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구조물의 시간에 따른 거동 해석을 위한 유한요소법에 기초한 단일 스텝 시간 범주들의 비교연구

A Comparative Study on Single Time Schemes Based on the FEM for the Analysis of Structural Transient Problems

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Abstract

New time schemes based on the FEM were developed and their performances were tested with 2D wave equation. The least-squares and weighted residual methods are used to construct new time schemes based on traditional residual minimization method. To overcome some drawbacks that time schemes based on the least-squares and weighted residual methods have, ad-hoc method is considered to minimize residuals multiplied by others residuals as a new approach. And variational method is used to get necessary conditions of ad-hoc minimization. A-stability was chosen to check the stability of newly developed time schemes. Specific values of new time schemes are presented along with their numerical solutions which were compared with analytic solution.

Keywords : Finite Element Method, Transient Analysis, Single Time Schemes, Least-squares Method, Weighted Residual Method, Ad-hoc Method, Recurrence Method, Wave Equation, Variational Method

1. Introduction

1.1 Spatial discretization

A wave equation[1] is one of the simplest PDEs (partial differential equations) which can be used for dynamic analysis of vibrating membranes in multi dimensional space. It is so useful because some of known analytic solutions are available under specific ICs (initial conditions) and BCs (boundary conditions). They can be used as a benchmark problem for the performance of newly developed time schemes based on the FEM (finite element method). Typical 3D form of wave equation[1,2] can be given as

$$\rho\frac{\partial^2 u}{\partial t^2}-\frac{\partial}{\partial x}\left(T_x\frac{\partial u}{\partial x}\right)-\frac{\partial}{\partial y}\left(T_y\frac{\partial u}{\partial y}\right)-\frac{\partial}{\partial z}\left(T_z\frac{\partial u}{\partial z}\right)+c_f u = f(x,y,z,t) \tag{1}$$

where, T_x, T_y and T_z are tensions along axe, u is deflection, c_f ismodulusofelasticfoundation, ρ isdensityand

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f is distributed force. Eq. (1) can be simplified into 1D or 2D problems by simply omitting terms of unnecessary spatial dimensions.

Applying spatial discretization to original PDEs given in Eq. (1) yields so called semi-discrete form of ODEs (ordinary differential equations)[3] of second order in time which can be written as

$$[M]\{\ddot{u}\}+[C]\{\dot{u}\}+[K]\{u\}=\{f(t)\} \tag{2}$$

where $[M]$ is mass matrix, $[C]$ is damping matrix, $[K]$ is stiffness matrix, $\{u\}$ is displacement vector and $\{f(t)\}$ is force vector. In many cases of structural problems, $[C]$ is omitted, but we include it for the purpose of general development of new schemes. The ICs and BCs are given as

$$\begin{aligned}&\text{ICs : } \{\dot{u}\}=\{v_0\} \text{ and } \{u\}=\{u_0\} \text{ at } t=0 \text{ on } \Omega_e\\ &\text{BCs : } \{\dot{u}\}=\{v_b\} \text{ and } \{u\}=\{u_b\} \text{ at } 0<t\leq\infty \text{ on } \Gamma_e\end{aligned} \tag{3}$$

where Ω_e is spatial computational domain and Γ_e is boundary of spatial computational domain.

Most of PDEs which describe motions of other complicated structural members such as beams, plates and shells also can be transformed into exactly the same form of ODEs given in Eq. (2) when they are turned into semi discrete form with proper numerical methods. Then the initial and boundary value problem replaced by Eq. (2) and (3) can be solved by using various time schemes developed from various methods.

In current study we develop new time scheme based on the FEM. And their performance, stability and accuracy are also tested.

Developing procedures of new time schemes using the mixed least-squares[4], weighted residual[5] and ad-hoc method and specific values of schemes are presented to analyze 2D wave equation.

