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SOME EISENSTEIN SERIES IDENTITIES RELATED TO MODULAR EQUATION OF THE FOURTH ORDER

Bhaskar Srivastava

Abstract. We find some Eisenstein series related to modulus 4 using a theta function identity of McCullough and Shen and residue theorem for elliptic functions.

1. Introduction

The Eisenstein series $P(q)$, $Q(q)$ and $R(q)$ are defined for $|q| < 1$ by

$$P(q) := 1 - 24 \sum_{n=1}^{\infty} \frac{nq^n}{1-q^n}, \tag{1.1}$$

$$Q(q) := 1 + 240 \sum_{n=1}^{\infty} \frac{n^3 q^n}{1-q^n}, \tag{1.2}$$

and

$$R(q) := 1 - 504 \sum_{n=1}^{\infty} \frac{n^5 q^n}{1-q^n}. \tag{1.3}$$

This is the notation used by Ramanujan in his lost notebook [8, pp. 136–162], but in his ordinary notebooks, P, Q and R are replaced by L, M and N respectively. We shall be using L, M and N, respectively, for P, Q and R.

We studied the continued fraction of Ramanujan

$$C(q) = 1 + \frac{(1+q)}{1+}\,\frac{q^2}{1+}\,\frac{(q+q^3)}{1+}\,\frac{q^4}{1+\cdots} = \frac{(q^2;q^4)_\infty^2}{(q;q^4)_\infty (q^3;q^4)_\infty} \tag{1.4}$$

and called this continued fraction analogous to the celebrated Rogers-Ramanujan continued fraction $R(q)$

$$R(q) = \frac{q^{1/5}}{1+}\,\frac{q}{1+}\,\frac{q^2}{1+\cdots} = q^{1/5} \frac{(q;q^5)_\infty (q^4;q^5)_\infty}{(q^2;q^5)_\infty (q^3;q^5)_\infty}. \tag{1.5}$$

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Since the continued fraction $C(q)$ sums to q-hypergeometric series on base 4, motivated me to study Eisenstein series identities related to modular equations of order 4. In this paper we will prove the following identities:

$$2\left(\frac{\theta_1'}{\theta_1}\right)'\left(\frac{\pi}{4}|q\right)+\left(\frac{\theta_1'}{\theta_1}\right)'\left(\frac{\pi}{2}|q\right)=\frac{1}{3}\left(L(\tau)-4^2L(4\tau)\right), \tag{1.6}$$

$$2\left(\frac{\theta_1'}{\theta_1}\right)'''\left(\frac{\pi}{4}|q\right)+\left(\frac{\theta_1'}{\theta_1}\right)'''\left(\frac{2\pi}{4}|q\right)=\frac{2}{15}\left(M(\tau)-4^4M(4\tau)\right), \tag{1.7}$$

$$1-4\sum_{m=0}^{\infty}\left(\frac{(4m+1)^2q^{4m+1}}{1-q^{4m+1}}-\frac{(4m+3)^2q^{4m+3}}{1-q^{4m+3}}\right)=\frac{\eta^4(\tau)\eta^6(2\tau)}{\eta(4\tau)\eta^3(4\tau)}, \tag{1.8}$$

$$\begin{aligned}&\left[1+4\sum_{n=0}^{\infty}\left(\frac{q^{4n+1}}{1-q^{4n+1}}-\frac{q^{4n+3}}{1-q^{4n+3}}\right)\right]^2\\&=1+8\left[\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}-\sum_{n=1}^{\infty}\frac{4nq^n}{1-q^{4n}}\right],\end{aligned} \tag{1.9}$$

and

(1.10)

$$\begin{aligned}&160+128\sum_{n=0}^{\infty}\left(\frac{(4n+1)^4q^{4n+1}}{1-q^{4n+1}}-\frac{(4n+3)^4q^{4n+3}}{1-q^{4n+3}}\right)\\&=-2^5\left[1+4\sum_{n=0}^{\infty}\left(\frac{q^{4n+1}}{1-q^{4n+1}}-\frac{q^{4n+3}}{1-q^{4n+3}}\right)\right]^5\\&\quad+10\left[1+4\sum_{n=0}^{\infty}\left(\frac{q^{4n+1}}{1-q^{4n+1}}-\frac{q^{4n+3}}{1-q^{4n+3}}\right)\right]\left(\frac{16}{15}M(\tau)+\frac{2}{15}4^4M(4\tau)\right)\\&\quad+10\left[1+4\sum_{n=0}^{\infty}\left(\frac{q^{4n+1}}{1-q^{4n+1}}-\frac{q^{4n+3}}{1-q^{4n+3}}\right)\right]\left(\frac{4}{3}L(\tau)-\frac{16}{3}L(4\tau)\right)^2.\end{aligned}$$

The identity (1.10) is very interesting.

