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ON A CERTAIN CLASS OF p-VALENT UNIFORMLY CONVEX FUNCTIONS USING DIFFERENTIAL OPERATOR

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Abstract. In this paper, using differential operator, we have introduce new class of p-valent uniformly convex functions in the unit disc $U = \{z : |z| < 1\}$ and obtain the coefficient bounds, extreme bounds and radius of starlikeness for the functions belonging to this generalized class. Furthermore, partial sums $f_k(z)$ of functions $f(z)$ in the class $S^*(\lambda, \alpha, \beta)$ are considered. The various results obtained in this paper are sharp.

1. Introduction

Let S denote the class of functions of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n, \quad a_n \geq 0, \quad (p \in N) \tag{1.1}$$

which are analytic and p-valent in the unit disc $U = \{z : |z| < 1\}$. Also S^* denote the subclass of S consisting of functions of the form

$$f(z) = z^p - \sum_{n=p+1}^{\infty} a_n z^n, \quad a_n \geq 0, \quad (p \in N). \tag{1.2}$$

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G. Murugusundaramoorthy([1]), Goodman([3], [4]) and Ronning([5], [6]) have studied the following subclasses

i) A function $f(z) \in S$ is said to be in the class $S_p(\alpha, \beta)$ of uniformly β-starlike function if it satisfies the condition

$$Re\left\{1 + \frac{zf''(z)}{f'(z)} - \alpha\right\} > \beta\left|\frac{zf'(z)}{f(z)} - 1\right| \quad (z \in U), \tag{1.3}$$

where $-1 < \alpha \leq 1, \beta \geq 0$ and $p \in N$.

ii) A function $f(z) \in S$ is said to be in the class $UCV(\alpha, \beta)$ of uniformly β-convex function if it satisfies the condition

$$Re\left\{1 + \frac{zf''(z)}{f'(z)} - \alpha\right\} > \beta\left|\frac{zf''(z)}{f'(z)}\right| \quad (z \in U), \tag{1.4}$$

where $-1 < \alpha \leq 1, \beta \geq 0$ and $p \in N$. It follows from (1.3) and (1.4) that $f(z) \in UCV(\alpha, \beta)$ is equivalent to

$$zf'(z) \in S_p(\alpha, \beta). \tag{1.5}$$

For the function $f(z) \in S$ is given by (1.1) and $g(z) \in S$ is given by, $g(z) = z^p + \sum_{n=p+1}^{\infty} b_n z^n$. We define the Hadamard product(convolution) of $f(z)$ and $g(z)$ given by,

$$(f * g)(z) = z^p + \sum_{n=p+1}^{\infty} a_n b_n z^n \quad (z \in U, \quad p \in N). \tag{1.6}$$

For the function $f(z) \in S$, we define the following,

$$I^0 f(z) = f(z), \quad I^1 f(z) = zf'(z) + \frac{1+p}{z^p}$$

and for $k = 2, 3, \cdots$

$$I^k f(z) = z(I^{k-1} f(z))' + \frac{1+p}{z^p} = z^p + \sum_{n=p+1}^{\infty} n(k) a_n z^n \quad (p \in N), \tag{1.7}$$

where I^k is called as differential operator. Ghanim and Darus([2]) have studied this operator extensively.

Let $S^*(\alpha,\beta)$ be the subclass of S consisting of the functions of the form (1.1) and satisfying the condition

$$\left|\frac{\frac{z(I^kf(z))'}{I^kf(z)}-p}{\frac{\beta z(I^kf(z))'}{I^kf(z)}-\alpha p}\right|<\mu \quad (z\in U,\quad p\in N), \tag{1.8}$$

where $-1\leq\alpha<\beta\leq 1$ and $0<\mu\leq 1$. Also let $S^{**}(\alpha,\beta)=S^*(\alpha,\beta)\cap S^*$.

The main object of this paper is to study the coefficient estimates, extreme points and radius of starlikeness for the functions belonging to the generalized class $S^{**}(\alpha,\beta)$. Furthermore, partial sums $f_k(z)$ of functions $f(z)$ in the class $S^*(\alpha,\beta)$ are considered and sharp lower bounds for the ratios of real part of $f(z)$ to $f_k(z)$ and $f'(z)$ to $f'_k(z)$ are determined.

