Commun. Korean Math. Soc. **26** (2011), No. 1, pp. 37–49
DOI 10.4134/CKMS.2011.26.1.037

ASYMPTOTIC EQUIVALENCE FOR LINEAR DIFFERENTIAL SYSTEMS

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Abstract. We investigate the asymptotic equivalence for linear differential systems by means of the notions of t_∞-similarity and strong stability.

1. Introduction and preliminaries

Let $\mathfrak{M}$ be the set of all $n \times n$ real matrix functions defined and continuous on $\mathbb{R}_+ = [0, \infty)$ and $\mathfrak{S}$ be the subset of $\mathfrak{M}$ consisting of invertible matrices $S(t)$ that are of class C^1 with the property that $S(t)$ and $S^{-1}(t)$ are bounded. A matrix function $A \in \mathfrak{M}$ is said to be t_∞-*similar* to a matrix function $B \in \mathfrak{M}$ if there exists an $n \times n$ matrix $F(t)$ absolutely integrable over $\mathbb{R}_+$, i.e.,

$$\int_0^\infty |F(s)|ds < \infty,$$

such that

$$S'(t) + S(t)B(t) - A(t)S(t) = F(t), \ ' = \frac{d}{dt} \tag{1.1}$$

for some matrix function $S \in \mathfrak{S}$.

The notion of t_∞-similarity in the set of $n \times n$ continuous matrix functions defined on $\mathbb{R}_+$ was introduced by Conti [9]. It is deeply related to stability problems and is an equivalence relation. Also, it preserves strict, uniform and exponential stability of linear homogeneous differential systems. Hewer [12] studied the stability properties of the variational equation. Trench [15] weakened the condition of t_∞-similarity and introduced the notion of t_∞-quasisimilarity that is not symmetric or transitive, but still preserves stability properties. Also, in [16] he gave analogs of some of the results in [9, 15] for systems of difference

Received June 27, 2009.
2010 *Mathematics Subject Classification.* 34C41, 34D05, 34D23, 34G10.
Key words and phrases. linear differential system, adjoint system, strong stability, t_∞-similarity, linear asymptotic equilibrium, asymptotic equivalence.
This work was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education, Science and Technology(NRF-2010-0008835).

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equations. Choi et al. investigated h-stability of the differential systems via t_∞-similarity in [8].

Trench [16] introduced the notion of summable similarity which is a discrete analog of t_∞-similarity of matrix functions. Also, Choi and Koo [3] studied the stability properties for variational difference systems by means of the notion of n_∞-similarity which is quite different from Trench's definition in [16].

Let $x(t)$ be a solution of the differential system

$$x'(t) = h(t, x(t)),\ x(t_0) = x_0, \tag{1.2}$$

where $h : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous. The solution $x(t)$ is said to be *strongly stable* [11] if for each $\varepsilon > 0$ there is a corresponding $\delta = \delta(\varepsilon) > 0$ such that any solution $\bar{x}(t)$ of (1.2) which satisfies the inequality $|\bar{x}(t_1) - x(t_1)| < \delta$ for some $t_1 \geq t_0$ satisfies the inequality $|\bar{x}(t) - x(t)| < \varepsilon$ for all $t \geq t_0$.

This notion of strong stability was introduced by Ascoli [1]. Obviously, strong stability implies uniform stability, which in turn implies ordinary stability. For the linear system

$$x'(t) = C(t)x(t), \tag{1.3}$$

where $C \in \mathfrak{M}$, the system (1.3) is strongly stable if and only if there exists a positive constant M such that

$$|X(t)| \leq M,\ \ |X^{-1}(t)| \leq M \text{ for } \ \text{all } t \geq t_0, \tag{1.4}$$

where $X(t)$ is a fundamental matrix of (1.3) [11, Theorem 1, p. 54].

Two systems

$$x'(t) = f(t, x(t)),\ x(t_0) = x_0 \tag{1.5}$$

and

$$y'(t) = g(t, y(t)),\ y(t_0) = y_0 \tag{1.6}$$

are said to be *asymptotically equivalent* if, for every solution $x(t)$ of (1.5), there exists a solution $y(t)$ of (1.6) such that

$$x(t) = y(t) + o(1) \text{ as } t \to \infty \tag{1.7}$$

and conversely, for every solution $y(t)$ of (1.6), there exists a solution $x(t)$ of (1.5) satisfying (1.7).

If we know that two systems are asymptotically equivalent, and if we also know the asymptotic behavior of the solutions of one of the systems, then it is clear that we can obtain information about the asymptotic behavior of the solutions of the other systems. Choi et al. [5] studied the asymptotic equivalence for the nonlinear differential systems through their variational systems. Also, for the asymptotic equivalence between Volterra difference systems, see [4, 6].

