

ITERATIVE METHODS FOR GENERALIZED EQUILIBRIUM PROBLEMS AND NONEXPANSIVE MAPPINGS

SUN YOUNG CHO, SHIN MIN KANG, AND XIAOLONG QIN

ABSTRACT. In this paper, a composite iterative process is introduced for a generalized equilibrium problem and a pair of nonexpansive mappings. It is proved that the sequence generated in the purposed composite iterative process converges strongly to a common element of the solution set of a generalized equilibrium problem and of the common fixed point of a pair of nonexpansive mappings.

1. Introduction and preliminaries

Throughout this paper, we always assume that H is a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let C be a nonempty closed convex subset of H and $A : C \rightarrow H$ be a nonlinear mapping.

Recall the following definitions.

(1) A is said to be *monotone* if

$$\langle Ax - Ay, x - y \rangle \geq 0, \quad \forall x, y \in C.$$

(2) A is said to be *strongly monotone* if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|x - y\|^2, \quad \forall x, y \in C.$$

For such a case, T is also said to be α -*strongly-monotone*.

(3) A is said to be *inverse-strongly monotone* if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C.$$

For such a case, A is also said to be α -*inverse-strongly monotone*.

Recall that the classical variational inequality problem, denoted by $VI(C, A)$, is to find $x \in C$ such that

$$(1.1) \quad \langle Ax, y - x \rangle \geq 0, \quad \forall y \in C.$$

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It is well known that $z \in H$ and $x \in C$ satisfy the inequality

$$\langle x - z, y - x \rangle \geq 0, \quad \forall y \in C$$

if and only if $x = P_C z$, where P_C denotes the metric projection from H onto C . From the above, we see that $x \in C$ is a solution to the problem (1.1) if and only if x is a fixed point of the mapping $P_C(I - \rho A)$, where $\rho > 0$ is a constant and I is the identity mapping.

Let $A : C \rightarrow H$ be an α -inverse-strongly monotone mapping and F be a bifunction from $C \times C$ into \mathbb{R} , where \mathbb{R} denotes the set of real numbers. We consider the following generalized equilibrium problem:

$$(1.2) \quad \text{Find } x \in C \text{ such that } F(x, y) + \langle Ax, y - x \rangle \geq 0, \quad \forall y \in C.$$

In this paper, the set of such an $x \in C$ is denoted by $EP(F, A)$, i.e.,

$$EP(F, A) = \{x \in C : F(x, y) + \langle Ax, y - x \rangle \geq 0, \forall y \in C\}.$$

Next, we give some special cases of the problem (1.2).

(I) If $A \equiv 0$, the zero mapping, then the generalized equilibrium problem (1.2) is reduced to the following equilibrium problem:

$$(1.3) \quad \text{Find } x \in C \text{ such that } F(x, y) \geq 0, \quad \forall y \in C.$$

In this paper, the set of such an $x \in C$ is denoted by $EP(F)$, i.e.,

$$EP(F) = \{x \in C : F(x, y) \geq 0, \forall y \in C\}.$$

(II) If $F \equiv 0$, then the problem (1.2) is reduced to the classical variational inequality problem (1.1).

The problem (1.2) is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, mini-max problems, the Nash equilibrium problem in noncooperative games and others; see, for instance, [1, 4, 5, 11].

Recently, many authors considered iterative method for the problems (1.1), (1.2) and (1.3); see, for example, [1, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 14, 15, 17, 18] for more details.

To study the equilibrium problems (1.2) and (1.3), we may assume that F satisfies the following conditions:

- (A1) $F(x, x) = 0$ for all $x \in C$,
- (A2) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$,
- (A3) for each $x, y, z \in C$,

$$\limsup_{t \downarrow 0} F(tz + (1-t)x, y) \leq F(x, y),$$

- (A4) for each $x \in C$, $y \mapsto F(x, y)$ is convex and weakly lower semi-continuous.

Let $T : C \rightarrow C$ be a nonlinear mapping. In this paper, we use $F(T)$ to denote the set of fixed points of T . Recall the following definitions.

(4) The mapping T is said to be *contractive* if there exists a constant $\lambda \in (0, 1)$ such that

$$\|Tx - Ty\| \leq \lambda \|x - y\|, \quad \forall x, y \in C.$$

(5) The mapping T is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

In 2005, H. Iiduka and W. Takahashi [9] considered the classical variational inequality (1.1) and a nonexpansive mapping. To be more precise, they obtained the following results.

Theorem IT. *Let C be a closed convex subset of a real Hilbert space H . Let A be an α -inverse-strongly monotone mapping from C into H and T be a nonexpansive mapping from C into itself such that $F(T) \cap VI(C, A) \neq \emptyset$. Suppose that $x_1 = x \in C$ and $\{x_n\}$ is given by*

$$(1.4) \quad x_{n+1} = \alpha_n x + (1 - \alpha_n)TP_C(x_n - \lambda_n Ax_n), \quad \forall n \geq 1,$$

where $\{\alpha_n\}$ is a sequence in $[0, 1)$ and $\{\lambda_n\}$ is a sequence in $[0, 2\alpha]$. If $\{\alpha_n\}$ and $\{\lambda_n\}$ are chosen so that $\{\lambda_n\} \subset [a, b]$ for some a, b with $0 < a < b < 2\alpha$,

$$\begin{aligned} \lim_{n \rightarrow \infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty \quad \text{and} \\ \sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty, \end{aligned}$$

then $\{x_n\}$ converges strongly to $P_{F(T) \cap VI(C, A)}x$.

