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GENERALIZED QUADRATIC MAPPINGS IN $2d$ VARIABLES

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Abstract. Let X, Y be vector spaces. It is shown that if an even mapping $f : X \to Y$ satisfies $f(0) = 0$, and

$$2({}_{2d-2}C_{d-1} - {}_{2d-2}C_d) f\left(\sum_{j=1}^{2d} x_j\right) + \sum_{\iota(j)=0,1, \sum_{j=1}^{2d} \iota(j)=d} f\left(\sum_{j=1}^{2d} (-1)^{\iota(j)} x_j\right)$$
$$= 2({}_{2d-1}C_d + {}_{2d-2}C_{d-1} - {}_{2d-2}C_d) \sum_{j=1}^{2d} f(x_j)$$

for all $x_1, \cdots, x_{2d} \in X$, then the even mapping $f : X \to Y$ is quadratic.

Furthermore, we prove the Hyers-Ulam stability of the above functional equation in Banach spaces.

1. Introduction and preliminaries

In 1940, S.M. Ulam [14] raised the following question: Under what conditions does there exist an additive mapping near an approximately additive mapping?

Let X and Y be Banach spaces with norms $||\cdot||$ and $\|\cdot\|$, respectively. Hyers [3] showed that if $\epsilon > 0$ and $f : X \to Y$ such that

$$\|f(x + y) - f(x) - f(y)\| \le \epsilon$$

for all $x, y \in X$, then there exists a unique additive mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \le \epsilon$$

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for all $x \in X$.

Consider $f : X \to Y$ to be a mapping such that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$. Assume that there exist constants $\epsilon \geq 0$ and $p \in [0, 1)$ such that

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon(||x||^p + ||y||^p)$$

for all $x, y \in X$. Th.M. Rassias [6] showed that there exists a unique $\mathbb{R}$-linear mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \leq \frac{2\epsilon}{2 - 2^p}||x||^p$$

for all $x \in X$. Găvruta [4] generalized the Rassias' result.

A square norm on an inner product space satisfies the important parallelogram equality

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

is called a *quadratic functional equation*. In particular, every solution of the quadratic functional equation is said to be a *quadratic function*. The Hyers-Ulam stability problem of the quadratic functional equation was proved by Skof [13] for mappings $f : X \to Y$, where X is a normed space and Y is a Banach space. Cholewa [1] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group. In [2], Czerwik proved the Hyers-Ulam stability of the quadratic functional equation. Several functional equations have been investigated in [5]–[12].

In this paper, we solve the functional equation

$$2({}_{2d-2}C_{d-1} - {}_{2d-2}C_d) f\left(\sum_{j=1}^{2d} x_j\right) + \sum_{\iota(j)=0,1,\sum_{j=1}^{2d}\iota(j)=d} f\left(\sum_{j=1}^{2d}(-1)^{\iota(j)}x_j\right)$$

$$= 2({}_{2d-1}C_d + {}_{2d-2}C_{d-1} - {}_{2d-2}C_d)\sum_{j=1}^{2d} f(x_j), \tag{1.1}$$

and prove the Hyers-Ulam stability of the functional equation (1.1) in Banach spaces.

2. Stability of generalized quadratic mappings in $2d$ variables

Throughout this section, assume that X and Y are vector spaces.

LEMMA 2.1. *If an even mapping $f : X \to Y$ satisfies $f(0) = 0$ and* (1.1), *then the mapping $f : X \to Y$ is quadratic.*

Proof. Letting $x_1 = x$, $x_2 = y$ and $x_3 = \cdots = x_{2d} = 0$ in (1.1), we get

$$\begin{aligned} 2({}_{2d-2}C_{d-1} -_{2d-2} C_d) f(x+y) + 2_{2d-2}C_d f(x+y) + 2_{2d-2}C_{d-1} f(x-y) \\ = 2({}_{2d-1}C_d +_{2d-2} C_{d-1} -_{2d-2} C_d)(f(x)+f(y)) \end{aligned}$$

for all $x, y \in X$. So

$${}_{2d-2}C_{d-1}(f(x+y)+f(x-y)) = ({}_{2d-1}C_d +_{2d-2} C_{d-1} -_{2d-2} C_d)(f(x)+f(y))$$

for all $x, y \in X$. Since ${}_{2d-1}C_d +_{2d-2} C_{d-1} -_{2d-2} C_d = 2_{2d-2}C_{d-1}$,

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

for all $x, y \in X$. Thus the even mapping $f : X \to Y$ is quadratic. □

From now on, assume that X is a normed vector space with norm $||\cdot||$ and that Y is a Banach space with norm $\|\cdot\|$.
For a given mapping $f : X \to Y$, we define

$$\begin{aligned} Df(x_1, \cdots, x_{2d}) :&= 2({}_{2d-2}C_{d-1} -_{2d-2} C_d) f\left(\sum_{j=1}^{2d} x_j\right) \\ &+ \sum_{\iota(j)=0,1, \sum_{j=1}^{2d} \iota(j)=d} f\left(\sum_{j=1}^{2d} (-1)^{\iota(j)} x_j\right) \\ &- 2({}_{2d-1}C_d +_{2d-2} C_{d-1} -_{2d-2} C_d) \sum_{j=1}^{2d} f(x_j) \end{aligned}$$

for all $x_1, \cdots, x_{2d} \in X$.