2. Preliminaries

Throughout the paper $q=e^{2\pi i\tau}$, $\text{Im}(\tau)>0$ and the standard q-notaions are used:

$$(a;q)_\infty=\prod_{k=0}^{\infty}(1-aq^k), \tag{2.1}$$

$$(a;q)_n = \prod_{k=1}^{n}(1-aq^{k-1}), \tag{2.2}$$

and

$$(a)_0 = (a;q)_0 = 1.$$

The Dedekind eta-function is defined by

$$\eta(\tau) = q^{\frac{1}{24}}(q;q)_\infty. \tag{2.3}$$

Jacobi theta function is defined as follows, see [9, p. 464]

$$\theta_1(z|q) = -iq^{\frac{1}{8}}\sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{n(n+1)}{2}}e^{(2n+1)iz} \tag{2.4}$$

$$= 2q^{\frac{1}{8}}\sum_{n=0}^{\infty}(-1)^n q^{\frac{n(n+1)}{2}}\sin(2n+1)z. \tag{2.5}$$

In terms of infinite products

$$\theta_1(z|q) = 2q^{\frac{1}{8}}\sin z(q;q)_\infty(qe^{2iz};q)_\infty(qe^{-2iz};q)_\infty \tag{2.6}$$

$$= iq^{\frac{1}{8}}e^{-iz}(q;q)_\infty(e^{2iz};q)_\infty(qe^{-2iz};q)_\infty. \tag{2.7}$$

Differentiating partially with respect to z and then putting $z=0$, we have the identity

$$\theta_1'(q) = \theta_1'(0|q) = 2q^{\frac{1}{8}}(q;q)_\infty^3. \tag{2.8}$$

From the definition of $\theta_1(z|q)$,

$$\theta_1(z+n\pi|q) = (-1)^n\theta_1(z|q), \tag{2.9}$$

and

$$\theta_1(z+n\pi\tau|q) = (-1)^n q^{-\frac{n^2}{2}}e^{-2niz}\theta_1(z|q). \tag{2.10}$$

In this paper we shall be using the following residue theorem of elliptic functions:

Theorem. *The sum of all the residues of an elliptic function in the period parallelogram is zero.*

3. The proofs of (1.6) and (1.7)

By infinite product expansion for $\theta_1(z|q)$ given in (2.7) and by simple computation, we have

$$\theta_1(4z|q^4) = \frac{(q^4;q^4)_\infty}{(q;q)_\infty^4}\theta_1(z|q)\theta_1(z-\frac{\pi}{4}|q)\theta_1(z+\frac{\pi}{4}|q)\theta_1(z-\frac{\pi}{2}|q). \tag{3.1}$$

Taking logarithmic derivative of both the sides of (3.1), we obtain

$$4\frac{\theta_1'}{\theta_1}(4z|q^4) - \frac{\theta_1'}{\theta_1}(z|q) = \frac{\theta_1'}{\theta_1}(z-\frac{\pi}{4}|q) + \frac{\theta_1'}{\theta_1}(z+\frac{\pi}{4}|q) + \frac{\theta_1'}{\theta_1}(z-\frac{\pi}{2}|q). \tag{3.2}$$

We now use the following identity, [6, eq.(2.10), p. 109] to simplify the left hand side:

$$\frac{\theta_1'}{\theta_1}(z|q) = \frac{1}{z} - \frac{1}{3}L(\tau)z - \frac{1}{45}M(\tau)z^3 - \frac{2}{945}N(\tau)z^5 - \frac{1}{4725}\left(1 + 480\sum_{n=1}^{\infty}\frac{n^7q^n}{1-q^n}\right)z^7 + \cdots. \tag{3.3}$$

