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FUNCTIONAL EQUATIONS IN BANACH MODULES AND APPROXIMATE ALGEBRA HOMOMORPHISMS IN BANACH ALGEBRAS

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Abstract. We prove the Hyers-Ulam stability of partitioned functional equations in Banach modules over a unital C^*-algebra. It is applied to show the stability of algebra homomorphisms in Banach algebras associated with partitioned functional equations in Banach algebras.

1. Introduction and preliminaries

Recently, T. Trif [9, Theorem 2.1] proved that, for vector spaces V and W, a mapping $f : V \to W$ with $f(0) = 0$ satisfies the functional equation

$$\begin{aligned} n \ {}_{n-2}C_{k-2} f\left(\frac{x_1 + \cdots + x_n}{n}\right) &+ {}_{n-2}C_{k-1} \sum_{i=1}^{n} f(x_i) \\ &= k \sum_{1 \le i_1 < \cdots < i_k \le n} f\left(\frac{x_{i_1} + \cdots + x_{i_k}}{k}\right) \end{aligned}$$

for all $x_1, \cdots, x_n \in V$ if and only if the mapping $f : V \to W$ satisfies the additive Cauchy equation $f(x+y) = f(x) + f(y)$ for all $x, y \in V$.

In [6], Park conjectured the following, and gave a partial answer for the conjecture.

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Conjecture. A mapping $f : V \to W$ with $f(0) = 0$ satisfies the functional equation

$$
\begin{aligned}
p^n f(\frac{x_1 + \cdots + x_{p^n}}{p^n}) \quad &+ \quad (pk-p)\sum_{i=1}^{p^{n-1}} f(\frac{x_{pi-p+1} + \cdots + x_{pi}}{p}) \\
&= \quad k\sum_{i=1}^{p^n} f(\frac{x_i + \cdots + x_{i+k-1}}{k}) \qquad (1.1)
\end{aligned}
$$

for all $x_1 = x_{p^n+1}, \cdots, x_{k-1} = x_{p^n+k-1}, x_k, \cdots, x_{p^n} \in V$ if and only if the mapping $f : V \to W$ satisfies the additive Cauchy equation $f(x+y) = f(x) + f(y)$ for all $x, y \in V$.

Throughout this paper, let A be a unital C^*-algebra with norm $|\cdot|$ and $U(A)$ the unitary group of A. Let ${}_AB$ and ${}_AC$ be left Banach A-modules with norms $||\cdot||$ and $\|\cdot\|$, respectively. Let d, r and p be positive integers.

The following is useful to prove the stability of linear functional equations in Banach modules over a unital C^*-algebra.

Lemma 1.1. ([5, Theorem 1]) *Let $a \in A$ and $|a| < 1 - \frac{2}{m}$ for some integer m greater than 2. Then there are m elements $u_1, \cdots, u_m \in U(A)$ such that $ma = u_1 + \cdots + u_m$.*

The main purpose of this paper is to prove the Hyers-Ulam stability of the functional equation (1.1) in Banach modules over a unital C^*-algebra for a special case, and to prove the Hyers-Ulam stability of algebra homomorphisms in Banach algebras.

2. Partitioned functional equations

In this section, we solve the conjecture for a special case.

Theorem 2.1. *Let V and W be vector spaces. A mapping $f : V \to W$ with $f(0) = 0$ satisfies the functional equation*

$$
\begin{aligned}
(4p)^n f(\frac{x_1 + \cdots + x_{(4p)^n}}{(4p)^n}) \quad &+ \quad 4p \sum_{i=1}^{(4p)^{n-1}} f(\frac{x_{4pi-4p+1} + \cdots + x_{4pi}}{4p}) \\
&= \quad 2\sum_{i=1}^{(4p)^n} f(\frac{x_i + x_{i+1}}{2}) \qquad (2.1)
\end{aligned}
$$

for all $x_1 = x_{(4p)^n+1}, x_2, \cdots, x_{(4p)^n} \in V$ *if and only if the mapping* $f : V \to W$ *satisfies the additive Cauchy equation* $f(x+y) = f(x) + f(y)$ *for all* $x, y \in V$.

Proof. Assume that a mapping $f : V \to W$ satisfies (2.1). Put $x_{4i-3} = x$, $x_{4i-2} = y$, $x_{4i-1} = z$ and $x_{4i} = -z$ in (2.1) for all $i = 1, \cdots, p(4p)^{n-1}$. Then

$$\begin{aligned} & (4p)^n f(\frac{x+y}{4}) + (4p)^n f(\frac{x+y}{4}) \\ & \qquad = 2(4p)^{n-1}(pf(\frac{x+y}{2}) + pf(\frac{y+z}{2}) + pf(\frac{-z+x}{2})) \end{aligned} \tag{2.2}$$

for all $x, y, z \in V$. Put $x = y$ and $z = x$ in (2.2). Then we get

$$2\ (4p)^n f(\frac{x}{2}) = 4p\ (4p)^{n-1} f(x)$$

for all $x \in V$. So we have

$$f(\frac{1}{2}x) = \frac{1}{2} f(x) \tag{2.3}$$

for all $x \in V$. Put $z = 0$ in (2.2). By (2.3), we get

$$2p\ (4p)^{n-1} f(x+y) = (4p)^{n-1}(pf(x+y) + pf(y) + pf(x))$$

for all $x, y \in V$. So the mapping $f : V \to W$ satisfies the additive Cauchy equation $f(x+y) = f(x) + f(y)$ for all $x, y \in V$.

The converse is obvious. □

THEOREM 2.2. *Let* V *and* W *be vector spaces. A mapping* $f : V \to W$ *with* $f(0) = 0$ *satisfies the functional equation*

$$\begin{aligned} (4p)^n f(\frac{x_1 + \cdots + x_{(4p)^n}}{(4p)^n}) \quad &+ \quad 8p \sum_{i=1}^{(4p)^{n-1}} f(\frac{x_{4pi-4p+1} + \cdots + x_{4pi}}{4p}) \\ &= \quad 3 \sum_{i=1}^{(4p)^n} f(\frac{x_i + x_{i+1} + x_{i+2}}{3}) \end{aligned} \tag{2.4}$$

for all $x_1 = x_{(4p)^n+1}, x_2 = x_{(4p)^n+2}, x_3, \cdots, x_{(4p)^n} \in V$ *if and only if the mapping* $f : V \to W$ *satisfies the additive Cauchy equation* $f(x+y) = f(x) + f(y)$ *for all* $x, y \in V$.