Recently, S. Takahashi and W. Takahashi [17] considered the equilibrium problem (1.3) by an iterative method. To be more precise, they proved the following theorem.

Theorem TT1. *Let C be a nonempty closed convex subset of a real Hilbert space H . Let F be a bifunction from $C \times C$ into \mathbb{R} satisfying (A1)-(A4) and T be a nonexpansive mapping of C into H such that $F(T) \cap EP(F) \neq \emptyset$. Let f be a contraction from H into itself and let $\{x_n\}$ and $\{y_n\}$ be sequences generated by $x_1 \in H$ and*

$$\begin{cases} F(y_n, u) + \frac{1}{r_n} \langle u - y_n, y_n - x_n \rangle \geq 0, & \forall u \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Ty_n, & \forall n \geq 1, \end{cases}$$

where $\{\alpha_n\} \in [0, 1]$ and $\{r_n\} \subset (0, \infty)$ satisfy

$$\begin{aligned} \lim_{n \rightarrow \infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \\ \liminf_{n \rightarrow \infty} r_n > 0 \quad \text{and} \quad \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty. \end{aligned}$$

Then $\{x_n\}$ and $\{y_n\}$ converge strongly to $z \in F(T) \cap EP(F)$, where $z = P_{F(T) \cap EP(F)}f(z)$.

Very recently, S. Takahashi and W. Takahashi [18] further considered the generalized equilibrium problem (1.2). They obtained the following result in a real Hilbert space.

Theorem TT2. *Let C be a closed convex subset of a real Hilbert space H and $F : C \times C \rightarrow \mathbb{R}$ be a bifunction satisfying (A1)-(A4). Let A be an α -inverse-strongly monotone mapping from C into H and S be a non-expansive mapping from C into itself such that $F(S) \cap EP(F, A) \neq \emptyset$. Let $u \in C$ and $x_1 \in C$ and let $\{z_n\} \subset C$ and $\{x_n\} \subset C$ be sequences generated by*

$$\begin{cases} F(z_n, y) + \langle Ax_n, y - z_n \rangle + \frac{1}{r_n} \langle y - z_n, z_n - x_n \rangle \geq 0, & \forall y \in C, \\ x_{n+1} = \beta_n x_n + (1 - \beta_n) S[\alpha_n u + (1 - \alpha_n) z_n], & \forall n \geq 1, \end{cases}$$

where $\{\alpha_n\} \subset [0, 1]$, $\{\beta_n\} \subset [0, 1]$ and $\{r_n\} \subset [0, 2\alpha]$ satisfy

$$0 < c \leq \beta_n \leq d < 1, \quad 0 < a \leq \lambda_n \leq b < 2\alpha,$$

$$\lim_{n \rightarrow \infty} |\lambda_n - \lambda_{n+1}| = 0, \quad \lim_{n \rightarrow \infty} \alpha_n = 0 \quad \text{and} \quad \sum_{n=1}^{\infty} \alpha_n = \infty.$$

Then $\{x_n\}$ converges strongly to $z = P_{F(S) \cap EP(F, A)}u$.

In this paper, motivated by the research going on in this direction, we introduce a composite iterative algorithm for the problem of finding a common element in the solution set of the generalized equilibrium problem (1.2) and in the common fixed point set of a pair of nonexpansive mappings. Strong convergence theorems are established in the framework of Hilbert spaces. The results presented in this paper improve and extend the corresponding results announced by H. Iiduka and W. Takahashi [9] and S. Takahashi and W. Takahashi [17, 18] and some others.

In order to prove our main results, we also need the following definitions and lemmas.

Recall that a Hilbert space H is said to satisfy *Opial's condition* [12] if, for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, then the inequality

$$\liminf_{n \rightarrow \infty} \|x_n - x\| < \liminf_{n \rightarrow \infty} \|x_n - y\|,$$

holds for any $y \in H$ such that $y \neq x$.

The following lemma can be found in [1, 8].

Lemma 1.1. *Let C be a nonempty closed convex subset of a real Hilbert space H and $F : C \times C \rightarrow \mathbb{R}$ be a bifunction satisfying (A1)-(A4). Then for any $r > 0$ and $x \in H$, there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \quad \forall y \in C.$$

Further, define

$$T_r x = \left\{ z \in C : F(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \forall y \in C \right\}$$

for all $r > 0$ and $x \in H$. Then the following hold:

- (a) T_r is single-valued,
- (b) T_r is firmly nonexpansive, i.e., for any $x, y \in H$,

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y \rangle,$$

- (c) $F(T_r) = EP(F)$,
- (d) $\|T_s x - T_r x\|^2 \leq \frac{s-r}{s} \langle T_s x - T_r x, T_s x - x \rangle$,
- (e) $EP(F)$ is closed and convex.

Lemma 1.2 ([16]). Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space E and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with

$$0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1.$$

Suppose that $x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$ for all integers $n \geq 0$ and

$$\limsup_{n \rightarrow \infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Then $\lim_{n \rightarrow \infty} \|y_n - x_n\| = 0$.