In this paper we investigate the asymptotic equivalence between linear differential systems by means of t_∞-similarity and strong stability, and give the several illustrative examples. Throughout this paper $\mathbb{R}^n$ is the n-dimensional real Euclidean space with norm

$$|x| = \sum_{i=1}^{n} |x_i|,\ x = (x_1, \ldots, x_n) \in \mathbb{R}^n$$

and for an $n \times n$ matrix $A = (a_{ij})$, we define its norm $|A|$ by

$$|A| = \max_{j=1,\ldots,n} \sum_{i=1}^{n} |a_{ij}|.$$

2. Main results

We consider two linear differential systems

$$x'(t) = A(t)x(t) \tag{2.1}$$

and

$$y'(t) = B(t)y(t), \tag{2.2}$$

where $A, B \in \mathfrak{M}$. Firstly, we investigate stability properties between (2.1) and its adjoint system

$$w'(t) = -A^T(t)w(t), \tag{2.3}$$

where A^T is the transpose of A.

System (2.1) is said to be *restrictively stable* [2] if (2.1) is stable and its adjoint system (2.3) is stable. Restrictive stability implies uniform stability, and this, in turn, implies ordinary stability. The converses are not true [2]. We note that for linear system (2.1), restrictive stability and strong stability are equivalent.

Also, (2.1) is said to be *reducible* (*reducible to zero*, respectively) [2] if there exists an $n \times n$ absolutely continuous matrix function $L(t)$ which, together with its inverse $L^{-1}(t)$, is defined and bounded such that

$$L^{-1}(t)A(t)L(t) - L^{-1}(t)L'(t)$$

is a constant matrix (the zero matrix, respectively) on $\mathbb{R}_+$.

Theorem 2.1. *The following statements are equivalent*:

(i) *System* (2.1) *is strongly stable.*

(ii) *Adjoint system* (2.3) *is strongly stable.*

(iii) *System* (2.1) *is stable and* $\int_{t_0}^{t} \mathrm{trA(s)ds}$ *is bounded from below, where* $\mathrm{tr}A(t)$ *is the trace of* $A(t)$.

(iv) *System* (2.1) *is reducible to zero.*

Proof. Let $X(t)$ be a fundamental matrix of (2.1). Then a fundamental matrix $W(t)$ of (2.3) is given by $W(t) = (X^{-1}(t))^T$. Thus (2.1) is strongly stable if and only if there exists a positive constant M such that

$$|X(t)| \leq M,\ \ |X^{-1}(t)| \leq M \ \text{ for all } t \in \mathbb{R}_+,$$

by (1.4), if and only if there exists a positive constant M such that

$$|W(t)| = |(X^{-1}(t))^T| = |X^{-1}(t)| \le M$$

and

$$|W^{-1}(t)| = |X^T(t)| = |X(t)| \le M,$$

by the property of $|X^{-1}(t)| = |(X^{-1}(t))^T|$ for each $t \in \mathbb{R}_+$, if and only if the adjoint system (2.3) is strongly stable. The remaining part of proof follows from [2, (3.9.iii), p. 45], [2, (3.9.v), p. 46] and [11, p. 57]. ☐

We say that linear system (2.1) has *linear asymptotic equilibrium* if for every vector $\xi \in \mathbb{R}^n$ there exists a solution $x(t) = x(t, t_0, x_0)$ of (2.1) such that

$$\lim_{t\to\infty} x(t) = \xi. \tag{2.4}$$

Lemma 2.2 ([7, Theorem 3.2])**.** *(2.1) has linear asymptotic equilibrium if and only if (2.1) has a fundamental matrix $X(t)$ such that $\lim_{t\to\infty} X(t)$ exists and is invertible.*

Corollary 2.3. *System (2.1) has linear asymptotic equilibrium if and only if its adjoint system (2.3) has linear asymptotic equilibrium.*

Proof. We easily see that

$$\begin{aligned} \lim_{t\to\infty} W(t) &= [(\lim_{t\to\infty} X^{-1}(t))]^T \\ &= [X_\infty^{-1}]^T \equiv W_\infty, \end{aligned}$$

where $\lim_{t\to\infty} X(t) = X_\infty$ exists and it is invertible. Thus $\lim_{t\to\infty} X(t) = X_\infty$ exists, and X_∞ is invertible if and only if $\lim_{t\to\infty} W(t) = W_\infty$ exists and W_∞ is invertible. The result follows from Lemma 2.2. ☐

We can easily obtain the following asymptotic relation between solutions of (2.1) and (2.3).