So (3.2) can be written as

$$\frac{\theta_1'}{\theta_1}(z - \frac{\pi}{4}|q) + \frac{\theta_1'}{\theta_1}(z + \frac{\pi}{4}|q) + \frac{\theta_1'}{\theta_1}(z - \frac{\pi}{2}|q) = \frac{1}{3}\left(L(\tau) - 4^2L(4\tau)\right)z + \frac{1}{45}\left(M(\tau) - 4^4M(4\tau)\right)z^3 + O(z^5). \tag{3.4}$$

Differentiate both side of (3.4) with respect to z and then put $z = 0$ to get

$$2\left(\frac{\theta_1'}{\theta_1}\right)'\left(\frac{\pi}{4}|q\right) + \left(\frac{\theta_1'}{\theta_1}\right)'\left(\frac{\pi}{2}|q\right) = \frac{1}{3}\left(L(\tau) - 4^2L(4\tau)\right), \tag{3.5}$$

which proves (1.6).

Differentiate thrice both side of (3.4) with respect to z and then put $z = 0$ to get

$$2\left(\frac{\theta_1'}{\theta_1}\right)'''\left(\frac{\pi}{4}|q\right) + \left(\frac{\theta_1'}{\theta_1}\right)'''\left(\frac{\pi}{2}|q\right) = \frac{2}{15}\left(M(\tau) - 4^4M(4\tau)\right),$$

which proves (1.7).

4. The proof of (1.8)

Recall the following identity [6, eq.(8.1), p. 117]

$$\cot^2 y - \cot^2 x + 8\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}(\cos 2nx - \cos 2ny) = \theta_1'(0|q)^2\,\frac{\theta_1(x-y|q)\theta_1(x+y|q)}{\theta_1^2(x|q)\theta_1^2(y|q)}. \tag{4.1}$$

There is a slight misprint which has been corrected.

Differentiate (4.1) partially with respect to x then putting $y = x$ to obtain

$$2\cot x \operatorname{cosec}^2 x - 16\sum_{n=1}^{\infty}\frac{n^2q^n}{1-q^n}\sin 2nx = \frac{\theta_1'(0|q)^3\theta_1(2x|q)}{\theta_1^4(x|q)}. \tag{4.2}$$

Putting $x = \frac{\pi}{4}$ in (4.2), we have

$$4 - 16\sum_{n=1}^{\infty}\frac{n^2q^n}{1-q^n}\sin\frac{n\pi}{2} = \theta_1'(0|q)^3\left[\frac{\theta_1\left(\frac{\pi}{2}|q\right)}{\theta_1^4\left(\frac{\pi}{4}|q\right)}\right],$$

or

$$4 - 16\sum_{n=1}^{\infty} \frac{n^2 q^n}{1-q^n} \sin\frac{n\pi}{2} = \frac{4(q;q)_\infty^4 (q^2;q^2)_\infty^6}{(q^4;q^4)_\infty^4},$$

so

$$1 - 4\sum_{m=0}^{\infty} \left(\frac{(4m+1)^2 q^{4m+1}}{1-q^{4m+1}} - \frac{(4m+3)^2 q^{4m+3}}{1-q^{4m+3}} \right) = \frac{(q;q)_\infty^4 (q^2;q^2)_\infty^6}{(q^4;q^4)_\infty^4}.$$

Using the definition of Dedekind eta-function defined in (2.3), we have

$$1 - 4\sum_{m=0}^{\infty} \left(\frac{(4m+1)^2 q^{4m+1}}{1-q^{4m+1}} - \frac{(4m+3)^2 q^{4m+3}}{1-q^{4m+3}} \right) = \frac{\eta^4(\tau)\eta^6(2\tau)}{\eta(4\tau)\eta^3(4\tau)},$$

which proves (1.8).

5. The proof of (1.9)

For proving (1.9) we first construct the following elliptic function:

(5.1) $$f(z) = \frac{\theta_1^2(z+\frac{\pi}{4}|q)\theta_1(z+\frac{\pi}{2}|q)}{\theta_1^3(z|q)}$$

and use residue theorem of elliptic functions.

It is easy to see that $f(z)$ is an elliptic function of periods π and $\pi\tau$, and has a pole of order 3 at $z=0$. We now compute residue of $f(z)$ at $z=0$.