Proof. Assume that a mapping $f : V \to W$ satisfies (2.4). Put $x_{4i-3} = x_{4i-1} = x$ and $x_{4i-2} = x_{4i} = y$ in (2.4) for all $i = 1, \cdots, p(4p)^{n-1}$. Then

$$(2.5)\qquad (4p)^n f(\frac{x+y}{2}) + 2\,(4p)^n f(\frac{x+y}{2}) = 6p\,(4p)^{n-1}(f(\frac{2x+y}{3}) + f(\frac{x+2y}{3}))$$

for all $x, y \in V$. Replacing x and y by $2x$ and $-x$ in (2.5), respectively, we get

$$3\,(4p)^n f(\frac{x}{2}) = 6p\,(4p)^{n-1} f(x)$$

for all $x \in V$. So

$$(2.6)\qquad f(\frac{1}{2}x) = \frac{1}{2}f(x)$$

for all $x \in V$. Put $x_{4i-3} = x$, $x_{4i-2} = y$, $x_{4i-1} = z$ and $x_{4i} = -z$ in (2.4) for all $i = 1, \cdots, p(4p)^{n-1}$. Then

$$\begin{aligned}(4p)^n f(\frac{x+y}{4}) + 2\,(4p)^n f(\frac{x+y}{4}) &= 3(4p)^{n-1}(pf(\frac{x+y+z}{3}) + pf(\frac{y}{3}) \\ (2.7)\qquad &+ pf(\frac{x}{3}) + pf(\frac{x+y-z}{3}))\end{aligned}$$

for all $x, y, z \in V$. Put $x = y$ and $z = x$ in (2.7). Then we get

$$3\,(4p)^n f(\frac{x}{2}) = 3p\,(4p)^{n-1}(f(x) + 3f(\frac{x}{3}))$$

for all $x \in V$. By (2.6), we get

$$6p\,(4p)^{n-1} f(x) = 3p\,(4p)^{n-1}(f(x) + 3f(\frac{x}{3}))$$

for all $x \in V$. So we have

$$(2.8)\qquad f(\frac{1}{3}x) = \frac{1}{3}f(x)$$

for all $x \in V$. Put $z = 0$ in (2.7). By (2.6) and (2.8), we get

$$3p\,(4p)^{n-1} f(x+y) = (4p)^{n-1}(pf(x+y) + pf(y) + pf(x) + pf(x+y))$$

for all $x, y \in V$. So the mapping $f : V \to W$ satisfies the additive Cauchy equation $f(x+y) = f(x) + f(y)$ for all $x, y \in V$.

The converse is obvious. □

3. Stability of partitioned functional equations in Banach modules over a C^*-algebra associated with its unitary group

We prove the Hyers-Ulam stability of the functional equation (1.1) in Banach modules over a unital C^*-algebra associated with its unitary group for a special case.

THEOREM 3.1. *Let $f : {}_AB \to {}_AC$ be a mapping with $f(0) = 0$ for which there exists a function $\varphi : {}_AB^{(4p)^n} \to [0,\infty)$ such that*

$$\widetilde{\varphi}(x_1, \cdots, x_{(4p)^n}) : = \sum_{j=0}^{\infty} (\frac{d}{(4p)^n r})^j \varphi((\frac{(4p)^n r}{d})^j x_1, \cdots, (\frac{(4p)^n r}{d})^j x_{(4p)^n}) < \infty, \tag{3.1}$$

$$\begin{aligned} \|D_u f(x_1, \cdots, x_{(4p)^n})\| &= \|\frac{d}{r} u f(\frac{rx_1 + \cdots + rx_{(4p)^n}}{d}) \\ &+ 4p \sum_{i=1}^{(4p)^{n-1}} uf(\frac{x_{4pi-4p+1} + \cdots + x_{4pi}}{4p}) \\ &- 2\sum_{i=1}^{(4p)^n} f(\frac{ux_i + ux_{i+1}}{2})\| \le \varphi(x_1, \cdots, x_{(4p)^n}) \end{aligned} \tag{3.2}$$

for all $u \in U(A)$ and all $x_1 = x_{(4p)^n+1}, x_2, \cdots, x_{(4p)^n} \in {}_AB$. Then there exists a unique A-linear mapping $T : {}_AB \to {}_AC$ such that

$$\|f(x) - T(x)\| \le \frac{1}{(4p)^n} \widetilde{\varphi}(x, \cdots, x) \tag{3.3}$$

for all $x \in {}_AB$.

Proof. Put $u = 1 \in U(A)$. Let $x_1 = \cdots = x_{(4p)^n} = x$ in (3.2). Then we get

$$\|\frac{d}{r}\, f(\frac{(4p)^n r}{d} x) + (4p)^n f(x) - 2(4p)^n f(x)\| \le \varphi(x, \cdots, x)$$

for all $x \in {}_AB$. So one can obtain that

$$\|f(x) - \frac{d}{(4p)^n r} f(\frac{(4p)^n r}{d} x)\| \le \frac{1}{(4p)^n} \varphi(x, \cdots, x)$$

for all $x \in {}_AB$. We prove by induction on j that

$$\begin{aligned}
&\|(\frac{d}{(4p)^n r})^j f((\frac{(4p)^n r}{d})^j x) - (\frac{d}{(4p)^n r})^{j+1} f((\frac{(4p)^n r}{d})^{j+1} x)\| \\
&\qquad \le \frac{1}{(4p)^n}(\frac{d}{(4p)^n r})^j \varphi((\frac{(4p)^n r}{d})^j x, \cdots, (\frac{(4p)^n r}{d})^j x)
\end{aligned}$$

for all $x \in {}_AB$. So we get

$$\begin{aligned}
(3.4)\quad &\|f(x) - (\frac{d}{(4p)^n r})^j f((\frac{(4p)^n r}{d})^j x)\| \\
&\qquad \le \frac{1}{(4p)^n}\sum_{m=0}^{j-1}(\frac{d}{(4p)^n r})^m \varphi((\frac{(4p)^n r}{d})^m x, \cdots, (\frac{(4p)^n r}{d})^m x)
\end{aligned}$$

for all $x \in {}_AB$.