Theorem 2.4. *Assume that (2.1) is strongly stable. Then for any solution $x(t, t_0, x_0)$ of (2.1) there exists a solution $w(t, t_0, w_0)$ of the adjoint system (2.3) such that the following asymptotic relationship holds:*

$$x(t) = w(t) + O(M) \quad \text{as} \quad t \to \infty$$

for some positive constant M, where O is the " big oh".

In view of Corollary 2.3, we obtain the following:

Corollary 2.5. *Assume that (2.1) has linear asymptotic equilibrium. Then (2.1) and (2.3) are asymptotically equivalent.*

Lemma 2.6 ([14, Theorem 21, p. 497])**.** *For systems (2.1) and (2.2) assume that a matrix function $A \in \mathfrak{M}$ is t_∞-similar to a matrix function $B \in \mathfrak{M}$. Then (2.1) is strongly stable if and only if (2.2) is also strongly stable.*

Note that Lemma 2.6 can be easily proved by using the notion of reducibility.

The following theorem shows that the asymptotic property between two linear systems is preserved under strong stability and t_∞-similarity.

Theorem 2.7. *Suppose that a matrix function $A \in \mathfrak{M}$ is t_∞-similar to a matrix function $B \in \mathfrak{M}$ and* (2.1) *is strongly stable. Let $X(t)$ and $Y(t)$ be fundamental matrices of* (2.1) *and* (2.2)*, and choose $S \in \mathfrak{S}$ so that* (1.1) *holds with $\int_0^\infty |F(s)|ds < \infty$. Then we have*

$$Y(t) = S^{-1}(t)X(t)[D(t_0) + o(1)] \text{ as } t \to \infty,$$

where $D(t_0) = X^{-1}(t_0)S(t_0)Y(t_0) + \int_{t_0}^\infty X^{-1}(s)F(s)Y(s)ds$ for each $t \ge t_0 \ge 0$, and o is the "little oh".

Proof. From [14, pp. 492–493], we have

$$S(t) = X(t)[X^{-1}(t_0)S(t_0)Y(t_0) + \int_{t_0}^t X^{-1}(s)F(s)Y(s)ds]Y^{-1}(t), \quad t \ge t_0 \ge 0. \tag{2.5}$$

Thus we have

$$Y(t) = S^{-1}(t)X(t)[X^{-1}(t_0)S(t_0)Y(t_0) + \int_{t_0}^t X^{-1}(s)F(s)Y(s)ds].$$

It follows from Lemma 2.6 that (2.2) is also strongly stable. Thus there exists a positive constant M such that

$$|X(t)| \le M, \quad |X^{-1}(t)| \le M,$$
$$|Y(t)| \le M, \quad |Y^{-1}(t)| \le M \text{ for all } t \ge t_0.$$

Putting $Q(t) = \int_{t_0}^t X^{-1}(s)F(s)Y(s)ds$ for any $t \ne s$ ($t > s$), we have

$$\begin{aligned} |Q(t) - Q(s)| &\le |\int_s^t X^{-1}(\tau)F(\tau)Y(\tau)d\tau| \\ &\le M^2 \int_s^t |F(\tau)|d\tau. \end{aligned}$$

Hence $Q(t)$ is convergent from the absolute integral of $\int_0^\infty F(\tau)d\tau$, i.e., $\lim_{t\to\infty} Q(t) = Q_\infty(t_0)$ exists. Therefore

$$\begin{aligned} Y(t) &= S^{-1}(t)X(t)[X^{-1}(t_0)S(t_0)Y(t_0) + \int_{t_0}^t X^{-1}(s)F(s)Y(s)ds] \\ &= S^{-1}(t)X(t)[X^{-1}(t_0)S(t_0)Y(t_0) + \int_{t_0}^\infty X^{-1}(s)F(s)Y(s)ds \\ &\quad - \int_t^\infty X^{-1}(s)F(s)Y(s)ds] \\ &= S^{-1}(t)X(t)[D(t_0) + o(1)] \text{ as } t \to \infty, \end{aligned}$$

where $D(t_0) = X^{-1}(t_0)S(t_0)Y(t_0) + \int_{t_0}^{\infty} X^{-1}(s)F(s)Y(s)ds$. This completes the proof. □

Corollary 2.8. *Assume that* $\int_0^\infty |A(t) - B(t)|dt < \infty$ *and* (2.1) *is strongly stable. Then we have the following asymptotic formula*:

$$Y(t) = X(t)[D(t_0) + o(1)] \text{ as } t \to \infty,$$

where $X(t)$ *and* $Y(t)$ *are fundamental matrices of* (2.1) *and* (2.2), *respectively. Moreover,* (2.1) *and* (2.2) *are asymptotically equivalent.*

