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COMPOSITE IMPLICIT RANDOM ITERATIONS FOR APPROXIMATING COMMON RANDOM FIXED POINT FOR A FINITE FAMILY OF ASYMPTOTICALLY NONEXPANSIVE RANDOM OPERATORS

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Abstract. In the present work we construct a composite implicit random iterative process with errors for a finite family of asymptotically nonexpansive random operators and discuss a necessary and sufficient condition for the convergence of this process in an arbitrary real Banach space. It is also proved that this process converges to the common random fixed point of the finite family of asymptotically nonexpansive random operators in the setting of uniformly convex Banach spaces. The present work also generalizes a recently established result in Banach spaces.

1. Introduction

Random fixed point theory was initiated by the Prague school of probabilists in the works of Hans [10] and Spacek [21] as stochastic generalization of deterministic fixed point theory. After that till recent times a large number of research papers focussing on various aspects of random fixed point theory have appeared in recent literatures. Some of these references are noted in [2], [6], [12], [13], [14] and [16].

Fixed point iteration schemes for nonlinear operators on Banach and Hilbert spaces have been developed and studied by many authors in recent times. The book of Berinde [3] gives a comprehensive survey of the development in this field.

The development of random fixed point iterations was initiated by Choudhury in [5] where random Ishikawa iteration scheme was defined and its strong convergence to a random fixed point in Hilbert spaces was discussed. After that several authors have worked on random fixed point iterations some of which are noted in [1], [4], [7], [8], [9], [17], [18]. In 2005 Beg and Abbas

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[1] constructed and studied different random iterative algorithms for weakly contractive and asymptotically nonexpansive random operators on arbitrary Banach spaces. They also established the convergence of an implicit random iteration process to a common random fixed point for a finite family of asymptotically quasi-nonexpansive random operators. Very recently Plubtieng et al. [17] constructed and established the convergence of an implicit random iteration process with errors for a common random fixed point of a finite family of asymptotically quasi-nonexpansive random operators in the setting of uniformly convex Banach spaces.

The purpose of this paper is to construct a composite implicit random iterative scheme with errors for a finite family of asymptotically nonexpansive random operators and to study the convergence of this iterative process in Banach spaces. Our results extend and improve some recent results in the existing literature.

2. Preliminaries

Throughout this paper, (Ω, Σ) denotes a measurable space and X stands for a real Banach space. For any function $T : \Omega \times X \to X$ we denote the n-th iterate $T(t, T(t, \ldots, T(t, x))))$ of T by $T^n(t, x)$.

Definition 2.1. A function $f : \Omega \to X$ is said to be measurable if $f^{-1}(B) \in \Sigma$ for every Borel subset B of X.

Definition 2.2. An operator $T : \Omega \times X \to X$ is called a random operator if $T(\cdot, x) : \Omega \to X$ is measurable for every $x \in X$.

Definition 2.3. A random operator $T : \Omega \times X \to X$ is continuous if $T(t, \cdot) : X \to X$ is continuous for each $t \in \Omega$.

Definition 2.4. A measurable function $p : \Omega \to X$ is said to be a random fixed point of the random operator $T : \Omega \times X \to X$ if $T(t, p(t)) = p(t), \forall t \in \Omega$. The set of all random fixed points of T is denoted by $RF(T)$.

Definition 2.5 ([1])**.** Let C be a nonempty subset of a separable Banach space X and $T : \Omega \times C \to C$ be a random operator. Then T is said to be

(i) Nonexpansive random operator if

$$\|T(t, x) - T(t, y)\| \leq \|x - y\| \text{ for all } x, y \in C \text{ and for each } t \in \Omega.$$

(ii) Asymptotically nonexpansive random operator if there exists a sequence of measurable functions $r_n : \Omega \to [1, \infty)$ with $\lim_{n\to\infty} r_n(t) = 1$ for each $t \in \Omega$ such that

$$\|T^n(t, x) - T^n(t, y)\| \leq r_n(t)\|x - y\|, \forall x, y \in C, n \in N \text{ and for each } t \in \Omega.$$

(iii) Asymptotically quasi-nonexpansive random operator if there exists a sequence of measurable functions $r_n : \Omega \to [0, \infty)$ with $\lim_{n\to\infty} r_n(t) = 0, \forall t \in \Omega$ such that

$$\|T^n(t, \eta(t)) - p(t)\| \leq (1 + r_n(t))\|\eta(t) - p(t)\| \text{ for each } t \in \Omega,$$

where $p : \Omega \to C$ is a random fixed point of T and $\eta : \Omega \to C$ is any measurable map.

(iv) Uniformly L-Lipschitzian random operator if for any $x, y \in C$ and for each $t \in \Omega$

$$\|T^n(t,x) - T^n(t,y)\| \leq L\|x - y\|,$$

where, $n \geq 1$ and L is a positive constant.

(v) Semi-compact random operator if for a sequence of measurable mappings $\{\xi_n\}$ from Ω to C, with $\lim_{n\to\infty} \|\xi_n(t) - T(t, \xi_n(t))\| = 0$ for all $t \in \Omega$, we have a subsequence $\{\xi_{n_k}\}$ of $\{\xi_n\}$ such that $\xi_{n_k}(t) \to \xi(t)$ for each $t \in \Omega$, where ξ is a measurable mapping from Ω to C.

An asymptotically nonexpansive random operator is uniformly L-Lipschitzian random operator.

Definition 2.6. A finite family $\{T_i : i \in I\}$ of N continuous random operators from $\Omega \times C \to C$ with $F = \bigcap_{i=1}^N RF(T_i) \neq \emptyset$ is said to satisfy Condition(B) if there is a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0,\infty)$ such that for all $t \in \Omega$

$$f(d(x(t), F)) \leq \max_{1\leq i\leq N}\{\|x(t) - T_i(t, x(t))\|\} \quad \text{for all } x,$$

where $x : \Omega \to C$ is a measurable function.