Now

(5.2) $$\operatorname{res}(f;0) = \frac{1}{2}\left[\frac{d^2}{dz^2}\left(z^3 f(z)\right)\right]_{z=0}.$$

Let

(5.3) $$F(z) = z^3 f(z) \text{ and } \varphi(z) = \frac{F'(z)}{F(z)}.$$

By logarithmic differentiation

(5.4) $$\operatorname{res}(f;0) = \frac{1}{2}\left[\frac{d^2}{dz^2}\left(z^3 f(z)\right)\right]_{z=0} = \frac{1}{2}F(0)\left[\varphi(0)^2 + \varphi'(0)\right].$$

Now

(5.5) $$\begin{aligned} \varphi(z) &= \frac{3z^2 f(z) + z^3 f'(z)}{z^3 f(z)} = \frac{3}{z} + \frac{f'(z)}{f(z)} \\ &= \frac{3}{z} - 3\frac{\theta_1'}{\theta_1}(z|q) + 2\frac{\theta_1'}{\theta_1}(z+\frac{\pi}{4}|q) + \frac{\theta_1'}{\theta_1}(z+\frac{\pi}{2}|q) \\ &= L(\tau)z + 2\frac{\theta_1'}{\theta_1}(z+\frac{\pi}{4}|q) + \frac{\theta_1'}{\theta_1}(z+\frac{\pi}{2}|q) + O(z^3). \end{aligned}$$

Now

$$\frac{\theta_1'}{\theta_1}(\frac{\pi}{4}|q) = \cot\frac{\pi}{4} + 4\sum_{n=1}^{\infty}\frac{q^n}{1-q^n}\sin\frac{n\pi}{2} \tag{5.6}$$
$$= 1 + 4\sum_{n=0}^{\infty}\left(\frac{q^{4n+1}}{1-q^{4n+1}} - \frac{q^{4n+3}}{1-q^{4n+3}}\right)$$

and

$$\frac{\theta_1'}{\theta_1}(\frac{\pi}{2}|q) = 4\sum_{n=1}^{\infty}\frac{q^n}{1-q^n}\sin n\pi = 0. \tag{5.7}$$

So

$$2\frac{\theta_1'}{\theta_1}(\frac{\pi}{4}|q) + \frac{\theta_1'}{\theta_1}(\frac{\pi}{2}|q) = 2\left[1 + 4\sum_{n=0}^{\infty}\left(\frac{q^{4n+1}}{1-q^{4n+1}} - \frac{q^{4n+3}}{1-q^{4n+3}}\right)\right]. \tag{5.8}$$

Putting $z = 0$ in (5.5) and using (5.8), we have

$$\varphi(0) = 2\frac{\theta_1'}{\theta_1}(\frac{\pi}{4}|q) + \frac{\theta_1'}{\theta_1}(\frac{\pi}{2}|q) \tag{5.9}$$
$$= 2\left[1 + 4\sum_{n=0}^{\infty}\left(\frac{q^{4n+1}}{1-q^{4n+1}} - \frac{q^{4n+3}}{1-q^{4n+3}}\right)\right].$$

Differentiating (5.5) with respect to z and then putting $z = 0$ and using (3.5), we have

$$\varphi'(0) = L(\tau) + 2\left(\frac{\theta_1'}{\theta_1}\right)'\left(\frac{\pi}{4}|q\right) + \left(\frac{\theta_1'}{\theta_1}\right)'\left(\frac{\pi}{2}|q\right) \tag{5.10}$$
$$= \frac{1}{3}\left(4L(\tau) - 16L(4\tau)\right)$$
$$= \frac{1}{3}\left[4\left(1 - 24\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}\right) - 16\left(1 - 24\sum_{n=1}^{\infty}\frac{nq^{4n}}{1-q^{4n}}\right)\right]$$
$$= -4 - 32\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n} + 128\sum_{n=1}^{\infty}\frac{nq^{4n}}{1-q^{4n}}.$$

Also

$$F(0) = \frac{\theta_1^2(\frac{\pi}{4}|q)\theta_1(\frac{\pi}{2}|q)}{\theta_1'(0|q)^3} \neq 0.$$

Now by the residue theorem and using (5.4), we obtain

$$\varphi(0)^2 = -\varphi'(0). \tag{5.11}$$

Hence from (5.9) and (5.10)

$$\begin{aligned}&\left[1+4\sum_{n=0}^{\infty}\left(\frac{q^{4n+1}}{1-q^{4n+1}}-\frac{q^{4n+3}}{1-q^{4n+3}}\right)\right]^2\\ &=1+8\left[\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}-\sum_{n=1}^{\infty}\frac{4nq^n}{1-q^{4n}}\right],\end{aligned}\tag{5.12}$$

which is (1.9).