Proof. Let $y(t) = y(t, t_0, y_0)$ be any solution of (2.2). Since strong stability implies boundedness, the solution $y(t)$ given by $y(t) = Y(t)Y^{-1}(t_0)y(t_0)$ is bounded. Putting $F(t) = B(t) - A(t)$ and $S(t) = I$, it follows from (2.5) that there exists a solution $x(t) = x(t, t_0, x_0)$ of (2.1) such that

$$\begin{aligned} y(t) &= Y(t)Y^{-1}(t_0)y_0 \\ &= X(t)X^{-1}(t_0)[y_0 + \int_{t_0}^{t} X(t_0)X^{-1}(s)F(s)Y(s)Y^{-1}(t_0)y_0 ds] \\ &= X(t)X^{-1}(t_0)[y_0 + \int_{t_0}^{\infty} X(t_0)X^{-1}(s)F(s)y(s)ds] \\ &\quad - X(t)\int_t^\infty X^{-1}(s)F(s)Y(s)Y^{-1}(t_0)y_0 ds \\ &= x(t) + o(1) \text{ as } t \to \infty, \end{aligned}$$

where $x_0 = y_0 + \int_{t_0}^{\infty} X(t_0)X^{-1}(s)F(s)y(s)ds$. Note that $\int_{t_0}^{\infty} X(t_0)X^{-1}(s)F(s)y(s)ds$ exists by means of $\int_0^\infty |F(s)|ds < \infty$ and boundedness of the solution $y(t)$ of (2.2).

Also, the converse holds. This completes the proof. □

Remark 2.9. In addition to the assumptions of Corollary 2.8, we assume that $|I - D(t_0)| < 1$ with $X(t_0) = I = Y(t_0)$. Then for any solution $x(t)$ of (2.1) there exists the unique solution $y(t)$ of (2.2) such that

$$x(t) = y(t) + o(1) \text{ as } t \to \infty,$$

and the converse similarly holds. Because we can choose a number t_0 so sufficiently large that

$$|I - D(t_0)| = |\int_{t_0}^{\infty} X^{-1}(s)F(s)Y(s)ds| < 1.$$

We easily see that $D(t_0)$ is invertible for sufficiently large number $t \geq t_0$. Hence we have the following asymptotic formula

$$Y(t) = X(t)[D(t_0) + o(1)] \text{ as } t \to \infty.$$

Theorem 2.10. *Assume that* $\int_0^\infty |A(t) - B(t)|dt < \infty$ *and* (2.1) *has linear asymptotic equilibrium. Then* (2.2) *has linear asymptotic equilibrium.*

Proof. For any $t \neq s$ ($t > s$), it follows from [14, pp. 492–493] that we have

$$\begin{aligned}|Y(t) - Y(s)| \leq& |X(t) - X(s)||X^{-1}(t_0)Y(t_0)| \\ &+ |X(t) - X(s)|| \int_{t_0}^{s} X^{-1}(\tau)F(\tau)Y(\tau)d\tau| \\ &+ |X(t)|| \int_{s}^{t} X^{-1}(\tau)F(\tau)Y(\tau)d\tau|,\end{aligned}$$

where $F(t) = B(t) - A(t)$. Since $\lim_{t\to\infty} X(t) = X_\infty$ exists and $\int_0^\infty |F(\tau)|d\tau < \infty$, $Y(t)$ has a Cauchy property, so $\lim_{t\to\infty} Y(t) = Y_\infty$ exists. Put

$$P(t) = \int_t^\infty X(t_0)X^{-1}(\tau)F(\tau)Y(\tau)Y^{-1}(t_0)d\tau.$$

Since $P(t_0)$ exists for each t_0, we can choose a number t_0 so sufficiently large that

$$|P(t_0)| < 1.$$

It follows that $I + P(t_0)$ has an invertible matrix [13].

Thus we obtain

$$\begin{aligned}&\lim_{t\to\infty} Y(t)Y^{-1}(t_0) \\ &= \lim_{t\to\infty} \left(X(t)X^{-1}(t_0) \left[I + \int_{t_0}^{t} X(t_0)X^{-1}(\tau)F(\tau)Y(\tau)Y^{-1}(t_0)d\tau \right] \right) \\ &= X_\infty X^{-1}(t_0)[I + P(t_0)].\end{aligned}$$

Since $I + P(t_0)$ and X_∞ are invertible, Y_∞ is invertible. Hence (2.2) has linear asymptotic equilibrium by Lemma 2.2. □

We can obtain the following theorem by using the well-known variation of constants formula.