1.2 Rearranged semi discrete equations

To solve Eq. (2), one should be able to impose BCs and ICs which are stated as displacements and velocities at each step of recurrence[6,7]. In finite element relation, dealing with properties of displacements and velocities can be done simultaneously by splitting Eq. (2) into a set of first order systems or using special interpolation functions such as the Hermite type interpolation functions to deal with velocities.

In current study we consider a set of rewritten first order systems[8] which is given by

$$\begin{aligned}&[M]\{\dot{v}\}+[C]\{v\}+[K]\{u\}=\{f(t)\}\\ &\{\dot{u}\}=\{v\}\end{aligned} \tag{4}$$

The relations given in Eq. (4) are often called as mixed relations[4], because they use independent approximations not only for the displacements but also for the velocities. The ICs and BCs of original relations given Eq. (3) can be rewritten as

$$\begin{aligned}&\text{ICs : } \{v\}=\{v_0\} \text{ and } \{u\}=\{u_0\} \text{ at } t=0 \text{ on } \Omega_e\\ &\text{BCs : } \{v\}=\{v_b\} \text{ and } \{u\}=\{u_b\} \text{ at } 0<t\leq\infty \text{ on } \Gamma_e\end{aligned} \tag{5}$$

Note that calculation of accelerations are not necessary in finding numerical solutions of next steps because the time schemes based on the FEM directly relates solutions between neighboring steps by interpolating them with finite element approximations, while time schemes based on the FDM (finite difference methods) requires updated of accelerations from the initial point of recurrences to find velocities through relations given by the Taylor expansions. However finding accelerations are often very vague, thus the schemes based on the FEM can be considered as an alternative way to overcome the drawback of the schemes based on the FDM.

2. Development of time schemes

2.1 Finite element approximations

Since the Eq. (4) is rewritten as a set of first order equations, one can use the Lagrange type interpolation functions for the approximations of u and v as

$$u \cong \sum_{j=1}^{n}\left(\psi_j \mathrm{u}_j\right), \quad v \cong \sum_{j=1}^{n}\left(\psi_j \mathrm{v}_j\right) \tag{6}$$

where ψ_j is the Lagrange type interpolation function of the *j-th* node, u_j and v_j are nodal values of corresponding variables. The linear interpolations used in current study can be given by

$$\psi_1 = 1 - \frac{t}{\Delta t}, \ \psi_2 = \frac{t}{\Delta t} \tag{7}$$

where time interval Δt is treated as element length here.

The linear interpolation functions given in Eq. (7) can be used regardless of adopted minimizing method to get single step time schemes based on various FEMs. By simply increasing the order of the interpolation functions, one may construct multi step time schemes which will not be discussed in current study.

Due to finite element approximations, conditions given in Eq. (4) cannot be identically satisfied throughout entire time domain. There are introduced errors which are called as residuals. Residuals of Eq. (4) can be written as

$$\begin{aligned} r_1 &= [M]\{\dot{v}\} + [C]\{v\} + [K]\{u\} - \{f(t)\} \neq 0 \\ r_2 &= \{\dot{u}\} - \{v\} \neq 0 \end{aligned} \tag{8}$$

To get finite element relations which can be used to derive new time schemes of Eq. (4), one should minimize residuals given in Eq. (8).

2.2 Minimization of residuals

Minimizing Eq. (8) can be done by using various FEMs. However, their numerical and computational aspects depend on the chosen method. Here we consider general time schemes based on the least-squares and weighted residual method which are broadly accepted in many computational mechanics, and develop a new time scheme which is based on ad-hoc minimization of the residuals.