Lemma 2.1 ([11])**.** *Let (Ω, Σ) be a measurable space, X be a separable Banach space and $T : \Omega \times X \to X$ be a continuous random operator. Then for any measurable function $x : \Omega \to X$, the function $T(t, x(t))$ is also measurable.*

We define the composite implicit random iterative process with errors in the following:

Definition 2.7 (Composite implicit random iterative scheme with errors). Let $\{T_i : i \in I = \{1, 2, \dots, N\}\}$ be a finite family of N continuous random operators from $\Omega \times C$ to C where C be a nonempty closed convex subset of a separable Banach space X. Let $\xi_0 : \Omega \to C$ be any measurable function. Then composite implicit random iteration scheme with errors is defined as follows:

$$\begin{cases}
\xi_1(t) = \alpha_1\xi_0(t) + \beta_1 T_1(t, a_1\xi_1(t) + b_1 T_1(t, \xi_1(t)) + c_1 g_1(t)) + \gamma_1 f_1(t) \\
\xi_2(t) = \alpha_2\xi_1(t) + \beta_2 T_2(t, a_2\xi_2(t) + b_2 T_2(t, \xi_2(t)) + c_2 g_2(t)) + \gamma_2 f_2(t) \\
\quad \cdots \\
\xi_N(t) = \alpha_N\xi_{N-1}(t) + \beta_N T_N(t, a_N\xi_N(t) + b_N T_N(t, \xi_N(t)) + c_N g_N(t)) + \gamma_N f_N(t) \\
\xi_{N+1}(t) = \alpha_{N+1}\xi_N(t) + \beta_{N+1} T_1^2(t, a_{N+1}\xi_{N+1}(t) + b_{N+1} T_1^2(t, \xi_{N+1}(t)) \\
\qquad\qquad + c_{N+1} g_{N+1}(t)) + \gamma_{N+1} f_{N+1}(t) \\
\quad \cdots \\
\xi_{2N}(t) = \alpha_{2N}\xi_{2N-1}(t) + \beta_{2N} T_N^2(t, a_{2N}\xi_{2N}(t) + b_{2N} T_N^2(t, \xi_{2N}(t)) + c_{2N} g_{2N}(t)) \\
\qquad\qquad + \gamma_{2N} f_{2N}(t) \\
\xi_{2N+1}(t) = \alpha_{2N+1}\xi_{2N}(t) + \beta_{2N+1} T_1^3(t, a_{2N+1}\xi_{2N+1}(t) + b_{2N+1} T_1^3(t, \xi_{2N+1}(t)) \\
\qquad\qquad + c_{2N+1} g_{2N+1}(t)) + \gamma_{2N+1} f_{2N+1}(t) \\
\quad \cdots
\end{cases}$$

which can be written in the compact form as

$$(2.1)\quad \begin{cases} \xi_n(t) = \alpha_n \xi_{n-1}(t) + \beta_n T_{i(n)}^{k(n)}(t, \eta_n(t)) + \gamma_n f_n(t) \\ \eta_n(t) = a_n \xi_n(t) + b_n T_{i(n)}^{k(n)}(t, \xi_n(t)) + c_n g_n(t), \qquad n \geq 1,\ \forall t \in \Omega, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{a_n\}, \{b_n\}, \{c_n\}$ are sequences in $[0,1]$ with $\alpha_n + \beta_n + \gamma_n = a_n + b_n + c_n = 1$ and $\{f_n\}, \{g_n\}$ are bounded sequences of measurable functions from Ω to C.

Remark 2.1. By Lemma 2.1 the sequence $\{\xi_n\}$ defined in (2.1) is a sequence of measurable functions.

Lemma 2.2 ([19, Lemma 1])**.** *Let $\{a_n\}, \{b_n\}$ and $\{\delta_n\}$ be sequences of non-negative real numbers satisfying the inequality*

$$a_{n+1} \leq (1 + \delta_n) a_n + b_n, \forall n \geq 1.$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then

(i) $\lim_{n\to\infty} a_n$ *exists,*
(ii) $\lim_{n\to\infty} a_n = 0$ *whenever* $\liminf_{n\to\infty} a_n = 0$.

Lemma 2.3 ([20])**.** *Suppose that X is a uniformly convex Banach space and $0 < p \leq t_n \leq q < 1$ for all positive integers n. Also suppose that $\{x_n\}$ and $\{y_n\}$ are two sequences in X such that $\limsup_{n\to\infty} \|x_n\| \leq r, \limsup_{n\to\infty} \|y_n\| \leq r$ and $\lim_{n\to\infty} \|t_n x_n + (1-t_n) y_n\| = r$ hold for some $r \geq 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

3. Main results

In this section we discuss the convergence of the composite implicit random iteration scheme with errors to the common random fixed point of the finite family of asymptotically nonexpansive random operators.