Lemma 1.3 ([19]). Assume that $\{\alpha_n\}$ is a sequence of nonnegative real numbers such that

$$\alpha_{n+1} \leq (1 - \gamma_n)\alpha_n + \delta_n,$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence such that

- (a) $\sum_{n=1}^{\infty} \gamma_n = \infty$,
- (b) $\limsup_{n \rightarrow \infty} \frac{\delta_n}{\gamma_n} \leq 0$ or $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n \rightarrow \infty} \alpha_n = 0$.

2. Main results

Now, we are ready to give our main results.

Theorem 2.1. Let C be a nonempty closed convex subset of a real Hilbert space H , $F : C \times C \rightarrow \mathbb{R}$ be a bifunction satisfying (A1)-(A4) and $A : C \rightarrow H$ be an α -inverse-strongly monotone mapping. Let S and $T : C \rightarrow C$ be nonexpansive mappings such that $\mathcal{F} := F(S) \cap F(T) \cap EP(F, A) \neq \emptyset$ and $f : C \rightarrow C$ be a contractive mapping with the coefficient $\lambda \in (0, 1)$. Let $\{x_n\}$ be a sequence defined by the following algorithm: $x_1 \in C$ and

$$\begin{cases} F(w_n, y) + \langle Ax_n, y - w_n \rangle + \frac{1}{r_n} \langle y - w_n, w_n - x_n \rangle \geq 0, & \forall y \in C, \\ z_n = \gamma_n w_n + (1 - \gamma_n) T w_n, \\ y_n = \alpha_n f(x_n) + (1 - \alpha_n) z_n, \\ x_{n+1} = \beta_n x_n + (1 - \beta_n) S y_n, & \forall n \geq 1, \end{cases}$$

where $\{r_n\}$ is a positive sequence and $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0, 1]$. Assume that the following conditions are satisfied

- (a) $0 < c \leq \beta_n \leq d < 1$,
- (b) $0 < a \leq r_n \leq b < 2\alpha$ and $\lim_{n \rightarrow \infty} (r_n - r_{n+1}) = 0$,
- (c) $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,
- (d) $\lim_{n \rightarrow \infty} \gamma_n = 1$.

Then $\{x_n\}$ converges strongly to $q \in \mathcal{F}$, where $q = P_{\mathcal{F}}f(q)$.

Proof. Note that w_n can be rewritten as $w_n = T_{r_n}(x_n - r_n Ax_n)$ for each $n \geq 1$. Take $p \in \mathcal{F}$. Since $p = T_{r_n}(p - r_n Ap)$ and A is α -inverse-strongly monotone and $0 < r_n < 2\alpha$, we see that

$$\begin{aligned}
 \|w_n - p\|^2 &= \|T_{r_n}(x_n - r_n Ax_n) - T_{r_n}(p - r_n Ap)\|^2 \\
 &\leq \|(x_n - r_n Ax_n) - (p - r_n Ap)\|^2 \\
 &= \|(x_n - p) - r_n(Ax_n - Ap)\|^2 \\
 (2.1) \quad &= \|x_n - p\|^2 - 2r_n \langle x_n - p, Ax_n - Ap \rangle + r_n^2 \|Ax_n - Ap\|^2 \\
 &\leq \|x_n - p\|^2 - r_n(2\alpha - r_n) \|Ax_n - Ap\|^2 \\
 &\leq \|x_n - p\|^2.
 \end{aligned}$$

Note that

$$\begin{aligned}
 \|z_n - p\| &\leq \gamma_n \|w_n - p\| + (1 - \gamma_n) \|Tw_n - p\| \\
 (2.2) \quad &\leq \gamma_n \|w_n - p\| + (1 - \gamma_n) \|w_n - p\| \\
 &= \|w_n - p\|.
 \end{aligned}$$

On the other hand, we have

$$\begin{aligned}
 \|y_n - p\| &= \|\alpha_n(f(x_n) - p) + (1 - \alpha_n)(z_n - p)\| \\
 &\leq \alpha_n \lambda \|x_n - p\| + \alpha_n \|f(p) - p\| + (1 - \alpha_n) \|x_n - p\| \\
 &= [1 - \alpha_n(1 - \lambda)] \|x_n - p\| + \alpha_n \|f(p) - p\|.
 \end{aligned}$$

It follows that

$$\begin{aligned}
 \|x_{n+1} - p\| &\leq \beta_n \|x_n - p\| + (1 - \beta_n) \|Sy_n - p\| \\
 &\leq \beta_n \|x_n - p\| + (1 - \beta_n) \|y_n - p\| \\
 &\leq [1 - \alpha_n(1 - \lambda)(1 - \beta_n)] \|x_n - p\| + \alpha_n(1 - \beta_n) \|f(p) - p\|.
 \end{aligned}$$

Putting $M_1 = \max \left\{ \|x_1 - p\|, \frac{\|f(p) - p\|}{1 - \lambda} \right\}$, we have that $\|x_n - p\| \leq M_1$ for all $n \geq 1$. Indeed, it is obvious that $\|x_1 - p\| \leq M_1$. Suppose that $\|x_k - p\| \leq M_1$ for some positive integer k . Then we have that

$$\begin{aligned}
 \|x_{k+1} - p\| &\leq [1 - \alpha_k(1 - \lambda)(1 - \beta_k)] \|x_k - p\| + \alpha_k(1 - \beta_k) \|f(p) - p\| \\
 &\leq [1 - \alpha_k(1 - \lambda)(1 - \beta_k)] M_1 + \alpha_k(1 - \beta_k)(1 - \lambda) M_1 \\
 &= M_1.
 \end{aligned}$$