Let x be an element in ${}_AB$. For positive integers l and m with $l > m$,

$$\begin{aligned}
&\|(\frac{d}{(4p)^n r})^l f((\frac{(4p)^n r}{d})^l x) - (\frac{d}{(4p)^n r})^m f((\frac{(4p)^n r}{d})^m x)\| \\
&\qquad \le \frac{1}{(4p)^n}\sum_{j=m}^{l-1}(\frac{d}{(4p)^n r})^j \varphi((\frac{(4p)^n r}{d})^j x, \cdots, (\frac{(4p)^n r}{d})^j x),
\end{aligned}$$

which tends to zero as $m \to \infty$ by (3.1). So $\{(\frac{d}{(4p)^n r})^j f((\frac{(4p)^n r}{d})^j x)\}$ is a Cauchy sequence for all $x \in {}_AB$. Since ${}_AC$ is complete, the sequence $\{(\frac{d}{(4p)^n r})^j f((\frac{(4p)^n r}{d})^j x)\}$ converges for all $x \in {}_AB$. We can define a mapping $T : {}_AB \to {}_AC$ by

$$(3.5)\qquad T(x) = \lim_{j\to\infty}(\frac{d}{(4p)^n r})^j f((\frac{(4p)^n r}{d})^j x)$$

for all $x \in {}_AB$.

By (3.1) and (3.5), we get

$$\begin{aligned}
&\|D_1 T(x_1, \cdots, x_{(4p)^n})\| \\
&\qquad = \lim_{j\to\infty}(\frac{d}{(4p)^n r})^j \|D_1 f((\frac{(4p)^n r}{d})^j x_1, \cdots, (\frac{(4p)^n r}{d})^j x_{(4p)^n})\| \\
&\qquad \le \lim_{j\to\infty}(\frac{d}{(4p)^n r})^j \varphi((\frac{(4p)^n r}{d})^j x_1, \cdots, (\frac{(4p)^n r}{d})^j x_{(4p)^n}) \\
&\qquad = 0
\end{aligned}$$

for all $x_1, \cdots, x_{(4p)^n} \in {}_AB$. Hence $D_1T(x_1, \cdots, x_{(4p)^n}) = 0$ for all $x_1, \cdots, x_{(4p)^n} \in {}_AB$. Put $x_1 = \cdots = x_{(4p)^n} = x$ in $D_1T(x_1, \cdots, x_{(4p)^n}) = 0$. Then

$$\frac{d}{r}T(\frac{(4p)^n r}{d}x) + (4p)^n T(x) - 2(4p)^n T(x) = 0$$

for all $x \in {}_AB$. So

$$T(\frac{(4p)^n r}{d}x) = \frac{(4p)^n r}{d}T(x)$$

for all $x \in {}_AB$. Since

$$\begin{aligned} \frac{d}{r}T(\frac{rx_1 + \cdots + rx_{(4p)^n}}{d}) &= \frac{d}{r}T(\frac{(4p)^n r(x_1 + \cdots + x_{(4p)^n})}{(4p)^n d}) \\ &= \frac{d}{r}\frac{(4p)^n r}{d}T(\frac{x_1 + \cdots + x_{(4p)^n}}{(4p)^n}) \\ &= (4p)^n T(\frac{x_1 + \cdots + x_{(4p)^n}}{(4p)^n}) \end{aligned}$$

for all $x_1, \cdots, x_{(4p)^n} \in {}_AB$,

$$\begin{aligned} (4p)^n T(\frac{x_1 + \cdots + x_{(4p)^n}}{(4p)^n}) &+ 4p \sum_{i=1}^{(4p)^{n-1}} T(\frac{x_{4pi-4p+1} + \cdots + x_{4pi}}{4p}) \\ &= 2\sum_{i=1}^{(4p)^n} T(\frac{x_i + x_{i+1}}{2}) \end{aligned}$$

for all $x_1 = x_{(4p)^n+1}, x_2, \cdots, x_{(4p)^n} \in {}_AB$. By Theorem 2.1, T is additive. Moreover, by passing to the limit in (3.4) as $j \to \infty$, we get the inequality (3.3).

Now let $L : {}_AB \to {}_AC$ be another additive mapping satisfying

$$\|f(x) - L(x)\| \le \frac{1}{(4p)^n}\widetilde{\varphi}(x, \cdots, x)$$

for all $x \in {}_AB$.

$$\begin{aligned}\|T(x)-L(x)\| &= (\frac{d}{(4p)^n r})^j \|T((\frac{(4p)^n r}{d})^j x) - L((\frac{(4p)^n r}{d})^j x)\| \\ &\le (\frac{d}{(4p)^n r})^j \|T((\frac{(4p)^n r}{d})^j x) - f((\frac{(4p)^n r}{d})^j x)\| \\ &+ (\frac{d}{(4p)^n r})^j \|f((\frac{(4p)^n r}{d})^j x) - L((\frac{(4p)^n r}{d})^j x)\| \\ &\le \frac{2}{(4p)^n}(\frac{d}{(4p)^n r})^j \widetilde{\varphi}((\frac{(4p)^n r}{d})^j x, \cdots, (\frac{(4p)^n r}{d})^j x),\end{aligned}$$

which tends to zero as $j \to \infty$ by (3.1). Thus $T(x) = L(x)$ for all $x \in {}_AB$. This proves the uniqueness of T.

By the assumption, for each $u \in U(A)$,

$$\begin{aligned}&(\frac{d}{(4p)^n r})^j \|D_u f((\frac{(4p)^n r}{d})^j x, \cdots, (\frac{(4p)^n r}{d})^j x)\| \\ &\qquad \le (\frac{d}{(4p)^n r})^j \varphi((\frac{(4p)^n r}{d})^j x, \cdots, (\frac{(4p)^n r}{d})^j x)\end{aligned}$$

for all $x \in {}_AB$, and

$$(\frac{d}{(4p)^n r})^j \|D_u f((\frac{(4p)^n r}{d})^j x, \cdots, (\frac{(4p)^n r}{d})^j x)\| \to 0$$

as $j \to \infty$ for all $x \in {}_AB$. So

$$D_u T(x, \cdots, x) = \lim_{j\to\infty} (\frac{d}{(4p)^n r})^j D_u f((\frac{(4p)^n r}{d})^j x, \cdots, (\frac{(4p)^n r}{d})^j x) = 0$$

for all $u \in U(A)$ and all $x \in {}_AB$. Hence

$$D_u T(x, \cdots, x) = \frac{d}{r}\, uT(\frac{(4p)^n r}{d} x) + (4p)^n u T(x) - 2(4p)^n T(ux) = 0$$

for all $u \in U(A)$ and all $x \in {}_AB$. So

$$uT(x) = T(ux)$$

for all $u \in U(A)$ and all $x \in {}_AB$.

