a rank one perturbation of the unilateral shift S is not quasinormal. In [7, Problem 184] Halmos initiated that if U is the unilateral shift, does there exist a compact operator C such that $U + C$ is normal? He has proved that there is no rank one operator $u \otimes v$ such that $T = S + u \otimes v$ is normal where S is the unilateral shift. Next we will consider the case of quasinormality of T.

Proposition 2.5. *Let $T = S + u \otimes v$. Then T is not quasinormal.*

Proof. If $T = S + u \otimes v$ is quasinormal, then

$$\begin{aligned} S^*S &= T^*T - (T^*u \otimes v) - (v \otimes T^*u) + \| u \|^2 (v \otimes v) \\ &= T^*T - D, \end{aligned}$$

where $D = (T^*u \otimes v) + (v \otimes T^*u) - \| u \|^2 (v \otimes v)$. Then D is an operator of finite rank. Since $T^*T - I = D$, it ensures that $\sigma(T^*T)\backslash\{1\} \subset \sigma_p(T^*T)$ by [7]. Since a Hermitian operator on a separable Hilbert space can have only countably many eigenvalues, it holds that $\sigma_p(T^*T)$ is countable. Since $\sigma(S)$ is the closed unit disc and $\sigma_p(S) = \emptyset$, $\sigma(T)$ can differ from the closed unit disc by the set of eigenvalues of T only (i.e., $\sigma(T) \setminus \overline{\mathbb{D}} = \sigma_p(T)$) by [7]. It is known from [1] that if T is quasinormal, then T is unitarily equivalent to $N \oplus (P \otimes S)$ where P is positive and N is normal. Then $\sigma_p(T) = \sigma_p(N) \cup \sigma_p(P \otimes S)$. Since $\sigma_p(P \otimes S) = \sigma_p(P)\sigma_p(S)$ by [2] and $\sigma_p(S) = \emptyset$, it follows that $\sigma_p(P \otimes S) = \emptyset$. Thus $\sigma_p(T) = \sigma_p(N)$. Since a normal operator on a separable Hilbert space can have only countably many eigenvalues, $\sigma_p(N)$ is countable. Hence $\sigma(T) \setminus \overline{\mathbb{D}}$ is countable and $\sigma(T^*T) \setminus [0, 1]$ is countable. Thus $\sigma(T^*T) = [0, 1]$ module countable sets. It contradicts to the countability of $\sigma(T^*T)$. □

Corollary 2.6. *If $T = S + u \otimes v$, then* (i) *neither T^* nor T is spectral, and* (ii) *T is not hypercyclic.*

Proof. (i) Since $\sigma_p(T^*)$ contains $\mathbb{D}$ by [14], it is uncountable. Hence by [6], T^* is not spectral. Assume that $T = S + u \otimes v$ is spectral. Then $T^*T - I = F$ where $F = v \otimes T^*u + S^*u \otimes v$ is an operator of finite rank. With the same argument as in the proof of Theorem 2.5, we get a contradiction.

(ii) If T is hypercyclic, then $\sigma_p(T^*) = \emptyset$ by [10]. But since $\sigma_p(T^*)$ contains $\mathbb{D}$ by [14], we have a contradiction. □

An arbitrary operator $T \in \mathcal{L}(\mathcal{H})$ has a unique polar decomposition $T = U|T|$, where $|T| = (T^*T)^{\frac{1}{2}}$ and U is the appropriate partial isometry. Associated with T there is a related operator $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$, called the *Aluthge transform* of T (see [9]).

Corollary 2.7. *If $T = S + u \otimes v$, then $T \neq \tilde{T}$.*

Proof. If $T = \tilde{T}$, then T must be quasinormal by [9]. So we have a contradiction. □

Next we give a sufficient and necessary condition for rank one perturbations of the unilateral shift to be hyponormal.

Theorem 2.8. (i) *Let u and v be linearly dependent. Then $T = S + u \otimes v$ is hyponormal if and only if*

$$\mathrm{Re}(\langle x, v\rangle\langle w, x\rangle) \geq -\frac{1}{2}|\langle x, e_0\rangle|^2$$

holds where $w = \alpha S^* v - \bar{\alpha} S v$ *and* $\alpha = \frac{\langle u,v\rangle}{\|v\|^2}$.