2.2.1 Least squares linear single step schemes

In the least-squares method, residuals are minimized by minimizing integrated sum of squares of residuals over the time domain as

$$I(u,v) = \frac{1}{2}\int_{t_n}^{t_{n+1}} \left[r_1^T r_1 + r_2^T r_2 \right] dt \tag{9}$$

Variational method[9] can be used to get bilinear form of equation. The necessary condition for the minimum $I(u,v)$ is

$$0 = \delta I(\delta u, \delta v, u, v) = \int_{t_n}^{t_{n+1}} \left[\delta r_1^T r_1 + \delta r_2^T r_2 \right] dt \tag{10}$$

where δ is variational operator. By collecting coefficients of δu and δv of Eq. (10) and setting them as zero, the LSLSS (the least-squares linear single step) scheme can be obtained as

$$\begin{bmatrix} [L_{11}] & [L_{12}] \\ [L_{21}] & [L_{22}] \end{bmatrix} \begin{Bmatrix} \{\chi_1\} \\ \{\chi_2\} \end{Bmatrix} = \begin{Bmatrix} \{l_1\} \\ \{l_2\} \end{Bmatrix} \tag{11}$$

where $\{\chi_1\} = \{u_n, v_n\}^T$ and $\{\chi_2\} = \{u_{n+1}, v_{n+1}\}^T$. Then the second equations can be used to relate solutions at n and $n+1$ step as

$$\{\chi_2\} = [L_{22}]^{-1}\left(\{l_2\} - [L_{21}]\{\chi_1\}\right) \tag{12}$$

Note that the specific values of $[L_{21}]$, $[L_{22}]$ and $\{l_2\}$ are calculated from

$$\begin{aligned} [L_{21}] &= \begin{bmatrix} \left[\frac{K^T K \Delta t}{6} - \frac{I}{\Delta t}\right] & \left[\frac{K^T C \Delta t}{6} - \frac{K^T M}{2} - \frac{I}{2}\right] \\ \left[\frac{C^T K \Delta t}{6} + \frac{M^T K}{2} + \frac{I}{2}\right] & \left[\frac{C^T C \Delta t}{6} + \frac{I \Delta t}{6} + \frac{M^T C}{2} - \frac{C^T M}{2} - \frac{M^T M}{\Delta t}\right] \end{bmatrix} \\ [L_{22}] &= \begin{bmatrix} \left[\frac{K^T K \Delta t}{3} + \frac{I}{\Delta t}\right] & \left[\frac{K^T C \Delta t}{3} + \frac{K^T M}{2} - \frac{I}{2}\right] \\ \left[\frac{C^T M \Delta t}{3} + \frac{M^T K}{2} - \frac{I}{2}\right] & \left[\frac{C^T C \Delta t}{3} + \frac{I \Delta t}{3} + \frac{M^T C}{2} + \frac{C^T M}{2} + \frac{M^T M}{\Delta t}\right] \end{bmatrix} \\ \{l_2\} &= \begin{Bmatrix} \frac{K^T \Delta t}{6} f_n + \frac{K^T \Delta t}{3} f_{n+1} \\ \frac{C^T \Delta t}{6} f_n + \frac{M^T}{2} f_n + \frac{C^T \Delta t}{3} f_{n+1} + \frac{M^T}{2} f_{n+1} \end{Bmatrix} \end{aligned} \tag{13}$$

where the time interval is given as $\Delta t = t_{n+1} - t_n$ and f_n is force vectors of n-th time step. Note that sub matrix $[L_{22}]$ is always symmetric, thus overall computational cost of inverting it can be minimized.

However the LSLSS scheme need to properly rescale residuals to prevent so called element locking. Since the least-squares method minimized squares of residuals within the least-squares functional, importance of each residual may be reflected differently. If the importance or scale of one residual is excessively higher than the other one, the other one is ignored in the functional causing malfunction of whole scheme. To prevent this, one may nondimensionalize residuals by multiplying certain factors[4] as

$$I(u,v)=\int_{t_n}^{t_{n+1}}\left[(\lambda_1)^2 r_1^T r_1+(\lambda_2)^2 r_2^T r_2\right]dt \tag{14}$$

where λ_1 and λ_2 are proper weight factors.