Theorem 3.1. *Let X be a separable Banach space and C be a nonempty closed convex subset of X. Let $\{T_i : i \in I\}$ be N asymptotically nonexpansive random operators from $\Omega \times C$ to C with the sequence of measurable mappings $\{r_{i_n}\} : \Omega \to [1, \infty)$ satisfying $\sum_{n=1}^{\infty} (r_{i_n}(t) - 1) < \infty$ for each $t \in \Omega$ and for all $i \in I = \{1, 2, \ldots, N\}$. Suppose that $F = \bigcap_{i=1}^{N} RF(T_i) \neq \emptyset$. Let $\{\xi_n\}$ be the implicit random iterative sequence with errors defined by (2.1) with the additional assumption $0 < \alpha \leq \alpha_n, \beta_n \leq \beta < 1$ and $\sum_{n=1}^{\infty} \gamma_n < \infty, \sum_{n=1}^{\infty} c_n < \infty$. Then $\{\xi_n\}$ converges strongly to a common random fixed point of the random operators $\{T_i, i \in I\}$ if and only if for all $t \in \Omega$, $\liminf_{n\to\infty} d(\xi_n(t), F) = 0$, where $d(\xi_n(t), F) = \inf\{\|\xi_n(t) - \xi(t)\| : \xi \in F\}$.*

Proof. Let $\xi \in F$. Since $\{f_n\}, \{g_n\}$ are bounded sequences of measurable functions from Ω to C, we can put for each $t \in \Omega$,

$$M(t) = \sup_{n \geq 1} \|f_n(t) - \xi(t)\| \vee \sup_{n \geq 1} \|g_n(t) - \xi(t)\|.$$

Obviously $M(t) < \infty$ for each $t \in \Omega$. Also let for each $n \geq 1, r_n(t) = \max\{r_{i_n}(t) : i = 1, 2, \ldots, N\}$. Thus we have by the condition of the theorem $\sum_{n=1}^{\infty}(r_n(t) - 1) < \infty$ for each $t \in \Omega$. Now for $\xi \in F$ and for each $t \in \Omega$,

$$\begin{aligned}
\|\eta_n(t) - \xi(t)\| &= \|a_n\xi_n(t) + b_nT_{i(n)}^{k(n)}(t, \xi_n(t)) + c_ng_n(t) - \xi(t)\| \\
&\leq a_n\|\xi_n(t) - \xi(t)\| + b_n\|T_{i(n)}^{k(n)}(t, \xi_n(t)) - \xi(t)\| + c_n\|g_n(t) - \xi(t)\| \\
&\leq a_n\|\xi_n(t) - \xi(t)\| + b_nr_{k(n)}(t)\|\xi_n(t) - \xi(t)\| + c_nM(t) \\
&= a_n\|\xi_n(t) - \xi(t)\| + b_n(1 + \mu_n(t))\|\xi_n(t) - \xi(t)\| + c_nM(t), \\
&\quad \text{(where } \mu_n(t) = r_{k(n)}(t) - 1) \\
&\leq (1 + \mu_n(t))\|\xi_n(t) - \xi(t)\| + c_nM(t). \qquad (3.1)
\end{aligned}$$

Also,

$$\begin{aligned}
\|\xi_n(t) - \xi(t)\| &= \|\alpha_n\xi_{n-1}(t) + \beta_nT_{i(n)}^{k(n)}(t, \eta_n(t)) + \gamma_nf_n(t) - \xi(t)\| \\
&\leq \alpha_n\|\xi_{n-1}(t) - \xi(t)\| + \beta_n\|T_{i(n)}^{k(n)}(t, \eta_n(t)) - \xi(t)\| \\
&\quad + \gamma_n\|f_n(t) - \xi(t)\| \\
&\leq \alpha_n\|\xi_{n-1}(t) - \xi(t)\| + \beta_nr_{k(n)}(t)\|\eta_n(t) - \xi(t)\| + \gamma_nM(t) \\
&\leq \alpha_n\|\xi_{n-1}(t) - \xi(t)\| + \beta_n(1 + \mu_n(t))[(1 + \mu_n(t))\|\xi_n(t) - \xi(t)\| \\
&\quad + c_nM(t)] + \gamma_nM(t) \\
&= \alpha_n\|\xi_{n-1}(t) - \xi(t)\| + \beta_n(1 + \mu_n(t))^2\|\xi_n(t) - \xi(t)\| \\
&\quad + \beta_nc_n(1 + \mu_n(t))M(t) + \gamma_nM(t) \\
&\leq \alpha_n\|\xi_{n-1}(t) - \xi(t)\| + (1 - \alpha_n)(1 + p_n(t))\|\xi_n(t) - \xi(t)\| \\
&\quad + [\beta_nc_n(1 + \mu_n(t)) + \gamma_n]M(t), \text{(where } p_n(t) = 2\mu_n(t) + \mu_n(t)^2) \\
&\leq \alpha_n\|\xi_{n-1}(t) - \xi(t)\| + (1 - \alpha_n + p_n(t))\|\xi_n(t) - \xi(t)\| \\
&\quad + [\beta_nc_n(1 + \mu_n(t)) + \gamma_n]M(t).
\end{aligned}$$

By rearranging both sides we have that

$$\begin{aligned}
&\|\xi_n(t) - \xi(t)\| \\
&\leq \|\xi_{n-1}(t) - \xi(t)\| + \frac{p_n(t)}{\alpha_n}\|\xi_n(t) - \xi(t)\| + \frac{\beta_nc_n(1 + \mu_n(t)) + \gamma_n}{\alpha_n}M(t) \\
&\leq \|\xi_{n-1}(t) - \xi(t)\| + \frac{p_n(t)}{\alpha}\|\xi_n(t) - \xi(t)\| + \frac{\beta_nc_n(1 + \mu_n(t)) + \gamma_n}{\alpha}M(t)
\end{aligned}$$

which implies that

$$\begin{aligned}
\|\xi_n(t) - \xi(t)\| &\leq \frac{\alpha}{\alpha - p_n(t)}\|\xi_{n-1}(t) - \xi(t)\| + \frac{\beta_nc_n(1 + \mu_n(t)) + \gamma_n}{\alpha - p_n(t)}M(t) \\
&= (1 + \frac{p_n(t)}{\alpha - p_n(t)})\|\xi_{n-1}(t) - \xi(t)\| + \frac{\beta_nc_n(1 + \mu_n(t)) + \gamma_n}{\alpha - p_n(t)}M(t). \qquad (3.2)
\end{aligned}$$