6. The proof of (1.10)

For proving (1.10) we construct the following elliptic function and use the residue theorem of elliptic functions.

Let

$$f(z)=\frac{\theta_1(2z|q)\theta_1^2(z+\frac{\pi}{4}|q)\theta_1(z+\frac{\pi}{2}|q)}{\theta_1^7(z|q)}.\tag{6.1}$$

By (2.9) it is easily seen that $f(z)$ is an elliptic function of periods π and $\pi\tau$ with only one pole at $z=0$ of order 6.

Now

$$\operatorname{res}(f;0)=\frac{1}{120}\left[\frac{d^5}{dz^5}\left(z^6f(z)\right)\right]_{z=0}.\tag{6.2}$$

To compute $\operatorname{res}(f;0)$, let

$$F(z)=z^6f(z)\text{ and }\ \varphi(z)=\frac{F^{'}(z)}{F(z)}.\tag{6.3}$$

By logarithmic differentiation and elementary computation, we get

$$\begin{aligned}&\frac{d^5}{dz^5}F(z)\\ &=F(z)\Big[\varphi(z)^5+10\varphi(z)^3\varphi^{'}(z)+5\varphi(z)\varphi^{'''}(z)+10\varphi(z)^2\varphi^{''}(z)\\ &\qquad+5\varphi(z)\varphi^{'}(z)^2+10\varphi^{'}(z)\varphi^{''}(z)+\varphi^{(4)}(z)\Big].\end{aligned}\tag{6.4}$$

It is obvious that

$$F(0)=\lim_{z\to 0}z^6f(z)=\frac{\theta_1^2(\frac{\pi}{4}|q)\theta_1(\frac{\pi}{2}|q)}{\theta_1^{'}(0|q)^6}\neq 0.$$

We now calculate $\varphi(0)$.

From (6.3)

(6.5)

$$\begin{aligned}\varphi(z) &= \frac{6z^5 f(z) + z^6 f^{'}(z)}{z^6 f(z)} = \frac{6}{z} + \frac{f^{'}(z)}{f(z)} \\ &= \frac{6}{z} + 2\frac{\theta_1^{'}}{\theta_1}(2z|q) - 7\frac{\theta_1^{'}}{\theta_1}(z|q) + 2\frac{\theta_1^{'}}{\theta_1}(z+\frac{\pi}{4}|q) + \frac{\theta_1^{'}}{\theta_1}(z+\frac{\pi}{2}|q) \\ &= L(\tau)z - \frac{z^3}{5}M(\tau) + 2\frac{\theta_1^{'}}{\theta_1}(z+\frac{\pi}{4}|q) + \frac{\theta_1^{'}}{\theta_1}(z+\frac{\pi}{2}|q) + O(z^5) \text{ by (3.3).}\end{aligned}$$

Putting $z = 0$, we have

$$\varphi(0) = 2\frac{\theta_1^{'}}{\theta_1}(\frac{\pi}{4}|q) + \frac{\theta_1^{'}}{\theta_1}(\frac{\pi}{2}|q). \tag{6.6}$$

Hence by (5.8)

$$\varphi(0) = 2\left[1 + 4\sum_{n=0}^{\infty}\left(\frac{q^{4n+1}}{1-q^{4n+1}} - \frac{q^{4n+3}}{1-q^{4n+3}}\right)\right]. \tag{6.7}$$