Theorem 2.11. *Assume that* $\int_0^\infty |A(t) - B(t)|dt < \infty$ *and* (2.1) *is strongly stable. Then* (2.1) *and* (2.2) *are asymptotically equivalent.*

Proof. Let $x(t) = x(t, t_0, x_0)$ be any solution of (2.1). Since (2.2) can be written as

$$y'(t) = B(t)y(t) = A(t)y(t) + (B(t) - A(t))y(t),$$

the solution $y(t, t_0, y_0)$ of (2.2) is given by

$$y(t) = X(t)X^{-1}(t_0)[I + \int_{t_0}^{t} X(t_0)X^{-1}(\tau)F(\tau)Y(\tau)Y^{-1}(t_0)d\tau]y_0,$$

where $F(t) = B(t) - A(t)$. Thus we can find the unique solution y_0 of the linear system $(I + P(t_0))y_0 = x_0$ such that the solution of the linear system is given by $y_0 = (I + P(t_0))^{-1}x_0$. Hence there exists a solution $y(t, t_0, y_0)$ of (2.2) with $y_0 = (I + P(t_0))^{-1}x_0$ such that the following asymptotic formula holds:

$$\begin{aligned} y(t) &= X(t)X^{-1}(t_0)\,(I + P(t_0) - P(t))\,y_0 \\ &= x(t, t_0, x_0) - X(t)X^{-1}(t_0)P(t)(I + P(t_0))^{-1}x_0 \\ &= x(t) + o(1) \text{ as } t \to \infty, \end{aligned}$$

since $|P(t)| \to 0$ as $t \to \infty$. Also, the converse similarly holds. □

Corollary 2.12. *Assume that* $\int_0^\infty |A(t) - B(t)|dt < \infty$ *and* (2.1) *has linear asymptotic equilibrium. Then* (2.1) *and* (2.2) *are asymptotically equivalent.*

Conti [10] showed that linear asymptotic equilibrium for linear differential systems is preserved by the notion of t_∞-similarity under condition that $\lim_{t\to\infty} S(t)$ and $\lim_{t\to\infty} S^{-1}(t)$ both exist. Also, we can prove that (2.1) and (2.2) are asymptotically equivalent under the some conditions.

Lemma 2.13 ([7, Theorem 2.3])**.** *Suppose that a fundamental matrix* $X(t)$ *of* (2.1) *is bounded for each* $t \ge t_0$ *and* $|\det(X(t))| > \alpha > 0$ *for each* $t \ge t_0$ *and some positive constant* α*. Then* (2.1) *is strongly stable.*

Corollary 2.14. *Assume that a matrix function* $A \in \mathfrak{M}$ *is* t_∞*-similar to a matrix function* $B \in \mathfrak{M}$ *with* $S(t) = I$*. Suppose that* $\lim_{t\to\infty} X(t)$ *exists and* $|\det(X(t))| > \alpha > 0$ *for each* $t \ge t_0$ *and some positive constant* α*. Then* (2.1) *and* (2.2) *are asymptotically equivalent.*

Proof. It follows from Lemma 2.2 that (2.1) has linear asymptotic equilibrium, and so (2.1) and (2.2) are asymptotically equivalent by Theorem 2.11. □

Now, we consider the nonhomogenous differential system of (2.1)

$$y'(t) = A(t)y(t) + g(t), \tag{2.6}$$

where $g(t)$ is a continuous function on $\mathbb{R}_+$.

We prove that (2.1) and (2.6) are asymptotically equivalent when the condition on linear asymptotic equilibrium is replaced by the condition on strong stability.

Theorem 2.15. *Assume that* (2.1) *is strongly stable and* $\int_0^\infty |g(t)|dt < \infty$*. Then* (2.1) *and* (2.6) *are asymptotically equivalent.*

Proof. Let $x(t, t_0, x_0)$ be any solution (2.1). Put $p(t) = \int_{t_0}^t X(t_0)X^{-1}(s)g(s)ds$. Then we easily see that $p(t)$ is convergent because $\int_{t_0}^\infty |g(t)|dt < \infty$ and $X^{-1}(t)$ is bounded. Thus there exists the solution $y(t, t_0, y_0)$ of (2.6) with the initial value $y_0 = x_0 - p_\infty(t_0)$ satisfying

$$\begin{aligned} y(t) &= X(t)X^{-1}(t_0)y_0 + X(t)\int_{t_0}^t X^{-1}(s)g(s)ds \\ &= x(t, t_0, x_0) + X(t)X^{-1}(t_0)[-p_\infty(t_0) + p(t)] \\ &= x(t) + o(1) \text{ as } t \to \infty, \end{aligned}$$

where $\lim_{t\to\infty} p(t) = p_\infty(t_0)$.