Since $\sum_{n=1}^{\infty}(r_{k(n)}(t)-1)<\infty$ for each $t\in\Omega$, we have $\sum_{n=1}^{\infty}\mu_n(t)<\infty$ and hence $\sum_{n=1}^{\infty}p_n(t)<\infty$. Therefore $\lim_{n\to\infty}p_n(t)=0$ for each $t\in\Omega$. Thus for $t\in\Omega$, there exists $n_1\in N$ such that $p_n(t)<\frac{\alpha}{2}$ for all $n\geq n_1$. Thus from (3.2) we have that, for all $n\geq n_1$

$$\begin{aligned}\|\xi_n(t)-\xi(t)\| &\leq (1+2\frac{p_n(t)}{\alpha})\|\xi_{n-1}(t)-\xi(t)\|+\frac{\beta_n c_n(1+\mu_n(t))+\gamma_n}{\alpha}2M(t)\\ &= (1+\lambda_n(t))\|\xi_{n-1}(t)-\xi(t)\|+\sigma_n(t), \qquad (3.3)\end{aligned}$$

where $\lambda_n(t)=\frac{2}{\alpha}p_n(t)$ and $\sigma_n(t)=\frac{\beta_n c_n(1+\mu_n(t))+\gamma_n}{\alpha}2M(t)$. Thus $\sum_{n=1}^{\infty}\lambda_n(t)<\infty$, $\sum_{n=1}^{\infty}\sigma_n(t)<\infty$. This gives that

$$d(\xi_n(t),F)\leq(1+\lambda_n(t)d(\xi_{n-1}(t),F)+\sigma_n(t).$$

Hence by Lemma 2.2 we have $\lim_{n\to\infty}d(\xi_n(t),F)$ exists for each $t\in\Omega$. Further by the condition of the theorem we have for all $t\in\Omega$,

$$\lim_{n\to\infty}d(\xi_n(t),F)=0. \qquad (3.4)$$

Now from (3.3) we have that

$$\begin{aligned}(3.5)\quad &\|\xi_{n+m}(t)-\xi(t)\|\\ &\leq[1+\lambda_{n+m}(t)]\|\xi_{n+m-1}(t)-\xi(t)\|+\sigma_{n+m}(t)\\ &\leq e^{\lambda_{n+m}(t)}\|\xi_{n+m-1}(t)-\xi(t)\|+\sigma_{n+m}(t)\\ &\leq e^{\lambda_{n+m}(t)+\lambda_{n+m-1}(t)}\|\xi_{n+m-2}-\xi(t)\|+e^{\lambda_{n+m}(t)}\sigma_{n+m-1}(t)+\sigma_{n+m}(t)\\ &\;\;\vdots\\ &\leq e^{\sum_{i=n+1}^{n+m}\lambda_i(t)}\|\xi_n(t)-\xi(t)\|+\sum_{k=n+1}^{n+m-1}\sigma_k(t)e^{\sum_{i=k+1}^{n+m}\lambda_i(t)}+\sigma_{n+m}(t)\\ &\leq R(t)\|\xi_n(t)-\xi(t)\|+R(t)\sum_{k=n+1}^{\infty}\sigma_k(t)\end{aligned}$$

for each $t\in\Omega$ and for all natural numbers m,n where $R(t)=e^{\sum_{n=1}^{\infty}\lambda_n(t)}<\infty$. Therefore for any $\xi\in F$ we have that

$$\begin{aligned}\|\xi_{n+m}(t)-\xi_n(t)\| &\leq R(t)\|\xi_n(t)-\xi(t)\|+R(t)\sum_{k=n+1}^{\infty}\sigma_k(t)+\|\xi_n(t)-\xi(t)\|\\ &=(R(t)+1)\|\xi_n(t)-\xi(t)\|+R(t)\sum_{k=n+1}^{\infty}\sigma_k(t). \qquad (3.6)\end{aligned}$$

Since $\sum_{n=1}^{\infty}\sigma_n(t)<\infty$ and $\lim_{n\to\infty}d(\xi_n(t),F)=0$, there exists $n_2\in N$ such that for all $n\geq n_2$ we have $d(\xi_n(t),F)<\frac{\epsilon}{2(R(t)+1)}$ and $\sum_{k=n+1}^{\infty}\sigma_k(t)<\frac{\epsilon}{2R(t)}$. So there exists $q\in F$ such that $\|\xi_n(t)-q(t)\|<\frac{\epsilon}{2(R(t)+1)}$ for all $n\geq n_2$.

Therefore from (3.6) we have that for all $n \geq n_2$,

$$\begin{aligned}\|\xi_{n+m}(t) - \xi_n(t)\| &\leq (R(t)+1)\|\xi_n(t) - q(t)\| + R(t)\sum_{k=n+1}^{\infty}\sigma_k(t) \\ &< (R(t)+1)\frac{\epsilon}{2(R(t)+1)} + R(t)\frac{\epsilon}{2R(t)} = \epsilon\end{aligned}$$

which in turn implies that $\{\xi_n(t)\}$ is a cauchy sequence for each $t \in \Omega$. Therefore $\xi_n(t) \to p(t)$ as $n \to \infty$ for each $t \in \Omega$, where $p : \Omega \to F$, being the limit of the sequence of measurable functions is also measurable. Now we prove that $p \in F$. Since for each $t \in \Omega$, $\xi_n(t) \to p(t)$ as $n \to \infty$ there exists $n_3 \in N$ such that $\|\xi_n(t) - p(t)\| < \frac{\epsilon}{2(1+r_1(t))}$ for all $n \geq n_3$. Since $\lim_{n\to\infty} d(\xi_n(t), F) = 0$ for each $t \in \Omega$, there exists $n_4 \in N$ such that $d(\xi_n(t), F) < \frac{\epsilon}{2(1+r_1(t))}$ for all $n \geq n_4$. So there exists $\xi^* \in F$ such that $\|\xi_n(t) - \xi^*(t)\| \leq \frac{\epsilon}{2(1+r_1(t))}$ for all $n \geq n_4$. Let $n_5 = \max\{n_3, n_4\}$. Now for all $l \in I$ and for all $n \geq n_5$