(ii) *Let* u *and* v *be linearly independent. Then* $T = S + u \otimes v$ *is hyponormal if and only if for all* $x = y + z$ *with* $y \in \vee\{u, v\}$ *and* $z \in (\vee\{u, v\})^\perp$

$$\mathrm{Re}\langle t, x\rangle + \frac{1}{2}\left(\|u\|^2 |\langle y, v\rangle|^2 - \|v\|^2|\langle y, u\rangle|^2\right) \geq -\frac{1}{2}|\langle x, e_0\rangle|^2$$

holds where $t = \langle x, S^* u\rangle v - \langle x, u\rangle S v$.

Proof. (i) If u and v are linearly dependent, then there exists $\alpha \in \mathbb{C}$ such that $u = \alpha v$. Since $\alpha = \frac{\langle u,v\rangle}{\|v\|^2}$,

$$\| v \|^2 u \otimes u = | \alpha |^2 \| v \|^2 v \otimes v = \| u \|^2 v \otimes v.$$

Set $w = \alpha S^* v - \bar{\alpha} S v$. Then

$$\begin{aligned} \langle (T^*T - TT^*)x, x\rangle &= \langle (S^*S - SS^*)x, x\rangle + \langle (w \otimes v)x, x\rangle + \langle (v \otimes w)x, x\rangle \\ &= \langle (S^*S - SS^*)x, x\rangle + 2\mathrm{Re}\langle (w \otimes v)x, x\rangle \\ &= \langle (S^*S - SS^*)x, x\rangle + 2\mathrm{Re}(\langle x, v\rangle\langle w, x\rangle) \end{aligned}$$

for all $x \in \mathcal{H}$. Hence T is hyponormal if and only if

$$\mathrm{Re}(\langle x, v\rangle\langle w, x\rangle) \geq -\frac{1}{2}\langle (S^*S - SS^*)x, x\rangle = -\frac{1}{2}|\langle x, e_0\rangle|^2.$$

(ii) Suppose that u and v are linearly independent. Set $t = \langle x, S^*u\rangle v - \langle x, u\rangle S v$. Then $T = S + u \otimes v$ is hyponormal if and only if for any $x \in \mathcal{H}$

$$\begin{aligned} &\langle (T^*T - TT^*)x, x\rangle \\ &= \langle [(S^*S - SS^*)x + (S^*u \otimes v)x + (v \otimes S^*u)x + (\| u \|^2\, v \otimes v)x \\ &\quad - (Sv \otimes u)x - (u \otimes Sv)x - (\| v \|^2\, u \otimes u)x], x\rangle \\ &= \langle (S^*S - SS^*)x, x\rangle + \langle x, S^*u\rangle\langle v, x\rangle - \langle x, u\rangle\langle Sv, x\rangle \\ &\quad + \langle x, v\rangle\langle S^*u, x\rangle - \langle u, x\rangle\langle x, Sv\rangle + \|u\|^2|\langle x, v\rangle|^2 - \|v\|^2|\langle x, u\rangle|^2 \\ &= \langle (S^*S - SS^*)x, x\rangle + \langle \langle x, S^*u\rangle v - \langle x, u\rangle Sv, x\rangle \\ &\quad + \langle x, \overline{\langle S^*u, x\rangle} v - \overline{\langle u, x\rangle} Sv\rangle + \|u\|^2|\langle x, v\rangle|^2 - \|v\|^2|\langle x, u\rangle|^2 \\ &= \langle (S^*S - SS^*)x, x\rangle + 2Re\langle t, x\rangle + \left(\|u\|^2|\langle x, v\rangle|^2 - \|v\|^2|\langle x, u\rangle|^2\right) \\ &= |\langle x, e_0\rangle|^2 + 2Re\langle t, x\rangle + \left(\|u\|^2|\langle x, v\rangle|^2 - \|v\|^2|\langle x, u\rangle|^2\right) \geq 0. \end{aligned} \tag{3}$$