Now let $a \in A$ ($a \neq 0$) and M an integer greater than $4|a|$. Then

$$|\frac{a}{M}| = \frac{1}{M}|a| < \frac{|a|}{4|a|} = \frac{1}{4} < 1 - \frac{2}{3} = \frac{1}{3}.$$

By Lemma 1.1, there exist three elements $u_1, u_2, u_3 \in U(A)$ such that $3\frac{a}{M} = u_1 + u_2 + u_3$. And $T(x) = T(3 \cdot \frac{1}{3}x) = 3T(\frac{1}{3}x)$ for all $x \in {}_AB$. So $T(\frac{1}{3}x) = \frac{1}{3}T(x)$ for all $x \in {}_AB$. Thus

$$\begin{aligned} T(ax) &= T(\frac{M}{3} \cdot 3\frac{a}{M}x) = M \cdot T(\frac{1}{3} \cdot 3\frac{a}{M}x) = \frac{M}{3}T(3\frac{a}{M}x) \\ &= \frac{M}{3}T(u_1x + u_2x + u_3x) = \frac{M}{3}(T(u_1x) + T(u_2x) + T(u_3x)) \\ &= \frac{M}{3}(u_1 + u_2 + u_3)T(x) = \frac{M}{3} \cdot 3\frac{a}{M}T(x) \\ &= aT(x) \end{aligned}$$

for all $x \in {}_AB$. Obviously, $T(0x) = 0T(x)$ for all $x \in {}_AB$. Hence

$$T(ax + by) = T(ax) + T(by) = aT(x) + bT(y)$$

for all $a, b \in A$ and all $x, y \in {}_AB$. So the unique additive mapping $T : {}_AB \to {}_AC$ is an A-linear mapping, as desired. □

Applying the unital C^*-algebra $\mathbb{C}$ to Theorem 3.1, one can obtain the following.

Corollary 3.2. *Let E_1 and E_2 be complex Banach spaces with norms $|| \cdot ||$ and $\| \cdot \|$, respectively. Let $f : E_1 \to E_2$ be a mapping with $f(0) = 0$ for which there exists a function $\varphi : E_1^{(4p)^n} \to [0, \infty)$ such that*

$$\begin{aligned} &\widetilde{\varphi}(x_1, \cdots, x_{(4p)^n}) : \\ &\quad = \sum_{j=0}^{\infty} (\frac{d}{(4p)^n r})^j \varphi((\frac{(4p)^n r}{d})^j x_1, \cdots, (\frac{(4p)^n r}{d})^j x_{(4p)^n}) \\ &\quad < \infty, \\ &\|D_\lambda f(x_1, \cdots, x_{(4p)^n})\| \le \varphi(x_1, \cdots, x_{(4p)^n}) \end{aligned}$$

for all $\lambda \in \mathbb{T}^1 := \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$ and all $x_1, \cdots, x_{(4p)^n} \in E_1$. Then there exists a unique $\mathbb{C}$-linear mapping $T : E_1 \to E_2$ such that

$$\|f(x) - T(x)\| \le \frac{1}{(4p)^n}\widetilde{\varphi}(x, \cdots, x)$$

for all $x \in E_1$.

Theorem 3.3. *Let $f : {}_AB \to {}_AC$ be a mapping with $f(0) = 0$ for which there exists a function $\varphi : {}_AB^{(4p)^n} \to [0, \infty)$ such that*

$$\widetilde{\varphi}(x_1, \cdots, x_{(4p)^n}) : = \sum_{j=0}^{\infty} (\frac{d}{2r})^j \varphi((\frac{2r}{d})^j x_1, \cdots, (\frac{2r}{d})^j x_{(4p)^n}) < \infty, \tag{3.6}$$

$$\begin{aligned} &\|D_u f(x_1, \cdots, x_{(4p)^n})\| \\ &= \|(4p)^n u f(\frac{x_1 + \cdots + x_{(4p)^n}}{(4p)^n}) + 4p \sum_{i=1}^{(4p)^{n-1}} u f(\frac{x_{4pi-4p+1} + \cdots + x_{4pi}}{4p}) \\ &\quad - \sum_{i=1}^{(4p)^n} \frac{d}{r} f(\frac{urx_i + urx_{i+1}}{d})\| \le \varphi(x_1, \cdots, x_{(4p)^n}) \end{aligned} \tag{3.7}$$

for all $u \in U(A)$ and all $x_1 = x_{(4p)^n+1}, x_2, \cdots, x_{(4p)^n} \in {}_AB$. Then there exists a unique A-linear mapping $T : {}_AB \to {}_AC$ such that

$$\|f(x) - T(x)\| \le \frac{1}{2(4p)^n} \widetilde{\varphi}(x, \cdots, x) \tag{3.8}$$

for all $x \in {}_AB$.

Proof. Put $u = 1 \in U(A)$. Let $x_1 = \cdots = x_{(4p)^n} = x$ in (3.7). Then we get

$$\|(4p)^n f(x) + (4p)^n f(x) - \frac{(4p)^n d}{r} f(\frac{2r}{d} x)\| \le \varphi(x, \cdots, x)$$

for all $x \in {}_AB$. So one can obtain that

$$\|f(x) - \frac{d}{2r} f(\frac{2r}{d} x)\| \le \frac{1}{2(4p)^n} \varphi(x, \cdots, x)$$

for all $x \in {}_AB$. We prove by induction on j that

$$\begin{aligned} &\|(\frac{d}{2r})^j f((\frac{2r}{d})^j x) - (\frac{d}{2r})^{j+1} f((\frac{2r}{d})^{j+1} x)\| \\ &\qquad \le \frac{1}{2(4p)^n} (\frac{d}{2r})^j \varphi((\frac{2r}{d})^j x, \cdots, (\frac{2r}{d})^j x) \end{aligned}$$

for all $x \in {}_AB$. So we get

$$\|f(x) - (\frac{d}{2r})^j f((\frac{2r}{d})^j x)\| \le \frac{1}{2(4p)^n} \sum_{m=0}^{j-1} (\frac{d}{2r})^m \varphi((\frac{2r}{d})^m x, \cdots, (\frac{2r}{d})^m x) \tag{3.9}$$

for all $x \in {}_AB$.