$$\begin{aligned}\|T_l(t,p(t)) - p(t)\| &\leq \|T_l(t,p(t)) - \xi^*(t)\| + \|\xi^*(t) - p(t)\| \\ &\leq \|T_l(t,p(t)) - T_l(t,\xi^*(t))\| + \|\xi^*(t) - p(t)\| \\ &\leq r_1(t)\|\xi^*(t) - p(t)\| + \|\xi^*(t) - p(t)\| \\ &= (1+r_1(t))\|\xi^*(t) - p(t)\| \\ &\leq (1+r_1(t))\|\xi^*(t) - \xi_n(t)\| + (1+r_1(t))\|\xi_n(t) - p(t)\| \\ &< (1+r_1(t))\frac{\epsilon}{2(1+r_1(t))} + (1+r_1(t))\frac{\epsilon}{2(1+r_1(t))} = \epsilon\end{aligned}$$

which implies that $T_l(t,p(t)) = p(t)$ for all $l \in I$ and for each $t \in \Omega$. Therefore $p \in F$. Thus $\{\xi_n\}$ converges strongly to a common random fixed point of $\{T_i, i \in I\}$. □

Lemma 3.1. *Let X be a uniformly convex separable Banach space and C be a nonempty closed convex subset of X. Let $\{T_i : i \in I\}$ be N asymptotically nonexpansive random operators from $\Omega \times C$ to C with the sequence of measurable mappings $\{r_{i_n}\} : \Omega \to [1,\infty)$ satisfying $\sum_{n=1}^{\infty}(r_{i_n}(t) - 1) < \infty$ for each $t \in \Omega$ and for all $i \in I = \{1,2,\dots,N\}$. Suppose that $F = \bigcap_{i=1}^{N} RF(T_i) \neq \emptyset$. Let $\{\xi_n\}$ be the implicit random iterative sequence with errors defined by* (2.1) *with the additional assumption $0 < \alpha \leq \alpha_n, \beta_n \leq \beta < 1$ and $\sum_{n=1}^{\infty}\gamma_n < \infty, \sum_{n=1}^{\infty} c_n < \infty$. Then*

$$\lim_{n\to\infty}\|\xi_n(t) - T_l(t,\xi_n(t))\| = 0 \quad \text{for each } t \in \Omega \text{ and for all } l = 1,2,\dots,N.$$

Proof. Let $\xi \in F$ be arbitrary. Since $\{f_n\}, \{g_n\}$ are bounded sequences of measurable functions from Ω to C, so we can put for each $t \in \Omega$,

$$M(t) = \sup_{n\geq1}\|f_n(t) - \xi(t)\| \vee \sup_{n\geq1}\|g_n(t) - \xi(t)\|.$$

Obviously $M(t) < \infty$ for each $t \in \Omega$. Also let for each $n \geq 1, r_n(t) = \max\{r_{i_n}(t) : i = 1,2,\dots,N\}$. Thus we have $\sum_{n=1}^{\infty}(r_n(t) - 1) < \infty$ for each

$t \in \Omega$. From (3.3) we have that

$$\|\xi_n(t) - \xi(t)\| \le (1 + \lambda_n(t))\|\xi_{n-1}(t) - \xi(t)\| + \sigma_n(t),$$

where $\sum_{n=1}^{\infty} \lambda_n(t) < \infty, \sum_{n=1}^{\infty} \sigma_n(t) < \infty$. Hence by Lemma 2.2 we get that $\lim_{n\to\infty} \|\xi_n(t) - \xi(t)\|$ exists for all $\xi \in F$ and for each $t \in \Omega$. Thus $\{\xi_n(t)\}$ is a bounded sequence for each $t \in \Omega$. Let $\lim_{n\to\infty} \|\xi_n(t) - \xi(t)\| = a_t$ for some $a_t \ge 0$. From (3.1) we get that

$$\|\eta_n(t) - \xi(t)\| \le (1 + \mu_n(t))\|\xi_n(t) - \xi(t)\| + c_n M(t).$$

Taking limsup on the both sides of the above inequality we get that

$$\limsup_{n\to\infty} \|\eta_n(t) - \xi(t)\| \le a_t \quad \text{for each } t \in \Omega. \tag{3.7}$$

Now

$$\begin{aligned} a_t &= \lim_{n\to\infty} \|\xi_n(t) - \xi(t)\| \\ &= \lim_{n\to\infty} \|\alpha_n \xi_{n-1}(t) + \beta_n T_{i(n)}^{k(n)}(t, \eta_n(t)) + \gamma_n f_n(t) - \xi(t)\| \\ &= \lim_{n\to\infty} \|(1 - \beta_n)(\xi_{n-1}(t) - \xi(t) + \gamma_n(f_n(t) - \xi_{n-1}(t))) \\ &\qquad + \beta_n(T_{i(n)}^{k(n)}(t, \eta_n(t)) - \xi(t) + \gamma_n(f_n(t) - \xi_{n-1}(t)))\|. \end{aligned} \tag{3.8}$$

Now for each $t \in \Omega$,

$$\|\xi_{n-1}(t) - \xi(t) + \gamma_n(f_n(t) - \xi_{n-1}(t))\| \le \|\xi_{n-1}(t) - \xi(t)\| + \gamma_n\|f_n(t) - \xi_{n-1}(t)\|.$$