Diferentiating (6.5) with respect to z and then setting $z = 0$ and using (1.6), we have

$$\begin{aligned}\varphi^{'}(0) &= L(\tau) + 2\left(\frac{\theta_1^{'}}{\theta_1}\right)^{'}\left(\frac{\pi}{4}|q\right) + \left(\frac{\theta_1^{'}}{\theta_1}\right)^{'}\left(\frac{\pi}{2}|q\right) \\ &= \frac{4}{3}L(\tau) - \frac{16}{3}L(4\tau) \\ &= -4 - 32\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n} + 128\sum_{n=1}^{\infty}\frac{nq^{4n}}{1-q^{4n}} \text{ by (1.1).}\end{aligned} \tag{6.8}$$

Diferentiating (6.5) twice with respect to z and setting $z = 0$, we have

$$\varphi^{''}(0) = 2\left(\frac{\theta_1^{'}}{\theta_1}\right)^{''}\left(\frac{\pi}{4}|q\right) + \left(\frac{\theta_1^{'}}{\theta_1}\right)^{''}\left(\frac{\pi}{2}|q\right). \tag{6.9}$$

Now for determining the right hand side of (6.9) we use the following identity [9, p. 489]

$$\frac{\theta_1^{'}}{\theta_1}(z|\tau) = \cot z + 4\sum_{n=1}^{\infty}\frac{q^n}{1-q^n}\sin 2nz. \tag{6.10}$$

Diferentiating (6.10) twice with respect to z, we get

$$\left(\frac{\theta_1^{'}}{\theta_1}\right)^{''}(z|\tau) = 2\text{cosec}^2 z\cot z - 16\sum_{n=1}^{\infty}\frac{n^2q^n}{1-q^n}\sin 2nz. \tag{6.11}$$

Setting $z = \frac{\pi}{4}$ and $z = \frac{\pi}{2}$, respectively in (6.11), we have

$$\left(\frac{\theta_1^{'}}{\theta_1}\right)^{''}(\frac{\pi}{4}|\tau) = 4 - 16\sum_{n=1}^{\infty}\frac{n^2q^n}{1-q^n}\sin\frac{n\pi}{2}$$
$$= 4 - 16\sum_{n=0}^{\infty}\left[\frac{(4n+1)^2q^{4n+1}}{1-q^{4n+1}} - \frac{(4n+3)^2q^{4n+3}}{1-q^{4n+3}}\right]$$

and

$$\left(\frac{\theta_1^{'}}{\theta_1}\right)^{''}(\frac{\pi}{2}|\tau) = 0.$$

Hence

$$\varphi^{''}(0) = 8\left[1 - 4\sum_{n=0}^{\infty}\left(\frac{(4n+1)^2q^{4n+1}}{1-q^{4n+1}} - \frac{(4n+3)^2q^{4n+3}}{1-q^{4n+3}}\right)\right]. \tag{6.12}$$

Differentiating (6.5) thrice with respect to z and then putting $z = 0$ and using (1.7), we have

$$\begin{aligned}\varphi^{'''}(0) =& -\frac{6}{5}M(\tau) + 2\left(\frac{\theta_1^{'}}{\theta_1}\right)^{'''}\left(\frac{\pi}{4}|q\right) + \left(\frac{\theta_1^{'}}{\theta_1}\right)^{'''}\left(\frac{\pi}{2}|q\right)\\ =& -\frac{16}{15}M(\tau) - \frac{2}{15}4^4M(4\tau) \text{ by (1.7)}\end{aligned} \tag{6.13}$$

and

$$\varphi^{(4)}(0) = 2\left(\frac{\theta_1^{'}}{\theta_1}\right)^{(4)}\left(\frac{\pi}{4}|q\right) + \left(\frac{\theta_1^{'}}{\theta_1}\right)^{(4)}\left(\frac{\pi}{2}|q\right).$$

By (6.10)

$$\begin{aligned}\varphi^{(4)}(0) &= 2\cot^{(4)}\left(\frac{\pi}{4}\right) + \cot^{(4)}\left(\frac{\pi}{2}\right) + 128\sum_{n=1}^{\infty}\frac{n^4q^n}{1-q^n}\sin\frac{n\pi}{2}\\ &= 160 + 128\sum_{n=1}^{\infty}\frac{n^4q^n}{1-q^n}\sin\frac{n\pi}{2}\\ &= 160 + 128\sum_{n=0}^{\infty}\left(\frac{(4n+1)^4q^{4n+1}}{1-q^{4n+1}} - \frac{(4n+3)^4q^{4n+3}}{1-q^{4n+3}}\right).\end{aligned} \tag{6.14}$$