Let x be an element in ${}_AB$. For positive integers l and m with $l > m$,

$$\begin{aligned} \|(\frac{d}{2r})^l f((\frac{2r}{d})^l x) &- (\frac{d}{2r})^m f((\frac{2r}{d})^m x)\| \\ &\le \frac{1}{2(4p)^n} \sum_{j=m}^{l-1} (\frac{d}{2r})^j \varphi((\frac{2r}{d})^j x, \cdots, (\frac{2r}{d})^j x), \end{aligned}$$

which tends to zero as $m \to \infty$ by (3.6). So $\{(\frac{d}{2r})^j f((\frac{2r}{d})^j x)\}$ is a Cauchy sequence for all $x \in {}_AB$. Since the space ${}_AC$ is complete, the sequence $\{(\frac{d}{2r})^j f((\frac{2r}{d})^j x)\}$ converges for all $x \in {}_AB$. We can define a mapping $T : {}_AB \to {}_AC$ by

$$T(x) = \lim_{j\to\infty} (\frac{d}{2r})^j f((\frac{2r}{d})^j x) \tag{3.10}$$

for all $x \in {}_AB$.

By (3.6) and (3.10), we get

$$\begin{aligned} \|D_1 T(x_1, \cdots, x_{(4p)^n})\| &= \lim_{j\to\infty} (\frac{d}{2r})^j \|D_1 f((\frac{2r}{d})^j x_1, \cdots, (\frac{2r}{d})^j x_{(4p)^n})\| \\ &\le \lim_{j\to\infty} (\frac{d}{2r})^j \varphi((\frac{2r}{d})^j x_1, \cdots, (\frac{2r}{d})^j x_{(4p)^n}) = 0 \end{aligned}$$

for all $x_1, \cdots, x_{(4p)^n} \in {}_AB$. Hence $D_1 T(x_1, \cdots, x_{(4p)^n}) = 0$ for all $x_1, \cdots, x_{(4p)^n} \in {}_AB$. Put $x_1 = \cdots = x_{(4p)^n} = x$ in $D_1 T(x_1, \cdots, x_{(4p)^n}) = 0$. Then

$$(4p)^n T(x) + (4p)^n T(x) - \frac{(4p)^n d}{r} T(\frac{2r}{d} x) = 0$$

for all $x \in {}_AB$. So

$$T(\frac{2r}{d} x) = \frac{2r}{d} T(x)$$

for all $x \in {}_AB$. Since

$$\frac{d}{r} T(\frac{r x_i + r x_{i+1}}{d}) = \frac{d}{r} T(\frac{2r(x_i + x_{i+1})}{2d}) = \frac{d}{r}\frac{2r}{d} T(\frac{x_i + x_{i+1}}{2}) = 2T(\frac{x_i + x_{i+1}}{2})$$

for all $x_i, x_{i+1} \in {}_AB$,

$$(4p)^n T(\frac{x_1 + \cdots + x_{(4p)^n}}{(4p)^n}) + 4p \sum_{i=1}^{(4p)^{n-1}} T(\frac{x_{4pi-4p+1} + \cdots + x_{4pi}}{4p}) = 2 \sum_{i=1}^{(4p)^n} T(\frac{x_i + x_{i+1}}{2})$$

for all $x_1 = x_{(4p)^n+1}, x_2, \cdots, x_{(4p)^n} \in {}_AB$. By Theorem 2.1, T is additive. Moreover, by passing to the limit in (3.9) as $j \to \infty$, we get the inequality (3.8).

The rest of the proof is the same as the proof of Theorem 3.1. □

THEOREM 3.4. *Let $f : {}_AB \to {}_AC$ be a mapping with $f(0) = 0$ for which there exists a function $\varphi : {}_AB^{(4p)^n} \to [0, \infty)$ such that*

$$\widetilde{\varphi}(x_1, \cdots, x_{(4p)^n}) := \sum_{j=0}^{\infty} 2^j \varphi(-\frac{1}{2^j}x_1, \frac{1}{2^j}x_2, \frac{1}{2^j}x_3, \cdots, -\frac{1}{2^j}x_{(4p)^n-4p+1}, \frac{1}{2^j}x_{(4p)^n-4p+2}, \cdots, \frac{1}{2^j}x_{(4p)^n}) < \infty, \tag{3.11}$$

$$\begin{aligned} \|D_u f(x_1, \cdots, x_{(4p)^n})\| &= \|(4p)^n u f(\frac{x_1 + \cdots + x_{(4p)^n}}{(4p)^n}) \\ &\quad + 4p \sum_{i=1}^{(4p)^{n-1}} u f(\frac{x_{4pi-4p+1} + \cdots + x_{4pi}}{4p}) \\ &\quad - 2 \sum_{i=1}^{(4p)^n} f(\frac{ux_i + ux_{i+1}}{2})\| \\ &\le \varphi(x_1, \cdots, x_{(4p)^n}) \end{aligned} \tag{3.12}$$

for all $u \in U(A)$ and all $x_1 = x_{(4p)^n+1}, x_2, \cdots, x_{(4p)^n} \in {}_AB$. Then there exists a unique A-linear mapping $T : {}_AB \to {}_AC$ such that

$$\|f(x) - T(x)\| \le \frac{1}{(4p)^n} \widetilde{\varphi}(x, \cdots, x) \tag{3.13}$$

for all $x \in {}_AB$.

Proof. Put $u = 1 \in U(A)$. Let $x_{4i-3} = x$ and $x_{4i-2} = x_{4i-1} = x_{4i} = y$ in (3.12) for all $i = 1, \cdots, p(4p)^{n-1}$. Then we get

$$\begin{aligned} &\|(4p)^n f(\frac{x+3y}{4}) + (4p)^n f(\frac{x+3y}{4}) \\ &\qquad -4p\ (4p)^{n-1} f(\frac{x+y}{2}) - 4p\ (4p)^{n-1} f(y)\| \\ &\le \varphi(x, y, y, y, \cdots, x, y, y, y) \end{aligned} \tag{3.14}$$

for all $x, y \in {}_AB$. Replacing x and y by $-x$ and x in (3.14), respectively, we get

$$\|2\ (4p)^n f(\frac{x}{2}) - (4p)^n f(x)\| \le \varphi(-x, x, x, x, \cdots, -x, x, x, x)$$

for all $x \in {}_AB$. So one can obtain that

$$\|f(x) - 2f(\frac{x}{2})\| \le \frac{1}{(4p)^n}\varphi(-x, x, x, x, \cdots, -x, x, x, x)$$