Conversely, letting $x_0 = y_0 + p_\infty(t_0)$, the similar asymptotic formula holds. This completes the proof. □

Corollary 2.16. *Suppose that* (2.1) *has linear asymptotic equilibrium and* $\int_0^\infty |g(t)|dt < \infty$. *Then* (2.1) *and* (2.6) *are asymptotically equivalent.*

3. Examples

In this section we give several examples as the illustrations of Section 2.

Example 3.1. Let us consider linear differential system

$$x'(t) = A(t)x(t) = \begin{pmatrix} \frac{-e^{-t}}{2+e^{-t}} & 0 \\ 0 & 0 \end{pmatrix} x(t) \tag{3.1}$$

and its adjoint system

$$w'(t) = -A^T(t)w(t) = \begin{pmatrix} \frac{e^{-t}}{2+e^{-t}} & 0 \\ 0 & 0 \end{pmatrix} w(t). \tag{3.2}$$

Then (3.1) and its adjoint system (3.2) are asymptotically equivalent.

Proof. Note that a fundamental matrix $X(t)$ of (3.1) is given by $X(t) = \begin{pmatrix} \frac{2+e^{-t}}{3} & 0 \\ 0 & 1 \end{pmatrix}$. Thus $\lim_{t\to\infty} X(t) = \mathrm{X}_\infty$ exists and is invertible. Hence (3.1) has linear asymptotic equilibrium, and (3.1) and (3.2) are asymptotically equivalent by Corollary 2.5. □

However we note that linear differential system (2.1) and its adjoint system (2.3) are not asymptotically equivalent in general under the assumption of strong stability of (2.1). The following example shows this fact.

Example 3.2. Let us consider the linear differential system

$$x'(t) = A(t)x(t) = \begin{pmatrix} -\sin t & 0 \\ 0 & \cos t \end{pmatrix} x(t) \tag{3.3}$$

and its adjoint system

$$w'(t) = -A^T(t)w(t) = \begin{pmatrix} \sin t & 0 \\ 0 & -\cos t \end{pmatrix} w(t),\ t \ge t_0 \ge 0, \tag{3.4}$$

where $A(t) = \begin{pmatrix} -\sin t & 0 \\ 0 & \cos t \end{pmatrix}$ is a continuous 2×2 matrix and $-A^T(t) = \begin{pmatrix} \sin t & 0 \\ 0 & -\cos t \end{pmatrix}$. Then (3.3) and (3.4) are not asymptotically equivalent.

Proof. Note that $X(t)$ and $X^{-1}(t)$ of (3.3) are given by

$$X(t) = \begin{pmatrix} \exp(\cos t - 1) & 0 \\ 0 & \exp(\sin t) \end{pmatrix},\quad t \ge t_0 = 0,$$

$$X^{-1}(t) = \begin{pmatrix} \exp(1 - \cos t) & 0 \\ 0 & \exp(-\sin t) \end{pmatrix} = W(t)^T,$$

respectively, where $W(t)$ is a fundamental matrix of (3.4). Then there exists a positive constant $M \geq e$ such that

$$|X(t)| \leq M, \;\; |X^{-1}(t)| \leq M, \;\; t \geq 0.$$

Thus (3.3) is strongly stable.

Suppose that for the solution $x(t, t_0, x_0)$ of (3.3) with the initial value $x_1(t_0) \neq 0$ there exists a solution $w(t, t_0, w_0)$ of (3.4) such that the following asymptotic relationship holds:

$$x(t) = w(t) + o(1) \;\text{ as }\; t \to \infty.$$

Then we obtain

$$\begin{aligned}
x(t) &= X(t)\begin{pmatrix} x_1(t_0) \\ x_2(t_0) \end{pmatrix} \\
&= \begin{cases} \begin{pmatrix} e^{-2}x_1(t_0) \\ x_2(t_0) \end{pmatrix}, & t = (2k-1)\pi \\ \\ \begin{pmatrix} x_1(t_0) \\ x_2(t_0) \end{pmatrix}, & t = (2k)\pi, \; k \in \mathbb{Z} \end{cases} \\
&= w(t) = [X^{-1}(t)]^T \begin{pmatrix} w_1(t_0) \\ w_2(t_0) \end{pmatrix} \\
&= \begin{cases} \begin{pmatrix} e^{2}w_1(t_0) \\ w_2(t_0) \end{pmatrix}, & t = (2k-1)\pi \\ \\ \begin{pmatrix} w_1(t_0) \\ w_2(t_0) \end{pmatrix}, & t = (2k)\pi, \; k \in \mathbb{Z}. \end{cases}
\end{aligned}$$