THEOREM 2.2. *Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ for which there exists a function $\varphi : X^{2d} \to [0, \infty)$ such that*

$$\widetilde{\varphi}(x_1, \cdots, x_{2d}) : = \sum_{j=1}^{\infty} 9^j \varphi\left(\frac{x_1}{3^j}, \cdots, \frac{x_{2d}}{3^j}\right) < \infty, \tag{2.1}$$

$$\|Df(x_1, \cdots, x_{2d})\| \le \varphi(x_1, \cdots, x_{2d}) \tag{2.2}$$

for all $x_1, \cdots, x_{2d} \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \leq \frac{1}{18 {}_{2d-3}C_{d-1}} \widetilde{\varphi}(x, x, x, \underbrace{0, \cdots, 0}_{2d-3 \text{ times}}) \tag{2.3}$$

for all $x \in X$.

Proof. Letting $x_1 = x_2 = x_3 = x$ and $x_4 = \cdots = x_{2d} = 0$ in (2.2), we get

$$\begin{aligned} \|2({}_{2d-2}C_{d-1} &- {}_{2d-2}C_d + {}_{2d-3}C_d) f(3x) \\ &- 6({}_{2d-1}C_d + {}_{2d-2}C_{d-1} - {}_{2d-2}C_d - {}_{2d-3}C_{d-1}) f(x)\| \\ &\leq \varphi(x, x, x, \underbrace{0, \cdots, 0}_{2d-3 \text{ times}}) \end{aligned}$$

for all $x \in X$. Since

$$\begin{aligned} {}_{2d-1}C_d + {}_{2d-2}C_{d-1} - {}_{2d-2}C_d - {}_{2d-3}C_{d-1} &= 3({}_{2d-2}C_{d-1} - {}_{2d-2}C_d + {}_{2d-3}C_d) \\ &= 3 {}_{2d-3}C_{d-1}, \end{aligned}$$

$$\begin{aligned} &\|2 {}_{2d-3}C_{d-1} f(3x) - 18 {}_{2d-3}C_{d-1}) f(x)\| \\ &= \|2 {}_{2d-3}C_{d-1}(f(3x) - 9f(x))\| \\ &\leq \varphi(x, x, x, \underbrace{0, \cdots, 0}_{2d-3 \text{ times}}) \end{aligned} \tag{2.4}$$

for all $x \in X$. So

$$\left\|f(x) - 9f\left(\frac{x}{3}\right)\right\| \leq \frac{1}{2 {}_{2d-3}C_{d-1}} \varphi(\frac{x}{3}, \frac{x}{3}, \frac{x}{3}, \underbrace{0, \cdots, 0}_{2d-3 \text{ times}})$$

for all $x \in X$. Hence

$$\begin{aligned} &\left\|9^l f\left(\frac{x}{3^l}\right) - 9^m f\left(\frac{x}{3^m}\right)\right\| \\ &\leq \sum_{j=l}^{m-1} \frac{9^j}{2 {}_{2d-3}C_{d-1}} \varphi(\frac{x}{3^{j+1}}, \frac{x}{3^{j+1}}, \frac{x}{3^{j+1}}, \underbrace{0, \cdots, 0}_{2d-3 \text{ times}}) \end{aligned} \tag{2.5}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.1) and (2.5) that the sequence $\{9^n f(\frac{x}{3^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{9^n f(\frac{x}{3^n})\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{n\to\infty} 9^n f(\frac{x}{3^n})$$

for all $x \in X$.

By (2.1) and (2.2),

$$\begin{aligned}\|DQ(x_1, \cdots, x_{2d})\| &= \lim_{n\to\infty} 9^n \left\| Df\left(\frac{x_1}{3^n}, \cdots, \frac{x_{2d}}{3^n}\right)\right\| \\ &\le \lim_{n\to\infty} 9^n \varphi\left(\frac{x_1}{3^n}, \cdots, \frac{x_{2d}}{3^n}\right) = 0\end{aligned}$$

for all $x_1, \cdots, x_{2d} \in X$. So $DQ(x_1, \cdots, x_{2d}) = 0$. By Lemma 2.1, the mapping $Q : X \to Y$ is quadratic. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.5), we get the inequality (2.3).