But finding weight factors for the exact nondimensionalization of residuals requires additional information from original physical system which is burdensome from the mathematical viewpoint.

2.2.2 Weighted residual linear single step scheme

Unlike the least-squares FEM, the weighted residual FEM minimizes residuals in weighted integral sense as

$$\begin{aligned}&\int_{t_n}^{t_{n+1}} w_1(r_1)\,dt=0\\&\int_{t_n}^{t_{n+1}} w_2(r_2)\,dt=0\end{aligned} \tag{15}$$

where w_1 and w_2 are weight functions which can be chosen as the same functions as the Lagrange type interpolation functions used for the approximation of u and v respectively. Then it can be rewritten as

$$\begin{bmatrix}[G_{11}] & [G_{12}]\\ [G_{21}] & [G_{22}]\end{bmatrix}\begin{Bmatrix}\{\chi_1\}\\ \{\chi_2\}\end{Bmatrix}=\begin{Bmatrix}\{g_1\}\\ \{g_2\}\end{Bmatrix} \tag{16}$$

The solutions of $n+1$ step can be calculated from the second equation of Eq. (16) as

$$\{\chi_2\}=[G_{22}]^{-1}\left(\{g_2\}-[G_{21}]\{\chi_1\}\right) \tag{17}$$

Note that the specific values of $[G_{21}]$, $[G_{22}]$ and $\{g_2\}$ in the WRLSS (the weighted residual linear single step) can be calculated from

$$\begin{aligned}&[G_{21}]=\begin{bmatrix}\left[\dfrac{K\,\Delta t}{6}\right] & \left[\dfrac{C\,\Delta t}{6}-\dfrac{M}{2}\right]\\ \left[-\dfrac{I}{2}\right] & \left[-\dfrac{I\,\Delta t}{6}\right]\end{bmatrix}\\&[G_{22}]=\begin{bmatrix}\left[\dfrac{K\,\Delta t}{3}\right] & \left[\dfrac{C\,\Delta t}{3}+\dfrac{M}{2}\right]\\ \left[\dfrac{I}{2}\right] & \left[-\dfrac{I\,\Delta t}{3}\right]\end{bmatrix}\\&\{g_2\}=\begin{Bmatrix}\dfrac{\Delta t}{6}f_n+\dfrac{\Delta t}{3}f_{n+1}\\ 0\end{Bmatrix}\end{aligned} \tag{18}$$

Compared with Eq. (13), values given in Eq. (18) are much simpler while sub matrix $[G_{22}]$ is not symmetric, which is not good because it increases overall computational cost of inverting it.

2.2.3 Ad-hoc linear single step scheme

To overcome drawbacks of time schemes based on the least-squares and weighted residual FEM, we consider new minimizing method based on variational principal, which will be called as the ad-hoc method.

Since we are dealing with semi-discrete ODEs that additional velocity relations should be treated, residuals should be simultaneously minimized over the time domain. To conduct this, the problem can be restated by minimization of the functional $A(u,v)$ which is stated as

$$A(u,v)=\int_{t_n}^{t_{n+1}}\left[r_1^T r_2\right]dt \tag{19}$$

Since the necessary condition of the minimum $A(u,v)$ is $\delta A(u,v,\delta u,\delta v)=0$, time schemes based on the new minimizations which will be called as ad-hoc scheme can be obtained by taking first variation of $A(u,v)$ and setting them as zero as

$$\delta A(u,v,\delta u,\delta v)=\int_{t_n}^{t_{n+1}}\left[\delta r_1^T r_2+\delta r_2^T r_1\right]dt=0 \tag{20}$$

By collecting coefficients of δu and δv of Eq. (20) and setting them as zero, the AHLSS (the ad-hoc linear single step) scheme can be obtained as

$$\begin{bmatrix} [A_{11}] & [A_{12}] \\ [A_{21}] & [A_{22}] \end{bmatrix} \begin{Bmatrix} \{\chi_1\} \\ \{\chi_2\} \end{Bmatrix} = \begin{Bmatrix} \{a_1\} \\ \{a_2\} \end{Bmatrix} \tag{21}$$