Thus we have

$$x(t) = \begin{pmatrix} e^{-2}x_1(t_0) \\ x_2(t_0) \end{pmatrix} = \begin{pmatrix} e^{2}w_1(t_0) \\ w_2(t_0) \end{pmatrix}, \; t = (2k-1)\pi \tag{3.5}$$

$$= \begin{pmatrix} x_1(t_0) \\ x_2(t_0) \end{pmatrix} = \begin{pmatrix} w_1(t_0) \\ w_2(t_0) \end{pmatrix}, \; t = (2k)\pi, \; k \in \mathbb{Z} \tag{3.6}$$

$$= w(t) + o(1) \text{ as } k \to \infty.$$

From (3.5) and (3.6) we obtain

$$x_1(t_0) = e^4 w_1(t_0) = e^4 x_1(t_0), \; k \in \mathbb{Z}.$$

Thus we have $x_1(t_0) = 0$. This contradicts the fact $x_1(t_0) \neq 0$. Hence (3.3) and (3.4) are not asymptotically equivalent. □

Example 3.3. To illustrate Corollary 2.8, we consider two linear systems

$$x'(t) = A(t)x(t) = \begin{pmatrix} -\sin t & 0 \\ 0 & \cos t \end{pmatrix} x(t), \; t \geq t_0 = 0 \tag{3.7}$$

and

$$y'(t) = B(t)y(t) = \begin{pmatrix} -\sin t + \frac{ae^{at}}{2+e^{at}} & 0 \\ 0 & \cos t \end{pmatrix} y(t), \ t \ge t_0 = 0, \tag{3.8}$$

where $A(t) = \begin{pmatrix} -\sin t & 0 \\ 0 & \cos t \end{pmatrix}$ and $B(t) = \begin{pmatrix} -\sin t + \frac{ae^{at}}{2+e^{at}} & 0 \\ 0 & \cos t \end{pmatrix}$ with a negative constant a. Then (3.7) and (3.8) are asymptotically equivalent.

Proof. We note that $A(t)$ and $B(t)$ are t_∞-similar with $S(t) = I$ and $F(t) = \begin{pmatrix} \frac{ae^{at}}{2+e^{at}} & 0 \\ 0 & 0 \end{pmatrix}$. Also, since fundamental matrices $X(t)$ of (3.7) and $Y(t)$ of (3.8) are given by

$$X(t) = \begin{pmatrix} \exp(\cos t - 1) & 0 \\ 0 & \exp(\sin t) \end{pmatrix},$$
$$Y(t) = \begin{pmatrix} \frac{(2+e^{at})}{3} \exp(\cos t - 1) & 0 \\ 0 & \exp(\sin t) \end{pmatrix},$$

respectively, both (3.7) and (3.8) are strongly stable. Then we obtain the asymptotic formula

$$\begin{aligned} Y(t) &= X(t) \left[I + \int_0^t X^{-1}(s)F(s)Y(s)ds \right] \\ &= X(t) \left[I + \begin{pmatrix} \int_0^\infty (\frac{ae^{as}}{3})ds & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} \int_t^\infty (\frac{ae^{as}}{3})ds & 0 \\ 0 & 0 \end{pmatrix} \right] \\ &= X(t)[I + P(0) - P(t)] \\ &= X(t)[I + P(0) + o(1)], \end{aligned}$$

where $P(t) = \begin{pmatrix} \int_t^\infty (\frac{ae^{as}}{3})ds & 0 \\ 0 & 0 \end{pmatrix}$.

Furthermore, we have

$$\begin{aligned} |P(0)| &= \left| \lim_{t\to\infty} \begin{pmatrix} \int_0^t (\frac{ae^{as}}{3})ds & 0 \\ 0 & 0 \end{pmatrix} \right| = \lim_{t\to\infty} \left| \int_0^t \frac{ae^{as}}{3} ds \right| \\ &= \left| -\frac{1}{3} \right| = \frac{1}{3}. \end{aligned}$$

Since $|P(0)| < 1$, $I + P(0)$ is invertible, and so, for any solution $x(t, 0, x_0)$ of (3.7), there exists the unique solution $y(t, 0, y_0)$ of (3.8) with $y_0 = [I+P(0)]^{-1}x_0$ such that the asymptotic formula holds. Also, the converse holds by letting $x_0 = [I + P(0)]y_0$. □

We give an example to illustrate Theorem 2.15.

Example 3.4. Consider the homogeneous differential system

$$x'(t) = A(t)x(t) = \begin{pmatrix} -\sin t & 0 \\ 0 & \cos t \end{pmatrix} x(t) \tag{3.9}$$

and nonhomogeneous differential system

$$y'(t) = A(t)y(t) + g(t) = \begin{pmatrix} -\sin t & 0 \\ 0 & \cos t \end{pmatrix} y(t) + \begin{pmatrix} e^{-t} \\ e^{-t} \end{pmatrix}. \tag{3.10}$$

Then (3.9) and (3.10) are asymptotically equivalent.