This shows that $\|x_n - p\| \leq M_1$ for all $n \geq 1$. Hence $\{x_n\}$ is bounded. In view of Lemma 1.1, we see that

$$\begin{aligned}
& \|w_{n+1} - w_n\| \\
& \leq \|T_{r_{n+1}}(I - r_{n+1}A)x_{n+1} - T_{r_{n+1}}(I - r_nA)x_n\| \\
& \quad + \|T_{r_{n+1}}(I - r_nA)x_n - T_{r_n}(I - r_nA)x_n\| \\
& \leq \|(I - r_{n+1}A)x_{n+1} - (I - r_nA)x_n\| \\
(2.3) \quad & \quad + \|T_{r_{n+1}}(I - r_nA)x_n - T_{r_n}(I - r_nA)x_n\| \\
& \leq \|x_{n+1} - x_n\| + |r_{n+1} - r_n| \|Ax_n\| \\
& \quad + \frac{|r_{n+1} - r_n|}{r_{n+1}} \|T_{r_{n+1}}(I - r_nA)x_n - (I - r_nA)x_n\| \\
& \leq \|x_{n+1} - x_n\| + |r_{n+1} - r_n| \|Ax_n\| + \frac{|r_{n+1} - r_n|}{r_{n+1}} M_2,
\end{aligned}$$

where M_2 is an appropriate constant such that

$$M_2 = \sup\{\|T_{r_{n+1}}(I - r_nA)x_n - (I - r_nA)x_n\| : n \geq 1\}.$$

Note that

$$\begin{aligned}
(2.4) \quad & \|z_{n+1} - z_n\| \leq \gamma_{n+1} \|w_{n+1} - w_n\| + (1 - \gamma_{n+1}) \|Tw_{n+1} - Tw_n\| \\
& \quad + |\gamma_{n+1} - \gamma_n| (\|w_n\| + \|Tw_n\|) \\
& \leq \|w_{n+1} - w_n\| + |\gamma_{n+1} - \gamma_n| M_3,
\end{aligned}$$

where M_3 is an appropriate constant such that $M_3 = \sup\{\|w_n\| + \|Tw_n\| : n \geq 1\}$. Substituting (2.3) into (2.4), we obtain that

$$\begin{aligned}
(2.5) \quad & \|z_{n+1} - z_n\| \leq \|x_{n+1} - x_n\| + |r_{n+1} - r_n| \|Ax_n\| + \frac{|r_{n+1} - r_n|}{r_{n+1}} M_2 \\
& \quad + |\gamma_{n+1} - \gamma_n| M_3 \\
& \leq \|x_{n+1} - x_n\| + (2|r_{n+1} - r_n| + |\gamma_{n+1} - \gamma_n|) M_4,
\end{aligned}$$

where M_4 is an appropriate constant such that

$$M_4 = \max\left\{\sup\{\|Ax_n\| : n \geq 1\}, \frac{M_2}{a}, M_3\right\}.$$

On the other hand, we have

$$\begin{aligned}
(2.6) \quad & \|y_{n+1} - y_n\| \leq \alpha_{n+1} \|f(x_{n+1}) - f(x_n)\| + (1 - \alpha_{n+1}) \|z_{n+1} - z_n\| \\
& \quad + |\alpha_{n+1} - \alpha_n| \|f(x_n) - z_n\| \\
& \leq \lambda \alpha_{n+1} \|x_{n+1} - x_n\| + (1 - \alpha_{n+1}) \|z_{n+1} - z_n\| \\
& \quad + |\alpha_{n+1} - \alpha_n| \|f(x_n) - z_n\|.
\end{aligned}$$

Substituting (2.5) into (2.6), we obtain that

$$\begin{aligned}
(2.7) \quad & \|y_{n+1} - y_n\| \\
& \leq \|x_{n+1} - x_n\| + (2|r_{n+1} - r_n| + |\gamma_{n+1} - \gamma_n| + |\alpha_{n+1} - \alpha_n|) M_5,
\end{aligned}$$

where M_5 is an appropriate constant such that $M_5 = \max\{\sup\{\|f(x_n) - z_n\| : n \geq 1\}, M_4\}$. It follows from (2.7) that

$$\begin{aligned} & \|Sy_{n+1} - Sy_n\| \\ & \leq \|y_{n+1} - y_n\| \\ & \leq \|x_{n+1} - x_n\| + (2|r_{n+1} - r_n| + |\gamma_{n+1} - \gamma_n| + |\alpha_{n+1} - \alpha_n|)M_5. \end{aligned}$$

This in turn implies that

$$\begin{aligned} & \|Sy_{n+1} - Sy_n\| - \|x_{n+1} - x_n\| \\ & \leq (2|r_{n+1} - r_n| + |\gamma_{n+1} - \gamma_n| + |\alpha_{n+1} - \alpha_n|)M_5. \end{aligned}$$

From the conditions (b)-(d), we see that

$$\limsup_{n \rightarrow \infty} (\|Sy_{n+1} - Sy_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

In view of Lemma 1.2, we see that

$$(2.8) \quad \lim_{n \rightarrow \infty} \|Sy_n - x_n\| = 0.$$

Consequently, we obtain that

$$(2.9) \quad \lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = \lim_{n \rightarrow \infty} (1 - \beta_n) \|Sy_n - x_n\| = 0.$$