Let $\mathcal{H} = \vee\{u, v\} \oplus (\vee\{u, v\})^\perp$ where $\vee\{u, v\} := \mathrm{span}\{u, v\}$. If $x = y + z$ with some $y \in \vee\{u, v\}$ and $z \in (\vee\{u, v\})^\perp$, then it clear that $\langle u, z\rangle = \langle v, z\rangle = 0$. Hence from (3) we get that $T = S + u \otimes v$ is hyponormal if and only if

$$|\langle x, e_0\rangle|^2 + 2\mathrm{Re}\langle t, x\rangle + \left(\|u\|^2|\langle y, v\rangle|^2 - \|v\|^2|\langle y, u\rangle|^2\right) \geq 0.$$

So we complete our proof. □

Corollary 2.9. *Let* u *and* v *be linearly independent with* $\| u \| = \| v \| = 1$ *and let* $x = cu + dv + z$ *for some constant* c, d *with* $| d | \geq | c |$, *and* $z \in (\vee\{u, v\})^\perp$.

If $\langle x, S^*u\rangle v = \langle x, u\rangle Sv$ or $x = \langle x, S^*u\rangle v - \langle x, u\rangle Sv$, then $T = S + u \otimes v$ is hyponormal.

Proof. Suppose that u and v are linearly independent with $\| u \|=\| v \|= 1$. Let $x = cu + dv + z$ be with $| d |\geq| c |$ and $z \in (\vee\{u, v\})^{\perp}$. Then $c = \langle y, u\rangle$ and $d = \langle y, v\rangle$ where $y = cu + dv$. Since $| d |\geq| c |$, it clear that $|\langle y, v\rangle|^2 \geq |\langle y, u\rangle|^2$. Set $t = \langle x, S^*u\rangle v - \langle x, u\rangle Sv$. If $\langle x, S^*u\rangle v = \langle x, u\rangle Sv$ or $x = \langle x, S^*u\rangle v - \langle x, u\rangle Sv$, then $t = x$ or $t = 0$. Hence

$$\mathrm{Re}\langle t, x\rangle + \frac{1}{2}\left(|\langle y, v\rangle|^2 - |\langle y, u\rangle|^2\right) + \frac{1}{2}|\langle x, e_0\rangle|^2 \geq 0.$$

Therefore, T is hyponormal from Theorem 2.8. □

Corollary 2.10. *Let $T = S + u \otimes v$ be where u and v are linearly dependent. If $\gamma v = \alpha S^*v - \bar{\alpha} Sv$ with $\alpha = \frac{\langle u,v\rangle}{\|v\|^2}$, then $\gamma = i\frac{2}{\|v\|^4}\mathrm{Im}\{\langle u, v\rangle\langle v, Sv\rangle\}$ and T is hyponormal.*

Proof. If $\gamma v = \alpha S^*v - \bar{\alpha} Sv$ with $\alpha = \frac{\langle u,v\rangle}{\|v\|^2}$, then $\gamma\| v \|^2 = \alpha\langle S^*v, v\rangle - \overline{\alpha}\langle Sv, v\rangle$. It follows that

$$\begin{aligned} \gamma\| v \|^4 &= \langle u, v\rangle\langle S^*v, v\rangle - \overline{\langle u, v\rangle\langle v, Sv\rangle} \\ &= 2i\mathrm{Im}\{\langle u, v\rangle\langle v, Sv\rangle\}. \end{aligned}$$

So we have $\gamma = \frac{2i}{\|v\|^4}\mathrm{Im}\{\langle u, v\rangle\langle v, Sv\rangle\}$. Since u and v are linearly dependent, it suffices to show that $\mathrm{Re}\{\langle x, v\rangle\langle \gamma v, x\rangle\} \geq 0$. In fact, $\langle x, v\rangle\langle \gamma v, x\rangle = \gamma \mid \langle x, v\rangle \mid^2$. Hence $\mathrm{Re}\{\langle x, v\rangle\langle \gamma v, x\rangle\} = 0$. □

Example 2.11. Let $u = \sum_{n=0}^{\infty} \frac{1}{n+1} e_n$ and $v = \sum_{n=0}^{\infty} \frac{2i}{n+1} e_n$. Then

$$\gamma = i\frac{2}{\| v \|^4}\mathrm{Im}\{\langle u, v\rangle\langle v, Sv\rangle\} = -i\frac{6}{\pi^2}$$

and $S + \left(\sum_{n=0}^{\infty} \frac{1}{n+1} e_n\right) \otimes \left(\sum_{n=0}^{\infty} \frac{2i}{n+1} e_n\right)$ is hyponormal.