And the second equation can be used to relate solutions of n and $n+1$ step as

$$\{\chi_2\} = [A_{22}]^{-1}\left(\{a_2\} - [A_{21}]\{\chi_1\}\right) \tag{22}$$

where the specific values of sub matrices given in Eq. (22) can be calculates from

$$[A_{21}] = \begin{bmatrix} \left[\dfrac{-K^T + K}{2}\right] & \left[-\dfrac{K^T\,\Delta t}{6} + \dfrac{C}{2} - \dfrac{M}{\Delta t}\right] \\ \left[-\dfrac{K\,\Delta t}{6} + \dfrac{C^T}{2} - \dfrac{M^T}{\Delta t}\right] & \left[-\dfrac{C^T\,\Delta t}{6} - \dfrac{C\,\Delta t}{6} - \dfrac{M^T}{2} + \dfrac{M}{2}\right] \end{bmatrix}$$

$$[A_{22}] = \begin{bmatrix} \left[\dfrac{K^T + K}{2}\right] & \left[-\dfrac{K^T\,\Delta t}{6} + \dfrac{C}{2} + \dfrac{M}{\Delta t}\right] \\ \left[-\dfrac{K\,\Delta t}{6} + \dfrac{C^T}{2} + \dfrac{M^T}{\Delta t}\right] & \left[-\dfrac{C^T\,\Delta t}{3} - \dfrac{C\,\Delta t}{3} - \dfrac{M^T}{2} - \dfrac{M}{2}\right] \end{bmatrix} \tag{23}$$

$$\{a_2\} = \begin{Bmatrix} \dfrac{I}{2} f_n + \dfrac{I}{2} f_{n+1} \\ -\dfrac{I\,\Delta t}{6} f_n - \dfrac{I\,\Delta t}{3} f_{n+1} \end{Bmatrix}$$

Some mathematical drawbacks of the LSLSS and WRLSS schemes have been eliminated by using the ad-hoc method.

As one can notice from Eq. (20), each residual is multiplied by the variation of the other one, making the other one as weight function in integrated weighted residual sense. Because of this property, importance of each residual[4] is reflected as same level in the AHLSS scheme.

Also, the sub matrix $[A_{22}]$ is always symmetric regardless of symmetry of sub matrices given in Eq. (2) and (4).

Not only the mathematical advantages, computational advantages such as improved accuracy and stabilities can be achieved with the AHLSS, which will be discussed with numerical simulations using 2D wave equation.

3. Stabilities

For the stability test, we set $f(t)$ of Eq. (4) as zero to get homogeneous equation. Every recurrence scheme [8,10] finds solutions through the following form of calculation :

$$\begin{Bmatrix} u_{n+1} \\ v_{n+1} \end{Bmatrix} = \left[\bar{\mathrm{A}}\right] \begin{Bmatrix} u_n \\ v_n \end{Bmatrix} \tag{24}$$

where $\left[\bar{\mathrm{A}}\right]$ is called amplifier matrix. On the other hand, a general solution of any recurrence schemes can be written as

$$\begin{Bmatrix} u_{n+1} \\ v_{n+1} \end{Bmatrix} = \{\mu\}^T \begin{Bmatrix} u_n \\ v_n \end{Bmatrix} \tag{25}$$

where $\{\mu\}$ is corresponding coefficient vector. Then by substituting the (24) into the (25) we can obtain eigenvalue problem as follow :

$$\left[\bar{\mathrm{A}} - \mu\,\mathrm{I}\right] \begin{Bmatrix} u_n \\ v_n \end{Bmatrix} = 0 \tag{26}$$

where $[\mathrm{I}]$ is identical matrix. The eigenvalues obtained from Eq. (26) can be used to measure the stability of the system. For a stable condition, it is sufficient and necessary that the moduli of all eigenvalues satisfy

$$|\mu| \le 1 \tag{27}$$

In many cases, $\{\mu\}$ includes complex numbers, thus so called z-transformation can be used to investigate stability limits. The z-transformation and stability criterion is presented in Fig. 1.