In this paper, all the investigated results are motivated by Ronning([5], [6]), K. G. Subramanian ([10], [11]).

2. Basic properties

In this we obtain a necessary and sufficient condition for functions $f(z)$ in the classes $S^*(\alpha,\beta)$ and $S^{**}(\alpha,\beta)$.

THEOREM 2.1. *A function $f(z)$ of the form (1.1) is in $S^*(\alpha,\beta)$ if*

$$\sum_{n=p+1}^{\infty}[(n-p)+\mu(n\beta-\alpha p)]n(k)\,|a_n|\leq\mu p(\beta-\alpha), \tag{2.1}$$

where $-1\leq\alpha<\beta\leq 1, 0<\mu\leq 1$ and $p\in N$.

proof. Since $f(z)\in S^*(\alpha,\beta)$, it is sufficient to show that

$$\left|\frac{\frac{z(I^kf(z))'}{I^kf(z)}-p}{\frac{\beta z(I^kf(z))'}{I^kf(z)}-\alpha p}\right|<\mu.$$

We have

$$\left|\frac{\frac{z(I^kf(z))'}{I^kf(z)}-p}{\frac{\beta z(I^kf(z))'}{I^kf(z)}-\alpha p}\right| = \left|\frac{\frac{pz^p+\sum_{n=p+1}^{\infty}n(k)a_nnz^n}{z^p+\sum_{n=p+1}^{\infty}n(k)a_nz^n}-p}{\frac{\beta\left\{pz^p+\sum_{n=p+1}^{\infty}n(k)a_nnz^n\right\}}{z^p+\sum_{n=p+1}^{\infty}n(k)a_nz^n}-\alpha p}\right|$$

$$\leq \left|\frac{pz^p+\sum_{n=p+1}^{\infty}n(k)a_nnz^n-pz^p-p\sum_{n=p+1}^{\infty}n(k)a_nz^n}{\beta\left\{pz^p+\sum_{n=p+1}^{\infty}n(k)a_nnz^n\right\}-\alpha p[z^p+\sum_{n=p+1}^{\infty}n(k)a_nz^n]}\right|$$

$$\leq \left|\frac{\sum_{n=p+1}^{\infty}(n-p)n(k)a_nn}{\beta p-\alpha p+\sum_{n=p+1}^{\infty}(n\beta-\alpha p)n(k)a_n}\right| \leq \mu.$$

Allowing value of z tends to 1 minus on the real axis, we get

$$\sum_{n=p+1}^{\infty}[(n-p)+\mu(n\beta-\alpha p)]n(k)\,|a_n| \leq \mu p(\beta-\alpha).$$

□

THEOREM 2.2. *A necessary and sufficient condition for $f(z)$ of the form (1.2) to be in the $S^{**}(\alpha,\beta)$*

$$\sum_{n=p+1}^{\infty}[(n-p)+\mu(n\beta-\alpha p)]n(k)|a_n| \leq \mu p(\beta-\alpha), \tag{2.2}$$

where $-1 \leq \alpha < \beta \leq 1, 0 < \mu \leq 1$ and $p \in N$.

Proof. In view of theorem (2.1), we need only to prove that necessity. If $f(z) \in S^{**}(\alpha,\beta)$ and z is real then

$$\left|\frac{\frac{z(I^kf(z))'}{I^kf(z)}-p}{\frac{\beta z(I^kf(z))'}{I^kf(z)}-\alpha p}\right| < \mu.$$

We have

$$\left|\frac{\frac{z(I^kf(z))'}{I^kf(z)}-p}{\frac{\beta z(I^kf(z))'}{I^kf(z)}-\alpha p}\right| \leq \left|\frac{\frac{pz^p-\sum_{n=p+1}^{\infty}n(k)a_nnz^n}{z^p-\sum_{n=p+1}^{\infty}n(k)a_nz^n}-p}{\frac{\beta\left\{Pz^p-\sum_{n=p+1}^{\infty}n(k)a_nnz^n\right\}}{z^p-\sum_{n=p+1}^{\infty}n(k)a_nz^n}-\alpha p}\right|.$$

The above expression is bounded by μ then we obtain the inequality

$$\sum_{n=p+1}^{\infty} [(n-p)+\mu(n\beta-\alpha p)]n(k)|a_n| \leq \mu p(\beta-\alpha),$$

where $-1 \leq \alpha < \beta \leq 1, 0 < \mu \leq 1$ and $p \in N$. □

In the following theorem we show that the class $S^{**}(\alpha,\beta)$ is closed under convex linear combination.