Next we now turn to some properties of some operators in $\{T = S + u \otimes v\}'$ where $\{T\}' = \{A \in \mathcal{L}(\mathcal{H}) \mid TA = AT\}$. We start with the following theorem.

Proposition 2.12. *Let $T = S + u \otimes v$. If $A \in \{T\}'$ is a diagonal normal operator such that $Ae_k = \gamma_k e_k$ for all $k \geq 0$, then either $A = \gamma I$ for some constant γ or there are at most finitely many j satisfying $a_{j+1}\bar{b}_j = -1$.*

Proof. Since A and T commute, i.e., $AT = TA$, we obtain that for all $k \geq 0$

$$\langle (AS - SA)e_k, e_{k+1}\rangle = \langle e_k, A^*v\rangle\langle u, e_{k+1}\rangle - \langle e_k, v\rangle\langle Au, e_{k+1}\rangle.$$

Since $\langle (AS - SA)e_k, e_{k+1}\rangle = \langle Ae_{k+1}, e_{k+1}\rangle - \langle Ae_k, e_k\rangle$, it follows that

$$(1 + a_{k+1}\bar{b}_k)(\gamma_{k+1} - \gamma_k) = 0 \text{ for all } k \geq 0.$$

Then $\gamma_k = \gamma_{k+1}$ or $a_{k+1}\bar{b}_k = -1$ for all $k \geq 0$. If $\gamma_k = \gamma_{k+1}$ for all $k \geq 0$, then $A = \gamma I$ for some constant γ. Otherwise, $\gamma_{j+1} \neq \gamma_j$ for some j. If there are at most finitely many j such that $\gamma_{j+1} \neq \gamma_j$, then $a_{j+1}\bar{b}_j = -1$ for such j. If there are infinitely many j such that $\gamma_{j+1} \neq \gamma_j$, then $a_{j+1}\bar{b}_j = -1$ for such infinitely many j. But, since $\sum_{k=0}^{\infty} |a_k|^2 < \infty$ and $\sum_{k=0}^{\infty} |b_k|^2 < \infty$, $a_{j+1}\bar{b}_j \neq -1$ for sufficiently large j. So we have a contradiction. Thus there are at most finitely many j satisfying $a_{j+1}\bar{b}_j = -1$. So we complete the proof. □

The following example is an application of Theorem 2.12.

Example 2.13. Suppose that u and v are nonzero vectors in $\mathcal{H}$ and their expansions with respect to the orthonormal basis $\{e_k\}_{k=0}^{\infty}$ are $u = v = \sum_{k=0}^{\infty} \frac{1}{2^k} e_n$. Since $\frac{1}{2^{k+1}} \frac{1}{2^k} \neq -1$ for every nonnegative integer k, Theorem 2.12 implies that the commutant of $S + u \otimes v$ is γI for some constant γ.

The following proposition is very useful to decide whether the unilateral weighted shift in the commutant of $T = S + u \otimes v$ is hyponormal or not.