Taking limsup on the both sides of above we get for each $t \in \Omega$,

$$\begin{aligned} &\limsup_{n\to\infty} \|\xi_{n-1}(t) - \xi(t) + \gamma_n(f_n(t) - \xi_{n-1}(t))\| \\ &\le \limsup_{n\to\infty} (\|\xi_{n-1}(t) - \xi(t)\| + \gamma_n\|f_n(t) - \xi_{n-1}(t)\|) = a_t. \end{aligned} \tag{3.9}$$

Also,

$$\begin{aligned} &\|T_{i(n)}^{k(n)}(t, \eta_n(t)) - \xi(t) + \gamma_n(f_n(t) - \xi_{n-1}(t))\| \\ &\le \|T_{i(n)}^{k(n)}(t, \eta_n(t)) - \xi(t)\| + \gamma_n\|f_n(t) - \xi_{n-1}(t)\| \\ &\le r_{k(n)}(t)\|\eta_n(t) - \xi(t)\| + \gamma_n\|f_n(t) - \xi_{n-1}(t)\|. \end{aligned}$$

Taking limsup on the both sides of above we get

$$\limsup_{n\to\infty} \|T_{i(n)}^{k(n)}(t, \eta_n(t)) - \xi(t) + \gamma_n(f_n(t) - \xi_{n-1}(t))\| \le a_t. \tag{3.10}$$

From (3.8), (3.9), (3.10) and Lemma 2.3 we get

$$\lim_{n\to\infty} \|T_{i(n)}^{k(n)}(t, \eta_n(t)) - \xi_{n-1}(t)\| = 0 \text{ for each } t \in \Omega. \tag{3.11}$$

Again for each $t \in \Omega$,

$$\begin{aligned} \|\xi_n(t) - \xi_{n-1}(t)\| &= \|\alpha_n \xi_{n-1}(t) + \beta_n T_{i(n)}^{k(n)}(t, \eta_n(t)) + \gamma_n f_n(t) - \xi_{n-1}(t)\| \\ &\le \beta_n\|T_{i(n)}^{k(n)}(t, \eta_n(t)) - \xi_{n-1}(t)\| + \gamma_n\|f_n(t) - \xi_{n-1}(t)\| \end{aligned}$$

$$\to 0 \quad \text{as } n \to \infty. \tag{3.12}$$

Hence for each $t \in \Omega$,

$$\lim_{n\to\infty} \|\xi_n(t) - \xi_{n+l}(t)\| = 0 \text{ for each } t \in \Omega \text{ and for all } l \in I. \tag{3.13}$$

Since

$$\|\xi_n(t) - T_{i(n)}^{k(n)}(t, \eta_n(t))\| \le \|\xi_n(t) - \xi_{n-1}(t)\| + \|\xi_{n-1}(t) - T_{i(n)}^{k(n)}(t, \eta_n(t))\|,$$

by using (3.11), (3.12) we get

$$\lim_{n\to\infty} \|\xi_n(t) - T_{i(n)}^{k(n)}(t, \eta_n(t))\| = 0 \text{ for each } t \in \Omega. \tag{3.14}$$

Now

$$\begin{aligned}
&\|\eta_n(t) - \xi_n(t)\| \\
&= \|a_n\xi_n(t) + b_nT_{i(n)}^{k(n)}(t, \xi_n(t)) + c_ng_n(t) - \xi_n(t)\| \\
&\le b_n\|T_{i(n)}^{k(n)}(t, \xi_n(t)) - \xi_n(t)\| + c_n\|g_n(t) - \xi_n(t)\| \\
&\le b_n[\|T_{i(n)}^{k(n)}(t, \xi_n(t)) - T_{i(n)}^{k(n)}(t, \eta_n(t))\| + \|T_{i(n)}^{k(n)}(t, \eta_n(t)) - \xi_n(t)\|] \\
&\quad + c_n\|g_n(t) - \xi_n(t)\| \\
&\le b_n[r_{k(n)}(t)\|\xi_n(t) - \eta_n(t)\| + \|T_{i(n)}^{k(n)}(t, \eta_n(t)) - \xi_n(t)\|] \\
&\quad + c_n\|g_n(t) - \xi_n(t)\| \\
&\le (1 - a_n)(1 + \mu_n(t))\|\xi_n(t) - \eta_n(t)\| + (1 - a_n)\|T_{i(n)}^{k(n)}(t, \eta_n(t)) - \xi_n(t)\| \\
&\quad + c_n\|g_n(t) - \xi_n(t)\| \\
&\le (1 - a_n + \mu_n(t))\|\xi_n(t) - \eta_n(t)\| + (1 - a_n)\|T_{i(n)}^{k(n)}(t, \eta_n(t)) - \xi_n(t)\| \\
&\quad + c_n\|g_n(t) - \xi_n(t)\|
\end{aligned}$$

which implies that

$$\begin{aligned}
\|\eta_n(t) - \xi_n(t)\| \le \frac{\mu_n(t)}{\alpha}\|\xi_n(t) - \eta_n(t)\| + \left(\frac{1}{\alpha} - 1\right)\|T_{i(n)}^{k(n)}(t, \eta_n(t)) - \xi_n(t)\| \\
+ \frac{c_n}{\alpha}\|g_n(t) - \xi_n(t)\|
\end{aligned}$$

which implies that

$$\begin{aligned}
\|\xi_n(t) - \eta_n(t)\| \le \frac{1-\alpha}{\alpha - \mu_n(t)}\|T_{i(n)}^{k(n)}(t, \eta_n(t)) - \xi_n(t)\| \\
+ \frac{c_n}{\alpha - \mu_n(t)}\|g_n(t) - \xi_n(t)\|.
\end{aligned} \tag{3.15}$$

Since $\sum_{n=1}^{\infty} \mu_n(t) < \infty$ for each $t \in \Omega$, we have $\lim_{n\to\infty} \mu_n(t) = 0$ for each $t \in \Omega$. So for $t \in \Omega$, there exists $n_6 \in N$ such that $\mu_n(t) < \frac{\alpha}{2}$ for all $n \ge n_6$.