for all $x \in {}_AB$. We prove by induction on j that

$$\begin{aligned} \|2^j f(\frac{1}{2^j}x) \quad &- \quad 2^{j+1} f(\frac{1}{2^{j+1}}x)\| \\ &\le \quad \frac{2^j}{(4p)^n}\varphi(-\frac{1}{2^j}x, \frac{1}{2^j}x, \frac{1}{2^j}x, \frac{1}{2^j}x, \cdots, -\frac{1}{2^j}x, \frac{1}{2^j}x, \frac{1}{2^j}x, \frac{1}{2^j}x) \end{aligned}$$

for all $x \in {}_AB$. So we get

$$\begin{aligned} \|f(x) - 2^j f(\frac{1}{2^j}x)\| \quad &\le \quad \frac{1}{(4p)^n}\sum_{m=0}^{j-1} 2^m \varphi(-\frac{1}{2^m}x, \frac{1}{2^m}x, \frac{1}{2^m}x, \frac{1}{2^m}x, \cdots, \\ &- \quad \frac{1}{2^m}x, \frac{1}{2^m}x, \frac{1}{2^m}x, \frac{1}{2^m}x) \end{aligned} \tag{3.15}$$

for all $x \in {}_AB$.

Let x be an element in ${}_AB$. For positive integers l and m with $l > m$,

$$\begin{aligned} &\|2^l f(\frac{1}{2^l}x) - 2^m f(\frac{1}{2^m}x)\| \\ &\le \frac{1}{(4p)^n}\sum_{j=m}^{l-1} 2^j \varphi(-\frac{1}{2^j}x, \frac{1}{2^j}x, \frac{1}{2^j}x, \frac{1}{2^j}x, \cdots, -\frac{1}{2^j}x, \frac{1}{2^j}x, \frac{1}{2^j}x, \frac{1}{2^j}x), \end{aligned}$$

which tends to zero as $m \to \infty$ by (3.11). So $\{2^j f(\frac{1}{2^j}x)\}$ is a Cauchy sequence for all $x \in {}_AB$. Since ${}_AC$ is complete, the sequence $\{2^j f(\frac{1}{2^j}x)\}$

converges for all $x \in {}_AB$. We can define a mapping $T : {}_AB \to {}_AC$ by

$$T(x) = \lim_{j\to\infty} 2^j f(\frac{1}{2^j}x) \tag{3.16}$$

for all $x \in {}_AB$.

By (3.11) and (3.16), we get

$$\begin{aligned}\|D_1T(x_1, \cdots, x_{(4p)^n})\| &= \lim_{j\to\infty} 2^j \|D_1 f(\frac{1}{2^j}x_1, \cdots, \frac{1}{2^j}x_{(4p)^n})\| \\ &\le \lim_{j\to\infty} 2^j \varphi(\frac{1}{2^j}x_1, \cdots, \frac{1}{2^j}x_{(4p)^n}) = 0\end{aligned}$$

for all $x_1, \cdots, x_{(4p)^n} \in {}_AB$. Hence $D_1T(x_1, \cdots, x_{(4p)^n}) = 0$ for all $x_1, \cdots, x_{(4p)^n} \in {}_AB$. By Theorem 2.1, T is additive. Moreover, by passing to the limit in (3.15) as $j \to \infty$, we get the inequality (3.13).

The rest of the proof is the same as the proof of Theorem 3.1. □

THEOREM 3.5. *Let $f : {}_AB \to {}_AC$ be a mapping with $f(0) = 0$ for which there exists a function $\varphi : {}_AB^{(4p)^n} \to [0, \infty)$ such that*

$$\begin{aligned}&\widetilde{\varphi}(x_1, \cdots, x_{(4p)^n}) : \\ &\quad = \sum_{j=0}^{\infty} (\frac{d}{(4p)^n r})^j \varphi((\frac{(4p)^n r}{d})^j x_1, \cdots, (\frac{(4p)^n r}{d})^j x_{(4p)^n}) < \infty,\end{aligned}$$

$$\begin{aligned}&\|\frac{d}{r}\, uf(\frac{rx_1 + \cdots + rx_{(4p)^n}}{d}) + 8p \sum_{i=1}^{(4p)^{n-1}} uf(\frac{x_{4pi-4p+1} + \cdots + x_{4pi}}{4p}) \\ &\qquad -3 \sum_{i=1}^{(4p)^n} f(\frac{ux_i + ux_{i+1} + ux_{i+2}}{3})\| \le \varphi(x_1, \cdots, x_{(4p)^n})\end{aligned}$$

for all $u \in U(A)$ and all $x_1 = x_{(4p)^n+1}, x_2 = x_{(4p)^n+2}, x_3, \cdots, x_{(4p)^n} \in {}_AB$. Then there exists a unique A-linear mapping $T : {}_AB \to {}_AC$ such that

$$\|f(x) - T(x)\| \le \frac{1}{(4p)^n}\widetilde{\varphi}(x, \cdots, x)$$

for all $x \in {}_AB$.

Proof. The proof is similar to the proof of Theorem 3.1. □

THEOREM 3.6. *Let $f : {}_AB \to {}_AC$ be a mapping with $f(0) = 0$ for which there exists a function $\varphi : {}_AB^{(4p)^n} \to [0, \infty)$ such that*

$$\widetilde{\varphi}(x_1, \cdots, x_{(4p)^n}) := \sum_{j=0}^{\infty} (\frac{d}{3r})^j \varphi((\frac{3r}{d})^j x_1, \cdots, (\frac{3r}{d})^j x_{(4p)^n}) < \infty,$$

$$\|(4p)^n uf(\frac{x_1 + \cdots + x_{(4p)^n}}{(4p)^n}) + 8p \sum_{i=1}^{(4p)^{n-1}} uf(\frac{x_{4pi-4p+1} + \cdots + x_{4pi}}{4p})$$

$$-\frac{d}{r} \sum_{i=1}^{(4p)^n} f(\frac{rux_i + rux_{i+1} + rux_{i+2}}{d})\| \le \varphi(x_1, \cdots, x_{(4p)^n})$$

for all $u \in U(A)$ and all $x_1 = x_{(4p)^n+1}, x_2 = x_{(4p)^n+2}, x_3, \cdots, x_{(4p)^n} \in {}_AB$. Then there exists a unique A-linear mapping $T : {}_AB \to {}_AC$ such that

$$\|f(x) - T(x)\| \le \frac{1}{3\ (4p)^n} \widetilde{\varphi}(x, \cdots, x)$$

for all $x \in {}_AB$.