It follows from (2.2) that

$$\begin{aligned} & \|x_{n+1} - p\|^2 \\ & \leq \beta_n \|x_n - p\|^2 + (1 - \beta_n) \|Sy_n - p\|^2 \\ (2.10) \quad & \leq \beta_n \|x_n - p\|^2 + (1 - \beta_n) (\|\alpha_n(f(x_n) - p) + (1 - \alpha_n)(z_n - p)\|^2) \\ & \leq \beta_n \|x_n - p\|^2 + \alpha_n \|f(x_n) - p\|^2 + (1 - \beta_n) \|z_n - p\|^2 \\ & \leq \beta_n \|x_n - p\|^2 + \alpha_n \|f(x_n) - p\|^2 + (1 - \beta_n) \|w_n - p\|^2. \end{aligned}$$

In view (2.1), we see that

$$\begin{aligned} & \|x_{n+1} - p\|^2 \\ & \leq \|x_n - p\|^2 + \alpha_n \|f(x_n) - p\|^2 - r_n(1 - \beta_n)(2\alpha - r_n) \|Ax_n - Ap\|^2. \end{aligned}$$

From the conditions (a) and (b), we arrive at

$$\begin{aligned} & a(1 - d)(2\alpha - b) \|Ax_n - Ap\|^2 \\ & \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n \|f(x_n) - p\|^2 \\ & \leq (\|x_n - p\| + \|x_{n+1} - p\|) \|x_n - x_{n+1}\| + \alpha_n \|f(x_n) - p\|^2. \end{aligned}$$

It follows from (2.9) and the condition (c) that

$$(2.11) \quad \lim_{n \rightarrow \infty} \|Ax_n - Ap\| = 0.$$

On the other hand, we have

$$\begin{aligned}
& \|w_n - p\|^2 \\
&= \|T_{r_n}(I - r_n A)x_n - T_{r_n}(I - r_n A)p\|^2 \\
&\leq \langle (I - r_n A)x_n - (I - r_n A)p, w_n - p \rangle \\
&= \frac{1}{2}(\|(I - r_n A)x_n - (I - r_n A)p\|^2 + \|w_n - p\|^2 \\
&\quad - \|(I - r_n A)x_n - (I - r_n A)p - (w_n - p)\|^2) \\
&= \frac{1}{2}(\|x_n - p\|^2 + \|w_n - p\|^2 - \|x_n - w_n - r_n(Ax_n - Ap)\|^2) \\
&= \frac{1}{2}(\|x_n - p\|^2 + \|w_n - p\|^2 - (\|x_n - w_n\|^2 \\
&\quad - 2r_n \langle x_n - w_n, Ax_n - Ap \rangle + r_n^2 \|Ax_n - Ap\|^2)).
\end{aligned}$$

This implies that

$$(2.12) \quad \|w_n - p\|^2 \leq \|x_n - p\|^2 - \|x_n - w_n\|^2 + 2r_n \|x_n - w_n\| \|Ax_n - Ap\|.$$

Combining (2.10) with (2.12), we arrive at

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\leq \|x_n - p\|^2 + \alpha_n \|f(x_n) - p\|^2 - (1 - \beta_n) \|x_n - w_n\|^2 \\
&\quad + 2r_n \|x_n - w_n\| \|Ax_n - Ap\|.
\end{aligned}$$

From the condition (a), we see that

$$\begin{aligned}
& (1 - d) \|x_n - w_n\|^2 \\
&\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n \|f(x_n) - p\|^2 \\
&\quad + 2r_n \|x_n - w_n\| \|Ax_n - Ap\| \\
&\leq (\|x_n - p\| + \|x_{n+1} - p\|) \|x_n - x_{n+1}\| + \alpha_n \|f(x_n) - p\|^2 \\
&\quad + 2r_n \|x_n - w_n\| \|Ax_n - Ap\|.
\end{aligned}$$

In view of the condition (c), (2.9) and (2.11), we obtain that

$$(2.13) \quad \lim_{n \rightarrow \infty} \|x_n - w_n\| = 0.$$

Note that

$$\|z_n - w_n\| \leq (1 - \gamma_n) \|Tw_n - w_n\|.$$

From the condition (d), we obtain that

$$(2.14) \quad \lim_{n \rightarrow \infty} \|z_n - w_n\| = 0.$$

On the other hand, we have

$$\begin{aligned}
\|Sy_n - y_n\| &\leq \|Sy_n - x_n\| + \|x_n - w_n\| + \|w_n - y_n\| \\
&\leq \|Sy_n - x_n\| + \|x_n - w_n\| + \|w_n - z_n\| + \|z_n - y_n\| \\
&\leq \|Sy_n - x_n\| + \|x_n - w_n\| + \|w_n - z_n\| + \alpha_n \|f(x_n) - z_n\|.
\end{aligned}$$

In view of the condition (c), (2.8), (2.13) and (2.14), we see that

$$(2.15) \quad \lim_{n \rightarrow \infty} \|S y_n - y_n\| = 0.$$

Note that $P_{\mathcal{F}}f$ is a contractive mapping. It follows that $P_{\mathcal{F}}f$ has a unique fixed point. Next, we use q to denote the unique fixed point. Now, we claim that

$$\limsup_{n \rightarrow \infty} \langle f(q) - q, y_n - q \rangle \leq 0.$$

To show this inequality, take a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that

$$(2.16) \quad \limsup_{n \rightarrow \infty} \langle f(q) - q, y_n - q \rangle = \lim_{i \rightarrow \infty} \langle f(q) - q, y_{n_i} - q \rangle.$$