Thus from (3.15) we get for $t \in \Omega$ and for all $n \geq n_6$ that

$$\|\xi_n(t)-\eta_n(t)\| \leq \frac{2(1-\alpha)}{\alpha}\|T_{i(n)}^{k(n)}(t,\eta_n(t))-\xi_n(t)\|+\frac{2c_n}{\alpha}\|g_n(t)-\xi_n(t)\|. \tag{3.16}$$

From (3.14), (3.16) we get that for each $t \in \Omega$,

$$\|\xi_n(t)-\eta_n(t)\| \to 0 \text{ as } n \to \infty. \tag{3.17}$$

Now

$$\begin{aligned} &\|\xi_{n-1}(t)-T_n(t,\xi_n(t))\| \\ &\leq \|\xi_{n-1}(t)-T_{i(n)}^{k(n)}(t,\eta_n(t))\| + \|T_{i(n)}^{k(n)}(t,\eta_n(t))-T_n(t,\xi_n(t))\| \\ &\leq \|\xi_{n-1}(t)-T_{i(n)}^{k(n)}(t,\eta_n(t))\| + L\|T_{i(n)}^{k(n)-1}(t,\eta_n(t))-\xi_n(t)\| \\ &= \sigma_n(t) + L\|T_{i(n)}^{k(n)-1}(t,\eta_n(t))-\xi_n(t)\|, \end{aligned} \tag{3.18}$$

where $\sigma_n(t) = \|\xi_{n-1}(t) - T_{i(n)}^{k(n)}(t,\eta_n(t))\|$ for each $t \in \Omega$. From (3.11) we have for each $t \in \Omega$, $\sigma_n(t) \to 0$ as $n \to \infty$. Again

$$\begin{aligned} &\|T_{i(n)}^{k(n)-1}(t,\eta_n(t))-\xi_n(t)\| \\ &\leq \|T_{i(n)}^{k(n)-1}(t,\eta_n(t))-T_{i(n-N)}^{k(n)-1}(t,\xi_{n-N}(t))\| \\ &\quad + \|T_{i(n-N)}^{k(n)-1}(t,\xi_{n-N}(t))-T_{i(n-N)}^{k(n)-1}(t,\eta_{n-N}(t))\| \\ &\quad + \|T_{i(n-N)}^{k(n)-1}(t,\eta_{n-N}(t))-\xi_{(n-N)-1}(t)\| + \|\xi_{(n-N)-1}(t)-\xi_n(t)\|. \end{aligned} \tag{3.19}$$

Now for each $n > N, n = (n-N) \pmod N$. Again since $n = (k(n)-1)N+i(n)$, we have $k(n-N) = k(n)-1$ and $i(n-N) = i(n)$. So from (3.19) we have

$$\begin{aligned} &\|T_{i(n)}^{k(n)-1}(t,\eta_n(t))-\xi_n(t)\| \\ &\leq \|T_{i(n-N)}^{k(n-N)}(t,\eta_n(t))-T_{i(n-N)}^{k(n-N)}(t,\xi_{n-N}(t))\| \\ &\quad + \|T_{i(n-N)}^{k(n-N)}(t,\xi_{n-N}(t))-T_{i(n-N)}^{k(n-N)}(t,\eta_{n-N}(t))\| \\ &\quad + \|T_{i(n-N)}^{k(n-N)}(t,\eta_{n-N}(t))-\xi_{(n-N)-1}(t)\| + \|\xi_{(n-N)-1}(t)-\xi_n(t)\| \\ &\leq L\|\eta_n(t)-\xi_{n-N}(t)\| + L\|\xi_{n-N}(t)-\eta_{n-N}(t)\| + \sigma_{n-N}(t) \\ &\quad + \|\xi_{(n-N)-1}(t)-\xi_n(t)\|. \end{aligned} \tag{3.20}$$

So from (3.18) and (3.20) we have for each $t \in \Omega$,

(3.21)

$$\begin{aligned} &\|\xi_{n-1}(t)-T_n(t,\xi_n(t))\| \\ &\leq \sigma_n(t) + L^2\|\eta_n(t)-\xi_{n-N}(t)\| + L^2\|\xi_{n-N}(t)-\eta_{n-N}(t)\| + L\sigma_{n-N}(t) \\ &\quad + L\|\xi_{(n-N)-1}(t)-\xi_n(t)\| \\ &\leq \sigma_n(t) + L^2(\|\eta_n(t)-\xi_n(t)\| + \|\xi_n(t)-\xi_{n-N}(t)\|) + L^2\|\xi_{n-N}(t)-\eta_{n-N}(t)\| \\ &\quad + L\sigma_{n-N}(t) + L\|\xi_{(n-N)-1}(t)-\xi_n(t)\|. \end{aligned}$$

Now for each $t \in \Omega$, it follows that

$$\|\xi_{n-1}(t) - T_n(t, \xi_n(t))\| \to 0 \text{ as } n \to \infty. \tag{3.22}$$

Now by (3.22) and (3.12) we get that for each $t \in \Omega$

$$\begin{aligned} &\|\xi_n(t) - T_n(t, \xi_n(t))\| \\ &\leq \|\xi_n(t) - \xi_{n-1}(t)\| + \|\xi_{n-1}(t) - T_n(t, \xi_n(t))\| \to 0 \text{ as } n \to \infty. \end{aligned} \tag{3.23}$$