Proof. The proof is similar to the proofs of Theorems 3.1 and 3.3. □

THEOREM 3.7. *Let $f : {}_AB \to {}_AC$ be a mapping with $f(0) = 0$ for which there exists a function $\varphi : {}_AB^{(4p)^n} \to [0, \infty)$ such that*

$$(3.17) \quad \widetilde{\varphi}(x_1, \cdots, x_{(4p)^n}) : \sum_{j=0}^{\infty} 4^j \varphi(-\frac{2}{4^j}x_1, \frac{1}{4^j}x_2, \frac{1}{4^j}x_3, \cdots, -\frac{2}{4^j}x_{(4p)^n-4p+1}, \frac{1}{4^j}x_{(4p)^n-4p+2}, \cdots, \frac{1}{4^j}x_{(4p)^n}) < \infty,$$

$$\begin{aligned} \|D_u f(x_1, \cdots, x_{(4p)^n})\| &= \|(4p)^n uf(\frac{x_1 + \cdots + x_{(4p)^n}}{(4p)^n}) \\ &+ 8p \sum_{i=1}^{(4p)^{n-1}} uf(\frac{x_{4pi-4p+1} + \cdots + x_{4pi}}{4p}) \qquad (3.18) \\ &- 3 \sum_{i=1}^{(4p)^n} f(\frac{ux_i + ux_{i+1} + ux_{i+2}}{3})\| \\ &\le \varphi(x_1, \cdots, x_{(4p)^n}) \end{aligned}$$

for all $u \in U(A)$ *and all* $x_1 = x_{(4p)^n+1}, x_2 = x_{(4p)^n+2}, x_3, \cdots, x_{(4p)^n} \in {}_AB$. *Then there exists a unique* A*-linear mapping* $T : {}_AB \to {}_AC$ *such that*

$$\|f(x) - T(x)\| \leq \frac{1}{3p \cdot (4p)^{n-1}} \widetilde{\varphi}(x, \cdots, x) \tag{3.19}$$

for all $x \in {}_AB$.

Proof. Put $u = 1 \in U(A)$. Let $x_{4i-3} = x$ and $x_{4i-2} = x_{4i-1} = x_{4i} = y$ in (3.18) for all $i = 1, \cdots, p(4p)^{n-1}$. Then we get

$$\begin{aligned} \|(4p)^n f(\frac{x+3y}{4}) &+ 2\ (4p)^n f(\frac{x+3y}{4}) \\ &- 9p\ (4p)^{n-1} f(\frac{x+2y}{3}) - 3p\ (4p)^{n-1} f(y)\| \\ &\leq \varphi(x, y, y, y, \cdots, x, y, y, y) \end{aligned} \tag{3.20}$$

for all $x, y \in {}_AB$. Replacing x and y by $-2x$ and x in (3.20), respectively, we get

$$\|3\ (4p)^n f(\frac{x}{4}) - 3p\ (4p)^{n-1} f(x)\| \leq \varphi(-2x, x, x, x, \cdots, -2x, x, x, x)$$

for all $x \in {}_AB$. So one can obtain that

$$\|f(x) - 4f(\frac{x}{4})\| \leq \frac{1}{3p\ (4p)^{n-1}} \varphi(-2x, x, x, x, \cdots, -2x, x, x, x)$$

for all $x \in {}_AB$. We prove by induction on j that

$$\begin{aligned} &\|4^j f(\frac{1}{4^j}x) - 4^{j+1} f((\frac{1}{4^{j+1}}x)\| \\ &\quad \leq \frac{4^j}{3p\ (4p)^{n-1}} \varphi(-\frac{2}{4^j}x, \frac{1}{4^j}x, \frac{1}{4^j}x, \frac{1}{4^j}x, \cdots, -\frac{2}{4^j}x, \frac{1}{4^j}x, \frac{1}{4^j}x, \frac{1}{4^j}x) \end{aligned}$$

for all $x \in {}_AB$. So we get

$$\begin{aligned} \|f(x) - 4^j f(\frac{1}{4^j}x)\| \leq \frac{1}{3p\ (4p)^{n-1}} \sum_{m=0}^{j-1} 4^m \varphi(-\frac{2}{4^m}x, \frac{1}{4^m}x, \frac{1}{4^m}x, \frac{1}{4^m}x, \cdots, \\ -\frac{2}{4^m}x, \frac{1}{4^m}x, \frac{1}{4^m}x, \frac{1}{4^m}x) \end{aligned} \tag{3.21}$$

for all $x \in {}_AB$.

Let x be an element in ${}_AB$. For positive integers l and m with $l > m$,

$$\|4^l f(\frac{1}{4^l}x) - 4^m f(\frac{1}{4^m}x)\|$$
$$\leq \frac{1}{3p\ (4p)^{n-1}} \sum_{j=m}^{l-1} 4^j \varphi(-\frac{2}{4^j}x, \frac{1}{4^j}x, \frac{1}{4^j}x, \frac{1}{4^j}x, \cdots, -\frac{2}{4^j}x, \frac{1}{4^j}x, \frac{1}{4^j}x, \frac{1}{4^j}x),$$

which tends to zero as $m \to \infty$ by (3.17). So $\{4^j f(\frac{1}{4^j}x)\}$ is a Cauchy sequence for all $x \in {}_AB$. Since ${}_AC$ is complete, the sequence $\{4^j f(\frac{1}{4^j}x)\}$ converges for all $x \in {}_AB$. We can define a mapping $T : {}_AB \to {}_AC$ by

$$T(x) = \lim_{j\to\infty} 4^j f(\frac{1}{4^j}x) \tag{3.22}$$

for all $x \in {}_AB$.

By (3.17) and (3.22), we get

$$\begin{aligned} \|D_1T(x_1, \cdots, x_{(4p)^n})\| &= \lim_{j\to\infty} 4^j \|D_1 f(\frac{1}{4^j}x_1, \cdots, \frac{1}{4^j}x_{(4p)^n})\| \\ &\leq \lim_{j\to\infty} 4^j \varphi(\frac{1}{4^j}x_1, \cdots, \frac{1}{4^j}x_{(4p)^n}) = 0 \end{aligned}$$

for all $x_1, \cdots, x_{(4p)^n} \in {}_AB$. Hence $D_1T(x_1, \cdots, x_{(4p)^n}) = 0$ for all $x_1, \cdots, x_{(4p)^n} \in {}_AB$. By Theorem 2.2, T is additive. Moreover, by passing to the limit in (3.21) as $j \to \infty$, we get the inequality (3.19).