We may, without loss of generality, assume that $y_{n_i} \rightharpoonup \eta$. Since C is closed and convex, we see that C is weakly closed. So, we have $\eta \in C$. Next, we show that $\eta \in \mathcal{F}$. First, we show $\eta \in EP(F, A)$. Note that

$$\begin{aligned} \|y_n - w_n\| &\leq \|y_n - z_n\| + \|z_n - w_n\| \\ &\leq \alpha_n \|f(x_n) - z_n\| + \|z_n - w_n\|. \end{aligned}$$

In view of the condition (c) and (2.14), we obtain that $\lim_{n \rightarrow \infty} \|y_n - w_n\| = 0$. It follows that $w_{n_i} \rightharpoonup \eta$. Since $w_n = T_{r_n}(x_n - r_n A x_n)$, for any $y \in C$, we have

$$F(w_n, y) + \langle A x_n, y - w_n \rangle + \frac{1}{r_n} \langle y - w_n, w_n - x_n \rangle \geq 0.$$

From the condition (A2), we see

$$\langle A x_n, y - w_n \rangle + \frac{1}{r_n} \langle y - w_n, w_n - x_n \rangle \geq F(y, w_n).$$

Replacing n by n_i , we arrive at

$$(2.17) \quad \langle A x_{n_i}, y - w_{n_i} \rangle + \left\langle y - w_{n_i}, \frac{w_{n_i} - x_{n_i}}{r_{n_i}} \right\rangle \geq F(y, w_{n_i}).$$

For t with $0 < t \leq 1$ and $y \in C$, let $y_t = t y + (1 - t) \eta$. Since $y \in C$ and $\eta \in C$, we have $y_t \in C$. It follows from (2.17) that

$$\begin{aligned} &\langle y_t - w_{n_i}, A y_t \rangle \\ &\geq \langle y_t - w_{n_i}, A y_t \rangle - \langle A x_{n_i}, y_t - w_{n_i} \rangle - \left\langle y_t - w_{n_i}, \frac{w_{n_i} - x_{n_i}}{r_{n_i}} \right\rangle \\ (2.18) \quad &+ F(y_t, w_{n_i}) \\ &= \langle y_t - w_{n_i}, A y_t - A w_{n_i} \rangle + \langle y - w_{n_i}, A w_{n_i} - A x_{n_i} \rangle \\ &\quad - \left\langle y_t - w_{n_i}, \frac{w_{n_i} - x_{n_i}}{r_{n_i}} \right\rangle + F(y_t, w_{n_i}). \end{aligned}$$

From (2.13), we have $A w_{n_i} - A x_{n_i} \rightarrow 0$ as $i \rightarrow \infty$. On the other hand, from the monotonicity of A , we obtain that

$$\langle y_t - w_{n_i}, A y_t - A w_{n_i} \rangle \geq 0.$$

It follows from (A4) and (2.18) that

$$(2.19) \quad \langle y_t - \eta, Ay_t \rangle \geq F(y_t, \eta).$$

From (A1), (A4) and (2.19), we see

$$\begin{aligned} 0 &= F(y_t, y_t) \leq tF(y_t, y) + (1-t)F(y_t, \eta) \\ &\leq tF(y_t, y) + (1-t)\langle y_t - \eta, Ay_t \rangle \\ &= tF(y_t, y) + (1-t)t\langle y - \eta, Ay_t \rangle, \end{aligned}$$

which yields that

$$F(y_t, y) + (1-t)\langle y - \eta, Ay_t \rangle \geq 0.$$

Letting $t \rightarrow 0$ in the above inequality, we arrive at

$$F(\eta, y) + \langle y - \eta, A\eta \rangle \geq 0.$$

This shows that $\eta \in EP(F, A)$. Next, we show that $\eta \in F(S)$. If not, we have $\eta \neq S\eta$. In view of the Opial condition, we see that

$$\begin{aligned} \liminf_{i \rightarrow \infty} \|y_{n_i} - \eta\| &< \liminf_{i \rightarrow \infty} \|y_{n_i} - S\eta\| \\ &\leq \liminf_{i \rightarrow \infty} \{\|y_{n_i} - Sy_{n_i}\| + \|Sy_{n_i} - S\eta\|\} \\ &\leq \liminf_{i \rightarrow \infty} \{\|y_{n_i} - Sy_{n_i}\| + \|y_{n_i} - \eta\|\}. \end{aligned}$$