In current study, all schemes showed unconditionally stable revolution of solutions.

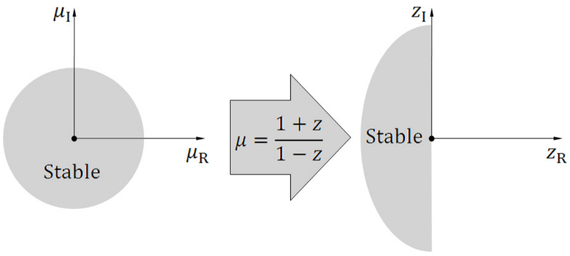


Fig. 1. A z-transformation for stability measure

4. Numerical results

For the verification of newly developed time schemes, numerical simulation was conducted with 2D wave equation which is obtained by simply eliminating terms related with z-direction from Eq. (1) as

$$\rho\frac{\partial^2 u}{\partial t^2}-\frac{\partial}{\partial x}\left(T_x\frac{\partial u}{\partial x}\right)-\frac{\partial}{\partial y}\left(T_y\frac{\partial u}{\partial y}\right)+c_f u=f(x,y,t) \quad (28)$$

where the coefficients are given as

$$T_x = 12.5lb/ft. \ , \quad T_y = 12.5lb/ft. \ , \quad c_f = 0 \ , \quad \rho = 2.5slugs/ft.^2 \quad f = 0 \quad (29)$$

Spatial domain is $4ft.\times 2ft.$ rectangular membrane. Linear rectangular mesh (4×4) was used as shown in the Fig. 2. Symmetry boundary conditions are used along the center lines.

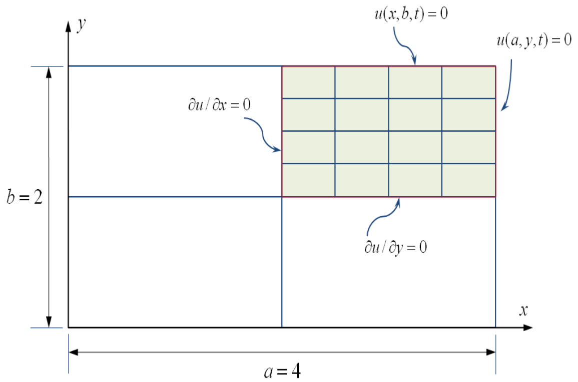


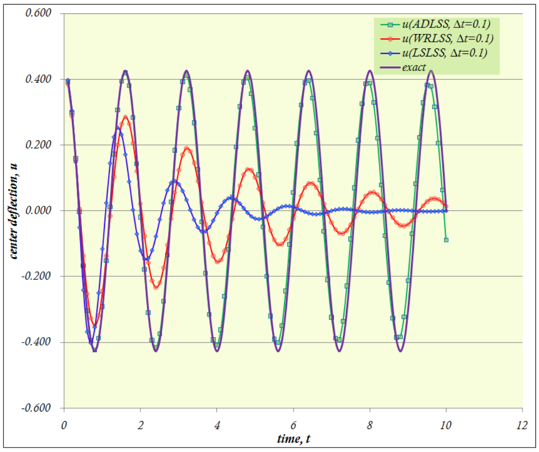
Fig. 2. Computational domain, ICs and BCs

BCs : $u(a,y,t)=0$, $u(x,b,t)=0$,
$\partial u(a/2,y,t)/\partial x=0$ and $\partial u(x,b/2,t)/\partial y=0$