Now, let $Q' : X \to Y$ be another quadratic mapping satisfying (2.3). Then we have

$$\begin{aligned}\|Q(x) - Q'(x)\| &= 9^n \left\|Q\left(\frac{x}{3^n}\right) - Q'\left(\frac{x}{3^n}\right)\right\| \\ &\le 9^n \left(\left\|Q\left(\frac{x}{3^n}\right) - f\left(\frac{x}{3^n}\right)\right\| + \left\|Q'\left(\frac{x}{3^n}\right) - f\left(\frac{x}{3^n}\right)\right\|\right) \\ &\le \frac{2\cdot 9^n}{18_{2d-3}C_{d-1}} \widetilde{\varphi}(\frac{x}{3^n}, \frac{x}{3^n}, {}^n\frac{x}{3}, \underbrace{0, \cdots, 0}_{2d-3 \text{ times}}),\end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in X$. So we can conclude that $Q(x) = Q'(x)$ for all $x \in X$. This proves the uniqueness of Q. □

Corollary 2.3. *Let $p > 2$ and θ be positive real numbers, and let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and*

$$\|Df(x_1, \cdots, x_{2d})\| \le \theta \sum_{j=1}^{2d} ||x_j||^p \tag{2.6}$$

for all $x_1, \cdots, x_{2d} \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \le \frac{3\theta}{2(3^p - 9)_{2d-3}C_{d-1}} ||x||^p$$

for all $x \in X$.

Proof. Defining $\varphi(x_1, \cdots, x_{2d}) = \theta \sum_{j=1}^{2d} ||x_j||^p$ in Theorem 2.2, we get the desired result, as desired. □

THEOREM 2.4. *Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ for which there exists a function $\varphi : X^{2d} \to [0,\infty)$ satisfying (2.2) and*

$$\widetilde{\varphi}(x_1, \cdots, x_{2d}) := \sum_{j=0}^{\infty} \frac{1}{9^j}\varphi(3^j x_1, \cdots, 3^j x_{2d}) < \infty \tag{2.7}$$

for all $x_1, \cdots, x_{2d} \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \le \frac{1}{18 {}_{2d-3}C_{d-1}} \widetilde{\varphi}(x, x, x, \underbrace{0, \cdots, 0}_{2d-3 \text{ times}}) \tag{2.8}$$

for all $x \in X$.

Proof. It follows from (2.4) that

$$\left\| f(x) - \frac{1}{9} f(3x) \right\| \le \frac{1}{18 {}_{2d-3}C_{d-1}} \varphi(x, x, x, \underbrace{0, \cdots, 0}_{2d-3 \text{ times}})$$

for all $x \in X$. Hence

$$\begin{aligned} &\left\| \frac{1}{9^l} f(3^l x) - \frac{1}{9^m} f(3^m x) \right\| \\ &\qquad \le \sum_{j=l}^{m-1} \frac{1}{9^j \cdot 18 {}_{2d-3}C_{d-1}} \varphi(3^j x, 3^j x, 3^j x, \underbrace{0, \cdots, 0}_{2d-3 \text{ times}}) \end{aligned} \tag{2.9}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.7) and (2.9) that the sequence $\{\frac{1}{9^n} f(3^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{9^n} f(3^n x)\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{n\to\infty} \frac{1}{9^n} f(3^n x)$$

for all $x \in X$.

By (2.2) and (2.7),

$$\begin{aligned} \|DQ(x_1, \cdots, x_{2d})\| &= \lim_{n\to\infty} \frac{1}{9^n} \|Df(3^n x_1, \cdots, 3^n x_{2d})\| \\ &\le \lim_{n\to\infty} \frac{1}{9^n} \varphi(3^n x_1, \cdots, 3^n x_{2d}) = 0 \end{aligned}$$

for all $x_1, \cdots, x_{2d} \in X$. So $DQ(x_1, \cdots, x_{2d}) = 0$. By Lemma 2.1, the mapping $Q : X \to Y$ is quadratic. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.9), we get the inequality (2.8).

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 2.5. *Let $p < 2$ and θ be positive real numbers, and let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.6). Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\| \leq \frac{3\theta}{2(9 - 3^p)({}_{2d-3}C_{d-1})}||x||^p$$

for all $x \in X$.

Proof. Defining $\varphi(x_1, \cdots, x_{2d}) = \theta \sum_{j=1}^{2d} ||x_j||^p$ in Theorem 2.4, we get the desired result, as desired. □

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