It follows that

$$\liminf_{i \rightarrow \infty} \|y_{n_i} - \eta\| < \liminf_{i \rightarrow \infty} \|y_{n_i} - \eta\|,$$

which derives a contradiction. Thus, we have $\eta \in F(S)$. Next, we show that $\eta \in F(T)$. If not, we have $\eta \neq T\eta$. Define a mapping R_n by $R_n x = \gamma_n x + (1 - \gamma_n)Tx$ for each $x \in C$. It follows that R_n is nonexpansive and $F(T) = F(R_n)$ for each $n \geq 1$. On the other hand, we from (2.14) know that $\|R_n w_n - w_n\| \rightarrow 0$ as $n \rightarrow \infty$. From the Opial condition, we see that

$$\begin{aligned} \liminf_{i \rightarrow \infty} \|w_{n_i} - \eta\| &< \liminf_{i \rightarrow \infty} \|w_{n_i} - R_{n_i}\eta\| \\ &\leq \liminf_{i \rightarrow \infty} \{\|w_{n_i} - R_{n_i}w_{n_i}\| + \|R_{n_i}w_{n_i} - R_{n_i}\eta\|\} \\ &\leq \liminf_{i \rightarrow \infty} \{\|w_{n_i} - R_{n_i}w_{n_i}\| + \|w_{n_i} - \eta\|\}. \end{aligned}$$

This also derives a contradiction. This implies that $\eta \in F(R_n) = F(T)$ for each $n \geq 1$. This shows that $\eta \in \mathcal{F}$. From (2.16), we see that

$$(2.20) \quad \limsup_{n \rightarrow \infty} \langle f(q) - q, y_n - q \rangle = \langle f(q) - q, \eta - q \rangle \leq 0.$$

Finally, we show that $x_n \rightarrow q$ as $n \rightarrow \infty$. Note that

$$\begin{aligned} &\|x_{n+1} - q\|^2 \\ &= \beta_n \langle x_n - q, x_{n+1} - q \rangle + (1 - \beta_n) \langle Sy_n - q, x_{n+1} - q \rangle \\ &\leq \beta_n \|x_n - q\| \|x_{n+1} - q\| + (1 - \beta_n) \|Sy_n - q\| \|x_{n+1} - q\| \\ &\leq \frac{\beta_n}{2} (\|x_n - q\|^2 + \|x_{n+1} - q\|^2) + \frac{1 - \beta_n}{2} (\|y_n - q\|^2 + \|x_{n+1} - q\|^2). \end{aligned}$$

It follows that

$$(2.21) \quad \|x_{n+1} - q\|^2 \leq \beta_n \|x_n - q\|^2 + (1 - \beta_n) \|y_n - q\|^2.$$

On the other hand, we have

$$\begin{aligned} \|y_n - q\|^2 &= \alpha_n \langle f(x_n) - q, y_n - q \rangle + (1 - \alpha_n) \langle z_n - q, y_n - q \rangle \\ &\leq \alpha_n \langle f(x_n) - q, y_n - q \rangle + (1 - \alpha_n) \|z_n - q\| \|y_n - q\| \\ &\leq \alpha_n \langle f(x_n) - f(q), y_n - q \rangle + \alpha_n \langle f(q) - q, y_n - q \rangle \\ &\quad + (1 - \alpha_n) \|x_n - q\| \|y_n - q\| \\ &\leq [1 - \alpha_n(1 - \lambda)] \|x_n - q\| \|y_n - q\| + \alpha_n \langle f(q) - q, y_n - q \rangle \\ &\leq \frac{1 - \alpha_n(1 - \lambda)}{2} (\|x_n - q\|^2 + \|y_n - q\|^2) + \alpha_n \langle f(q) - q, y_n - q \rangle. \end{aligned}$$

It follows that

$$(2.22) \quad \|y_n - q\|^2 \leq [1 - \alpha_n(1 - \lambda)] \|x_n - q\|^2 + 2\alpha_n \langle f(q) - q, y_n - q \rangle.$$

Substituting (2.22) into (2.21), we arrive at

$$\begin{aligned} \|x_{n+1} - q\|^2 &\leq [1 - \alpha_n(1 - \beta_n)(1 - \lambda)] \|x_n - q\|^2 \\ &\quad + 2\alpha_n(1 - \beta_n) \langle f(q) - q, y_n - q \rangle. \end{aligned}$$

In view of (2.16), we from Lemma 1.3 see that $x_n \rightarrow q$ as $n \rightarrow \infty$. This completes the proof. \square

Remark 2.2. Theorem 2.1 includes Theorem TT2 as a special case. To be more precise, if $f(x) = u \in C$ for each $x \in C$ and $\gamma_n = 1$ for each $n \geq 1$, then Theorem 2.1 is reduced to Theorem TT2.

From Theorem 2.1, we have the following results on the equilibrium problem (1.3).

Corollary 2.3. *Let C be a nonempty closed convex subset of a real Hilbert space H and $F : C \times C \rightarrow \mathbb{R}$ be a bifunction satisfying (A1)-(A4). Let $S : C \rightarrow C$ be a nonexpansive mapping such that $\mathcal{F} := F(S) \cap EP(F) \neq \emptyset$ and $f : C \rightarrow C$ be a contractive mapping with the coefficient $\lambda \in (0, 1)$. Let $\{x_n\}$ be a sequence defined by the following algorithm: $x_1 \in C$ and*

$$\begin{cases} F(w_n, y) + \frac{1}{r_n} \langle y - w_n, w_n - x_n \rangle \geq 0, & \forall y \in C, \\ y_n = \alpha_n f(x_n) + (1 - \alpha_n) w_n, \\ x_{n+1} = \beta_n x_n + (1 - \beta_n) S y_n, & \forall n \geq 1, \end{cases}$$

where $\{r_n\}$ is a positive sequence and $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0, 1]$. Assume that the following conditions are satisfied

- (a) $0 < c \leq \beta_n \leq d < 1$,
- (b) $0 < a \leq r_n \leq b < 2\alpha$ and $\lim_{n \rightarrow \infty} (r_n - r_{n+1}) = 0$,
- (c) $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$.