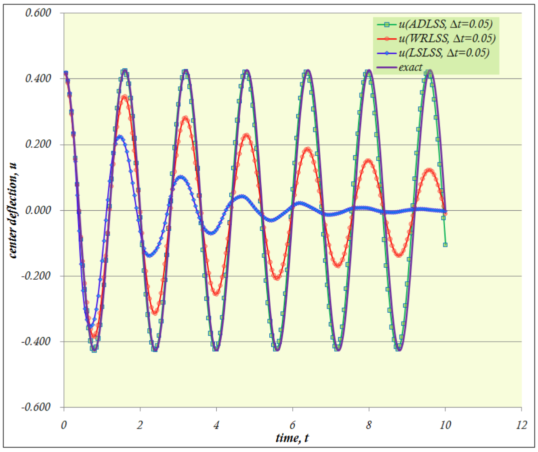
$$\text{ICs} : u_0(x,y)=\frac{409.6}{\pi^6}\sum_{m,n=1,3,\cdots}^{\infty}\left[\frac{1}{m^3n^3}\sin\left(\frac{m\pi x}{4}\right)\sin\left(\frac{n\pi y}{2}\right)\right]$$
$$v_0(x,y)=0 \quad (30)$$

The analytic solution which is used to measure the accuracy of new schemes is given as

$$u(x,y,t)=\frac{409.6}{\pi^6}\sum_{m,n=1,3,\cdots}^{\infty}\left[\frac{1}{m^3n^3}\cos(\varpi t)\sin\left(\frac{m\pi x}{4}\right)\sin\left(\frac{n\pi y}{2}\right)\right] \quad (31)$$



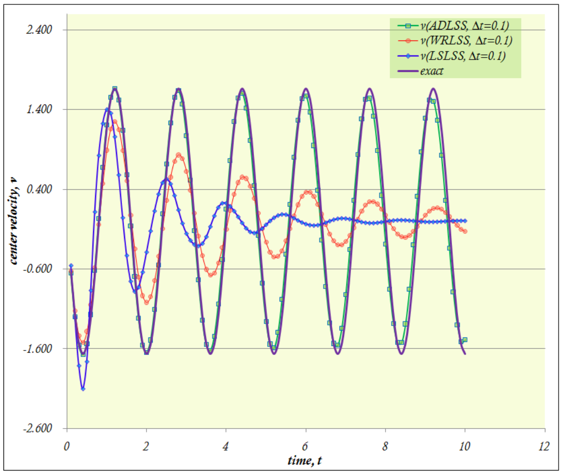
a. Center deflection of membrane with Δt = 0.1sec.



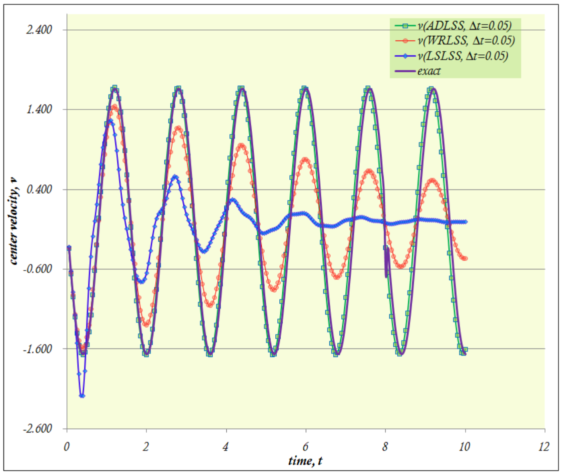
b. Center deflection of membrane with Δt = 0.05sec.

Fig. 3. Comparison of center displacement of various schemes with that of analytic solution

Comparison of numerical simulations of various time schemes shows that the ADLSS scheme is providing the best results as shown in Fig. 3.



a. Center velocity of membrane with Δt = 0.1sec.



a. Center velocity of membrane with Δt = 0.05sec.