Now by (6.2) and (6.4) the res$(f;0)$ is

(6.15)

$$\begin{aligned}\mathrm{res}(f;0) = \frac{1}{120}F(0)\Big[&\varphi(0)^5 + 10\varphi(0)^3\varphi^{'}(0) + 5\varphi(0)\varphi^{'''}(0) + 10\varphi(0)^2\varphi^{''}(0)\\ &+ 5\varphi(0)\varphi^{'}(0)^2 + 10\varphi^{'}(0)\varphi^{''}(0) + \varphi^{(4)}(0)\Big].\end{aligned}$$

We see that $\varphi(0)$ calculated in (5.9) is the same as in (6.7) and $\varphi^{'}(0)$ calculated in (5.10) is the same as in (6.8). Hence by (5.11)

$$\varphi(0)^2 = -\varphi^{'}(0).$$

Putting this in (6.15), $\operatorname{res}(f;0)$ simplifies to

$$\operatorname{res}(f;0) = \frac{1}{120}F(0)\left[\varphi(0)^5 + 5\varphi(0)\varphi^{'''}(0) - 5\varphi(0)\varphi^{'}(0)^2 + \varphi^{(4)}(0)\right].$$

Since $\operatorname{res}(f;0) = 0$ by the residue theorem, we have

$$\varphi^{(4)}(0) = -\left[\varphi(0)^5 + 5\varphi(0)\varphi^{'''}(0) - 5\varphi(0)\varphi^{'}(0)^2\right]. \tag{6.16}$$

Using (6.7), (6.8), (6.13) and (6.14) the identity in (6.16) simplifies to

$$\begin{aligned}
&160 + 128\sum_{n=0}^{\infty}\left(\frac{(4n+1)^4q^{4n+1}}{1-q^{4n+1}} - \frac{(4n+3)^4q^{4n+3}}{1-q^{4n+3}}\right)\\
= &-2^5\left[1 + 4\sum_{n=0}^{\infty}\left(\frac{q^{4n+1}}{1-q^{4n+1}} - \frac{q^{4n+3}}{1-q^{4n+3}}\right)\right]^5\\
&+10\left[1 + 4\sum_{n=0}^{\infty}\left(\frac{q^{4n+1}}{1-q^{4n+1}} - \frac{q^{4n+3}}{1-q^{4n+3}}\right)\right]\left(\frac{16}{15}M(\tau) + \frac{2}{15}4^4M(4\tau)\right)\\
&+10\left[1 + 4\sum_{n=0}^{\infty}\left(\frac{q^{4n+1}}{1-q^{4n+1}} - \frac{q^{4n+3}}{1-q^{4n+3}}\right)\right]\left(\frac{4}{3}L(\tau) - \frac{16}{3}L(4\tau)\right)^2,
\end{aligned}$$

which is (1.10).

References

[1] G. E. Andrews and B. C. Berndt, *Ramanujan's Lost Notebook. Part I*, Springer, New York, 2005.

[2] G. Gasper and M. Rahman, *Basic Hypergeometric Series*, Cambridge University Press, Cambridge, 1990.

[3] Z.-G. Liu, *Some Eisenstein series identities*, J. Number Theory **85** (2000), no. 2, 231–252.

[4] ———, *Residue theorem and theta function identities*, Ramanujan J. **5** (2001), no. 2, 129–151.

[5] ———, *Some theta functions identities associated with the modular equations of degree 5*, Integers **1** (2001), A3, 14 pp.

[6] ———, *Some Eisenstein series identities related to modular equations of the seventh order*, Pacific J. Math. **209** (2003), no. 1, 103–130.

[7] S. McCullough and L.-C. Shen, *On the Szegõ kernel of an annulus*, Proc. Amer. Math. Soc. **121** (1994), no. 4, 1111–1121.

[8] S. Ramanujan, *Collected Papers*, Cambridge University Press, Cambridge, 1927.

[9] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis. An Introduction to the General Theory of Infinite Processes and of Analytic Functions: with an account of the principal transcendental functions*, Cambridge University Press, New York 1962.

Department of Mathmatics and Astronomy
Lucknow University
226007 Lucknow, India
E-mail address: bhaskarsrivastav@yahoo.com