Proposition 2.14. *Let $W \in \mathcal{L}(\mathcal{H})$ be the unilateral weighted shift with positive weight sequence $\{\gamma_k\}$ defined by $We_k = \gamma_k e_{k+1}$ for $k = 0, 1, 2, \ldots$. If $W \in \{T\}'$ where $T = S+u\otimes v$, then for every nonnegative integer k, the following equation holds*:

$$\gamma_{k+1}(1 + a_{k+1}\bar{b}_k) = \gamma_k(1 + a_{k+2}\bar{b}_{k+1}).$$

Proof. Since the equation $WT = TW$ holds, we get that for every nonnegative integer k

$$\langle (WS - SW)e_k, e_{k+2}\rangle = \langle e_k, W^* v\rangle\langle u, e_{k+2}\rangle - \langle e_k, v\rangle\langle Wu, e_{k+2}\rangle.$$

And we get that

$$\begin{aligned} \langle (WS - SW)e_k, e_{k+2}\rangle &= \langle We_{k+1}, e_{k+2}\rangle - \langle S(\gamma_k e_{k+1}), e_{k+2}\rangle \\ &= \gamma_{k+1} - \gamma_k. \end{aligned}$$

Together,

$$\gamma_{k+1} - \gamma_k = \gamma_k a_{k+2}\bar{b}_{k+1} - \gamma_{k+1}a_{k+1}\bar{b}_k.$$

Thus we have

$$\gamma_{k+1}(1 + a_{k+1}\bar{b}_k) = \gamma_k(1 + a_{k+2}\bar{b}_{k+1})$$

for every nonnegative integer k. □

Corollary 2.15. *Let $W \in \mathcal{L}(\mathcal{H})$ be the unilateral weighted shift with positive weight sequence $\{\gamma_k\}$ defined by $We_k = \gamma_k e_{k+1}$ for $k = 0, 1, 2, \ldots$. Suppose that $T = S + u \otimes v$ satisfies $1 + a_{k+1}\bar{b}_k \neq 0$ for every nonnegative integer k and $W \in \{T\}'$. Then W is hyponormal if and only if $a_{k+2}\bar{b}_{k+1} \geq a_{k+1}\bar{b}_k$ for every nonnegative integer k.*

Proof. If W is hyponormal, it is known that the weight sequence $\{\gamma_k\}_{k=0}^{\infty}$ of W is increasing. Hence from Proposition 2.14, we get that $\gamma_{k+1} \geq \gamma_k$ if and only if $1 + a_{k+2}\bar{b}_{k+1} \geq 1 + a_{k+1}\bar{b}_k$ for every nonnegative integer k. So we complete the proof. □

Next examples show that the hyponormality of the unilateral weighted shift in the commutant of $T = S + u \otimes v$ depends on u and v.

Example 2.16. Suppose that $T = S + u \otimes v$ where $u = -\sum_{k=0}^{\infty} \frac{1}{k+1} e_k$ and $v = \sum_{k=0}^{\infty} \frac{1}{k+2} e_k$ and $W \in \mathcal{L}(\mathcal{H})$ is the unilateral weighted shift with positive weight sequence $\{\gamma_k\}$. If $W \in \{T\}'$, then from Proposition 2.14 we get that

$$\begin{aligned}\frac{\gamma_{k+1}}{\gamma_k} &= \frac{1 + a_{k+2}\bar{b}_{k+1}}{1 + a_{k+1}\bar{b}_k} \\ &= \frac{1 - (\frac{1}{k+3})^2}{1 - (\frac{1}{k+2})^2} > 1.\end{aligned}$$

Thus the weight sequence $\{\gamma_k\}$ is increasing. Hence W is hyponormal.

Example 2.17. Suppose that $T = S + u \otimes v$ where $u = \sum_{k=0}^{\infty} \frac{1}{2^k} e_k$ and $v = \sum_{k=0}^{\infty} \frac{1}{2^{k+1}} e_k$ and $W \in \mathcal{L}(\mathcal{H})$ is the unilateral weighted shift with positive weight sequence $\{\gamma_k\}$. If $W \in \{T\}'$, then from Proposition 2.14 we obtain that

$$\begin{aligned}\frac{\gamma_{k+1}}{\gamma_k} &= \frac{1 + a_{k+2}\bar{b}_{k+1}}{1 + a_{k+1}\bar{b}_k} \\ &= \frac{1 + (\frac{1}{2^{k+2}})^2}{1 + (\frac{1}{2^{k+1}})^2} < 1.\end{aligned}$$

Thus the weight sequence $\{\gamma_k\}$ is decreasing. Hence W is not hyponormal.

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