Fig. 4. Comparison of center velocity of various schemes with that of analytic solution

For velocities, the ADLSS scheme showed the best results almost superposing analytic solutions as shown in the Fig. 4.

The WRLSS and LSLSS schemes provided the poorest result with large time intervals (i.e., $\Delta t \geq 0.01$sec) while schemes with small intervals provided acceptable results. Among them the LSLSS schemes showed very unstable revolutions of velocities. One may improve the performance of the LSLSS by properly rescaling residuals in Eq. (8), however specific procedures and results are not discussed in current study.

For better understanding of the related physics, revolution of vibrating membrane is presented in Fig. 5 using the result of the analytic solutions given in Eq. (31), because numerical results obtained from the ADLSS scheme with time interval $\Delta t = 0.01$sec almost superposed analytic solution.

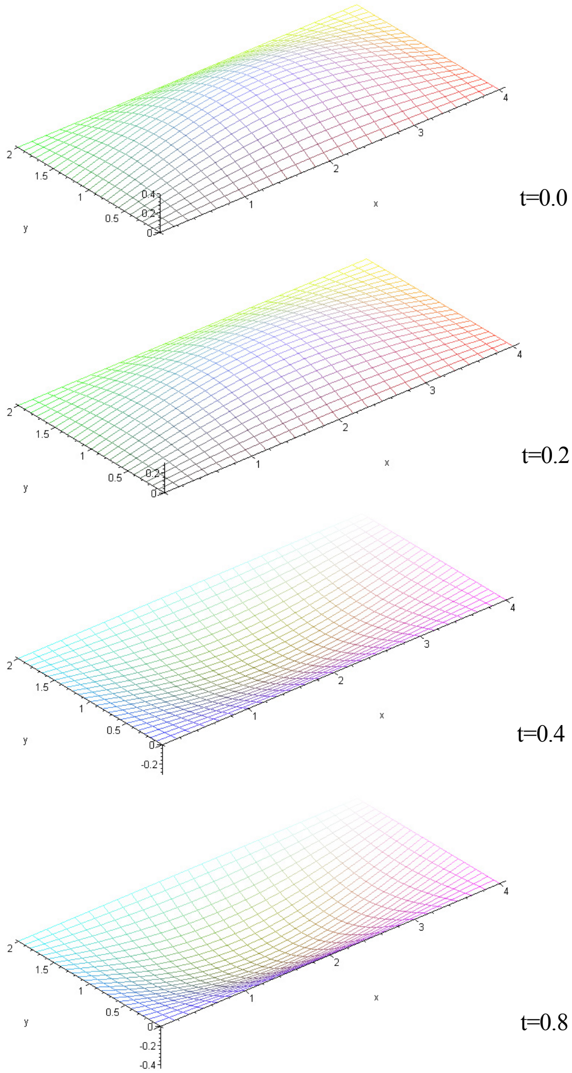


Fig. 5. Revolution of deflections with exact solution

5. Conclusion

Three new time schemes based on the FEM were developed and their performances were tested with 2D wave equation. To check the accuracy of new schemes, their numerical results were compared with available analytic solution.

To overcome some drawbacks that other time schemes based on the least squares and weighted residual FEM have, unconventional ad-hoc residual minimization was considered, and bilinear form of equation was obtained by using variational method.

Newly developed ad-hoc scheme provided symmetric relations which make overall computational cost cheaper. Also, they automatically equalized importance levels of residuals in the minimizing functional.

In numerical analysis of 2D wave equation, time scheme developed from ad-hoc relation provided the best results with unconditional stability and good accuracy, while the other schemes showed poor accuracy when large time step was used. But other schemes provided acceptable accuracy with smaller time intervals.

Thus, it is possible to use newly developed ADLSS time scheme in practical analysis of other structural dynamics problems with proper modifications of sub matrices given in Eq. (2) and (4).

For future studies, development of multi step procedures using higher order interpolations is in our consideration to improve accuracies of the ADLSS time scheme.

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