Now for each $l \in \{1, 2, \ldots, N\}$, by using (3.23) and (3.13) we get that

$$\begin{aligned} \|\xi_n(t) - T_{n+l}(t, \xi_n(t))\| &\leq \|\xi_n(t) - \xi_{n+l}(t)\| + \|\xi_{n+l}(t) - T_{n+l}(t, \xi_{n+l}(t))\| \\ &\quad + \|T_{n+l}(t, \xi_{n+l}(t)) - T_{n+l}(t, \xi_n(t))\| \\ &\leq \|\xi_n(t) - \xi_{n+l}(t)\| + \|\xi_{n+l}(t) - T_{n+l}(t, \xi_{n+l}(t))\| \\ &\quad + L\|\xi_{n+l}(t) - \xi_n(t)\| \\ &= (1 + L)\|\xi_n(t) - \xi_{n+l}(t)\| + \|\xi_{n+l}(t) - T_{n+l}(t, \xi_{n+l}(t))\| \\ &\to 0 \quad \text{as } n \to \infty \text{ for each } t \in \Omega. \end{aligned}$$

Consequently we have

$$\|\xi_n(t) - T_l(t, \xi_n(t))\| \to 0 \text{ as } n \to \infty \text{ for each } t \in \Omega \text{ and for each } l \in I. \qquad \square$$

Theorem 3.2. *Let X be a uniformly convex separable Banach space and C be a nonempty closed convex subset of X. Let $\{T_i : i \in I\}$ be N uniformly L-Lipschitzian asymptotically nonexpansive random operators from $\Omega \times C$ to C with the sequence of measurable mappings $\{r_{i_n}\} : \Omega \to [1, \infty)$ satisfying $\sum_{n=1}^{\infty}(r_{i_n}(t) - 1) < \infty$ for each $t \in \Omega$ and for all $i \in I = \{1, 2, \ldots, N\}$. Suppose that $F = \bigcap_{i=1}^{N} RF(T_i) \neq \emptyset$. Let $\{\xi_n\}$ be the random composite implicit iterative sequence with errors defined by (2.1) with the additional assumption $0 < \alpha \leq \alpha_n, \beta_n \leq \beta < 1$ and $\sum_{n=1}^{\infty} \gamma_n < \infty, \sum_{n=1}^{\infty} c_n < \infty$. If the family $\{T_i : i \in I\}$ satisfies Condition(B) for each $t \in \Omega$, then $\{\xi_n\}$ converges strongly to a common random fixed point of $\{T_i, i \in I\}$.*

Proof. By the proof of Theorem 3.1 we have $\lim_{n\to\infty} d(\xi_n(t), F)$ exists for each $t \in \Omega$. Again by Lemma 3.1 and Condition(B), we have that

$$\lim_{n\to\infty} f(d(\xi_n(t), F)) = 0.$$

Since $f : [0, \infty) \to [0, \infty)$ is a nondecreasing function with $f(0) = 0$ so we have $\lim_{n\to\infty} d(\xi_n(t), F) = 0$. Hence the result follows by Theorem 3.1. $\square$

Theorem 3.3. *Let X be a uniformly convex separable Banach space and C be a nonempty closed convex subset of X. Let $\{T_i : i \in I\}$ be N uniformly L-Lipschitzian asymptotically nonexpansive random operators from $\Omega \times C$ to C with the sequence of measurable mappings $\{r_{i_n}\} : \Omega \to [1, \infty)$ satisfying $\sum_{n=1}^{\infty}(r_{i_n}(t) - 1) < \infty$ for each $t \in \Omega$ and for all $i \in I = \{1, 2, \ldots, N\}$. Suppose that $F = \bigcap_{i=1}^{N} RF(T_i) \neq \emptyset$ and let one member of the family $\{T_i : i \in I\}$ to be semi-compact random operator. Let $\{\xi_n\}$ be the implicit random iterative* *sequence with errors defined by* (2.1) *with the additional assumption* $0 < \alpha \leq \alpha_n, \beta_n \leq \beta < 1$ *and* $\sum_{n=1}^{\infty} \gamma_n < \infty, \sum_{n=1}^{\infty} c_n < \infty$*, then* $\{\xi_n\}$ *converges strongly to a common random fixed point of* $\{T_i, i \in I\}$*.*

Proof. From Lemma 3.1 we get that $\lim_{n\to\infty} \|\xi_n(t) - T_l(t, \xi_n(t))\| = 0$ for each $t \in \Omega$ and for each $l \in I$. Let us assume that T_1 is semi-compact random operator. So there exists a subsequence $\{\xi_{n_k}(t)\}$ of $\{\xi_n(t)\}$ such that $\xi_{n_k}(t) \to \xi(t)$ for each $t \in \Omega$, where ξ is a measurable mapping from Ω to C. Now

$$\begin{aligned} \|\xi(t) - T_l(t, \xi(t))\| &= \lim_{k\to\infty} \|\xi_{n_k}(t) - T_l(t, \xi_{n_k}(t))\| \\ &= 0 \quad \text{for each } t \in \Omega \text{ and for each } l \in I. \end{aligned}$$

From above it follows that $\xi \in F$. Since $\{\xi_n(t)\}$ has a subsequence $\{\xi_{n_k}(t)\}$ such that $\xi_{n_k}(t) \to \xi(t)$ for each $t \in \Omega$, we have that $\liminf_{n\to\infty} d(\xi_n(t), F) = 0$. Hence the result follows by Theorem 3.1. □

Remark 3.1. (1) Theorem 3.1 and Theorem 3.3 extend and improve Theorem 4.1 and Theorem 4.2 of [1] respectively.

(2) Our results in this paper are also valid for the composite implicit random iterative process with errors considered in the sense of Liu [15]. In light of this remark our results extend and improve the corresponding results of [17].

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