The rest of the proof is the same as the proof of Theorem 3.1. □

4. Stability of partitioned functional equations in Banach algebras and approximate algebra homomorphisms

In this section, let A and B be Banach algebras with norms $|| \cdot ||$ and $\| \cdot \|$, respectively.

D.G. Bourgin [2] proved the stability of ring homomorphisms between Banach algebras. In [1], R. Badora generalized the Bourgin's result.

We prove the Hyers-Ulam stability of algebra homomorphisms in Banach algebras associated with the functional equation (1.1).

THEOREM 4.1. *Let A and B be complex Banach algebras. Let $f : A \to B$ be a mapping with $f(0) = 0$ for which there exists a function*

$\psi : A \times A \to [0, \infty)$ *such that*

$$\widetilde{\psi}(x,y) := \sum_{j=0}^{\infty} (\frac{d}{(4p)^n r})^j \psi((\frac{(4p)^n r}{d})^j x, y) < \infty, \tag{4.1}$$

$$\|D_\lambda f(x_1, \cdots, x_{(4p)^n})\| \leq \varphi(x_1, \cdots, x_{(4p)^n}), \tag{4.2}$$

$$\|f(x \cdot y) - f(x)f(y)\| \leq \psi(x,y) \tag{4.3}$$

for all $\lambda \in \mathbb{T}^1$ and all $x, y, x_1, \cdots, x_{(4p)^n} \in A$, where φ and $D_\lambda f$ satisfy the conditions given in the statement of Theorem 3.1. Then there exists a unique algebra homomorphism $T : A \to B$ satisfying (3.3).

Proof. Under the assumption (4.2), in Corollary 3.2, we showed that there exists a unique $\mathbb{C}$-linear mapping $T : A \to B$ satisfying (3.3). The $\mathbb{C}$-linear mapping $T : A \to B$ was given by

$$T(x) = \lim_{j\to\infty} (\frac{d}{(4p)^n r})^j f((\frac{(4p)^n r}{d})^j x)$$

for all $x \in A$. Let

$$R(x,y) = f(x \cdot y) - f(x)f(y)$$

for all $x, y \in A$. By (4.1), we get

$$\lim_{j\to\infty} (\frac{d}{(4p)^n r})^j R((\frac{(4p)^n r}{d})^j x, y) = 0$$

for all $x, y \in A$. So

$$\begin{aligned} T(x \cdot y) &= \lim_{j\to\infty} (\frac{d}{(4p)^n r})^j f((\frac{(4p)^n r}{d})^j (x \cdot y)) \\ &= \lim_{j\to\infty} (\frac{d}{(4p)^n r})^j f(((\frac{(4p)^n r}{d})^j x) y) \\ &= \lim_{j\to\infty} (\frac{d}{(4p)^n r})^j (f((\frac{(4p)^n r}{d})^j x) f(y) + R((\frac{(4p)^n r}{d})^j x, y)) \\ &= T(x)f(y) \end{aligned} \tag{4.4}$$

for all $x, y \in A$. Thus

$$\begin{aligned} T(x)f((\frac{(4p)^n r}{d})^j y) &= T(x((\frac{(4p)^n r}{d})^j y)) = T(((\frac{(4p)^n r}{d})^j x) y) \\ &= T((\frac{(4p)^n r}{d})^j x) f(y) = (\frac{(4p)^n r}{d})^j T(x) f(y) \end{aligned}$$

for all $x, y \in A$. Hence

$$T(x)(\frac{d}{(4p)^n r})^j f((\frac{(4p)^n r}{d})^j y) = T(x)f(y) \tag{4.5}$$

for all $x, y \in A$. Taking the limit in (4.5) as $j \to \infty$, we obtain

$$T(x)T(y) = T(x)f(y)$$

for all $x, y \in A$. Therefore,

$$T(x \cdot y) = T(x)T(y)$$

for all $x, y \in A$. So $T : A \to B$ is an algebra homomorphism. □

THEOREM 4.2. *Let A and B be complex Banach $*$-algebras. Let $f : A \to B$ be a mapping with $f(0) = 0$ for which there exists a function $\psi : A \times A \to [0, \infty)$ satisfying* (4.1) *and* (4.3) *such that*

$$\begin{aligned} \|D_\lambda f(x_1, \cdots, x_{(4p)^n})\| &\leq \varphi(x_1, \cdots, x_{(4p)^n}), \\ \|f(x^*) - f(x)^*\| &\leq \varphi(x, \cdots, x) \end{aligned}$$

for all $\lambda \in \mathbb{T}^1$ and all $x, x_1, \cdots, x_{(4p)^n} \in A$, where φ and $D_\lambda f$ satisfy the conditions given in the statement of Theorem 3.1. Then there exists a unique $*$-algebra homomorphism $T : A \to B$ satisfying (3.3).

Proof. By the same reasoning as the proof of Theorem 4.1, there exists a unique $\mathbb{C}$-linear mapping $T : A \to B$ satisfying (3.3).

Now

$$\begin{aligned} &(\frac{d}{(4p)^n r})^j \|f((\frac{(4p)^n r}{d})^j x^*) - f((\frac{(4p)^n r}{d})^j x)^*\| \\ &\qquad \leq (\frac{d}{(4p)^n r})^j \varphi((\frac{(4p)^n r}{d})^j x, \cdots, (\frac{(4p)^n r}{d})^j x) \end{aligned}$$

for all $x \in A$. Thus

$$(\frac{d}{(4p)^n r})^j \|f((\frac{(4p)^n r}{d})^j x^*) - f((\frac{(4p)^n r}{d})^j x)^*\| \to 0$$

as $j \to \infty$ for all $x \in A$. Hence

$$\begin{aligned} T(x^*) &= \lim_{j\to\infty} (\frac{d}{(4p)^n r})^j f((\frac{(4p)^n r}{d})^j x^*) \\ &= \lim_{j\to\infty} (\frac{d}{(4p)^n r})^j f((\frac{(4p)^n r}{d})^j x)^* = T(x)^* \end{aligned}$$

for all $x \in A$.

The rest of the proof is the same as the proof of Theorem 4.1. □

Similarly, for the other cases given in Section 3, one can obtain similar results to the theorems given above.

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