Then $\{x_n\}$ converges strongly to $q \in \mathcal{F}$, where $q = P_{\mathcal{F}} f(q)$.

Proof. Putting $A = 0$ in Theorem 2.1, we can obtain the desired conclusion immediately. \square

Corollary 2.4. *Let C be a nonempty closed convex subset of a real Hilbert space H and $A : C \rightarrow H$ be an α -inverse-strongly monotone mapping. Let S and $T : C \rightarrow C$ be nonexpansive mappings such that $\mathcal{F} := F(S) \cap F(T) \cap VI(C, A) \neq \emptyset$ and $f : C \rightarrow C$ be a contractive mapping with the coefficient $\lambda \in (0, 1)$. Let $\{x_n\}$ be a sequence defined by the following algorithm: $x_1 \in C$ and*

$$\begin{cases} z_n = \gamma_n P_C(x_n - r_n A x_n) + (1 - \gamma_n) T P_C(x_n - r_n A x_n), \\ y_n = \alpha_n f(x_n) + (1 - \alpha_n) z_n, \\ x_{n+1} = \beta_n x_n + (1 - \beta_n) S y_n, \quad \forall n \geq 1, \end{cases}$$

where $\{r_n\}$ is a positive sequence and $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0, 1]$. Assume that the following conditions are satisfied

- (a) $0 < c \leq \beta_n \leq d < 1$,
- (b) $0 < a \leq r_n \leq b < 2\alpha$ and $\lim_{n \rightarrow \infty} (r_n - r_{n+1}) = 0$,
- (c) $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,
- (d) $\lim_{n \rightarrow \infty} \gamma_n = 1$.

Then $\{x_n\}$ converges strongly to $q \in \mathcal{F}$, where $q = P_{\mathcal{F}} f(q)$.

Proof. In Theorem 2.1, putting $F(x, y) = 0$ for each $x, y \in C$, we see that

$$\langle A x_n, y - w_n \rangle + \frac{1}{r_n} \langle y - w_n, w_n - x_n \rangle \geq 0, \quad \forall y \in C$$

is equivalent to

$$\langle x_n - r_n A x_n - w_n, w_n - y \rangle \geq 0, \quad \forall y \in C.$$

This implies that $w_n = P_C(x_n - r_n A x_n)$. From Theorem 2.1, we can obtain the desired results immediately. \square

Recall that $T : C \rightarrow C$ is said to be a k -strictly pseudocontractive [2] if there exists a constant $k \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C.$$

From [2], we see that if T is k -strictly pseudocontractive, then $I - T$ is $\frac{1-k}{2}$ -inverse-strongly monotone.

The following theorem is not hard to derive.

Corollary 2.5. *Let C be a nonempty closed convex subset of a real Hilbert space H , $F : C \times C \rightarrow \mathbb{R}$ be a bifunction satisfying (A1)-(A4) and $T_A : C \rightarrow C$ be an k -strictly pseudocontractive mapping. Let S and $T : C \rightarrow C$ be nonexpansive mappings such that $\mathcal{F} := F(S) \cap F(T) \cap EP(F, I - T_A) \neq \emptyset$ and $f : C \rightarrow C$ be a contractive mapping with the coefficient $\lambda \in (0, 1)$. Let $\{x_n\}$ be a sequence*

defined by the following algorithm: $x_1 \in C$ and

$$\begin{cases} F(w_n, y) + \langle x_n - T_A x_n, y - w_n \rangle + \frac{1}{r_n} \langle y - w_n, w_n - x_n \rangle \geq 0, & \forall y \in C, \\ z_n = \gamma_n w_n + (1 - \gamma_n) T w_n, \\ y_n = \alpha_n f(x_n) + (1 - \alpha_n) z_n, \\ x_{n+1} = \beta_n x_n + (1 - \beta_n) S y_n, & \forall n \geq 1, \end{cases}$$

where $\{r_n\}$ is a positive sequence and $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0, 1]$. Assume that the following conditions are satisfied

- (a) $0 < c \leq \beta_n \leq d < 1$,
- (b) $0 < a \leq r_n \leq b < 1 - k$ and $\lim_{n \rightarrow \infty} (r_n - r_{n+1}) = 0$,
- (c) $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,
- (d) $\lim_{n \rightarrow \infty} \gamma_n = 1$.

Then $\{x_n\}$ converges strongly to $q \in \mathcal{F}$, where $q = P_{\mathcal{F}} f(q)$.

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SUN YOUNG CHO
DEPARTMENT OF MATHEMATICS
GYEONGSANG NATIONAL UNIVERSITY
JINJU 660-701, KOREA
E-mail address: ooly61@yahoo.co.kr

SHIN MIN KANG
DEPARTMENT OF MATHEMATICS AND RINS
GYEONGSANG NATIONAL UNIVERSITY
JINJU 660-701, KOREA
E-mail address: smkang@gnu.ac.kr

XIAOLONG QIN
DEPARTMENT OF MATHEMATICS
HANGZHOU NORMAL UNIVERSITY
HANGZHOU 310036, P. R. CHINA
E-mail address: ljhhqx1@yahoo.com.cn; qxlxajh@163.com