Proof. Since a fundamental matrix $X(t)$ of (3.9) is given by

$$X(t) = \begin{pmatrix} e^{(\cos t - 1)} & 0 \\ 0 & e^{\sin t} \end{pmatrix}, \ t \geq t_0 = 0,$$

(3.9) is strongly stable. Also, the solution $y(t, t_0, y_0)$ of (3.10) is given by

$$\begin{aligned} y(t) &= X(t,0)y_0 + \int_0^t X(t,s)g(s)ds \\ &= \begin{pmatrix} e^{(\cos t - 1)} & 0 \\ 0 & e^{\sin t} \end{pmatrix} \left[y_0 + \int_{t_0}^t \begin{pmatrix} e^{(1-\cos s)} & 0 \\ 0 & e^{-\sin s} \end{pmatrix} \begin{pmatrix} e^{-s} \\ e^{-s} \end{pmatrix} ds \right] \\ &= \begin{pmatrix} e^{(\cos t - 1)} & 0 \\ 0 & e^{\sin t} \end{pmatrix} \left[y_0 + \begin{pmatrix} e \int_0^t e^{(-\cos s - s)} ds \\ \int_0^t e^{(-\sin s - s)} ds \end{pmatrix} \right], \ t \geq t_0 = 0. \end{aligned}$$

Note that $\int_0^\infty e^{(-\cos s - s)} ds < \infty$ and $\int_0^\infty e^{(-\sin s - s)} ds < \infty$. Since $\int_{t_0}^\infty e^{-s} ds < \infty$ and (3.9) is strongly stable, (3.9) and (3.10) are asymptotically equivalent by Theorem 2.15. □

References

[1] G. Ascoli, *Osservazioni sopra alcune questioni di stabilità. I*, Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **9** (1950), 129–134.

[2] L. Cesari, *Asymptotic Behavior and Stability Problems in Ordinary Differential Equations*, Academic Press Inc., Publishers, New York; Springer-Verlag, Berlin-Gottingen-Heidelberg, 1963.

[3] S. K. Choi and N. J. Koo, *Variationally stable difference systems by n_∞-similarity*, J. Math. Anal. Appl. **249** (2000), no. 2, 553–568.

[4] ———, *Asymptotic equivalence between two linear Volterra difference systems*, Comput. Math. Appl. **47** (2004), no. 2-3, 461–471.

[5] S. K. Choi, N. J. Koo, and S. Dontha, *Asymptotic property in variation for nonlinear differential systems*, Appl. Math. Lett. **18** (2005), no. 1, 117–126.

[6] S. K. Choi, N. J. Koo, and Y. H. Goo, *Asymptotic property of nonlinear Volterra difference systems*, Nonlinear Anal. **51** (2002), no. 2, Ser. A: Theory Methods, 321–337.

[7] S. K. Choi, N. J. Koo, and D. M. Im, *Asymptotic equivalence between linear differential systems*, Bull. Korean Math. Soc. **42** (2005), no. 4, 691–701.

[8] S. K. Choi, N. J. Koo, and H. S. Ryu, *h-stability of differential systems via t_∞-similarity*, Bull. Korean Math. Soc. **34** (1997), no. 3, 371–383.

[9] R. Conti, *Sulla t-similitudine tra matrici e la stabilità dei sistemi differenziali lineari*, Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **19** (1955), 247–250.

[10] ———, *Linear differential equations asymptotically equivalent to $\dot{x} = 0$*, Riv. Mat. Univ. Parma (4) **5** (1979), part 2, 847–853.

[11] W. A. Coppel, *Stability and Asymptotic Behavior of Differential Equations*, D. C. Heath and Company, Boston, 1965.

[12] G. A. Hewer, *Stability properties of the equation of first variation by t_∞-similarity*, J. Math. Anal. Appl. **41** (1973), 336–344.
[13] L. A. Lusternik and W. S. Sobolev, *Functional Analysis*, Science, Moscov, 1965.
[14] G. Sansone and R. Conti, *Non-linear Differential Equatioins*, The Macmillan Company, New York, 1964.
[15] W. F. Trench, *On t_∞ quasisimilarity of linear systems*, Ann. Mat. Pura Appl. (4) **142** (1985), 293–302.
[16] ———, *Linear asymptotic equilibrium and uniform, exponential, and strict stability of linear difference systems*, Advances in difference equations, II. Comput. Math. Appl. **36** (1998), no. 10-12, 261–267.

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