THEOREM 2.3. *Let $f(z)$ defined by (1.2) and $g(z) = z^p - \sum_{n=p+1}^{\infty} b_n z^n$ be in the class $S^{**}(\alpha,\beta)$. Then the function*

$$h(z) = (1-\xi)f(z) + \xi g(z) = z^p - \sum_{n=p+1}^{\infty} \eta_n z^n$$

is also in the class $S^{**}(\alpha,\beta)$, where $\eta_n = (1-\xi)a_n + \xi b_n, \quad 0 \leq \xi < 1$.

Proof. Since the function $f(z)$ and $g(z)$ belongs to $S^{**}(\alpha,\beta)$, we have

$$\sum_{n=p+1}^{\infty} [(n-p)+\mu(n\beta-\alpha p)]n(k)\,|a_n| \leq \mu p(\beta-\alpha) \tag{2.3}$$

and

$$\sum_{n=p+1}^{\infty} [(n-p)+\mu(n\beta-\alpha p)]n(k)\,|b_n| \leq \mu p(\beta-\alpha). \tag{2.4}$$

Clearly,

$$\begin{aligned} h(z) &= (1-\xi)f(z) + \xi g(z) \\ &= (1-\xi)\left(z^p - \sum_{n=p+1}^{\infty} a_n z^n\right) + \xi\left(z^p - \sum_{n=p+1}^{\infty} b_n z^n\right) \\ &= z^p - \sum_{n=p+1}^{\infty} [(1-\xi)a_n + \xi b_n] z^n \\ &= z^p - \sum_{n=p+1}^{\infty} c_n z^n, \end{aligned}$$

where $c_n = (1-\xi)a_n + \xi b_n$. Using (2.3) and (2.4),

$$\begin{aligned}
&\sum_{n=p+1}^{\infty} [(n-p)+\mu(n\beta-\alpha p)]n(k)c_n \\
&\qquad = \sum_{n=p+1}^{\infty} [(n-p)+\mu(n\beta-\alpha p)]n(k)[(1-\xi)a_n+\xi b_n] \\
&\qquad = (1-\xi)\sum_{n=p+1}^{\infty} [(n-p)+\mu(n\beta-\alpha p)]n(k)a_n \\
&\qquad\qquad + \xi \sum_{n=p+1}^{\infty} [(n-p)+\mu(n\beta-\alpha p)]n(k)b_n \\
&\qquad = (1-\xi)\mu p(\beta-\alpha) + \xi\mu p(\beta-\xi) \\
&\qquad \leq \mu p(\beta-\alpha).
\end{aligned}$$

Thus we have $\sum_{n=p+1}^{\infty} [(n-p)+\mu(n\beta-\alpha p)]n(k)c_n \leq \mu p(\beta-\alpha)$. Hence $h(z) \in S^{**}(\alpha,\beta)$. □

THEOREM 2.4(EXTREME POINTS). *Let $f_1(z) = z^p$ and for $n = 2, 3, 4, \cdots$*

$$f_n(z) = z^p - \frac{\mu p(\beta-\alpha)}{[(n-p)+\mu(n\beta-\alpha p)]n(k)} z^n \tag{2.5}$$

Then $f(z) \in S^{**}(\alpha,\beta)$ if and only if $f(z)$ can be expressed in the form $f(z) = \sum_{n=1}^{\infty} \xi_n f_n(z)$, where $\xi_n \geq 0$ and $\sum_{n=1}^{\infty} \xi_n = 1$.

Proof. Suppose that $f(z) = \sum_{n=1}^{\infty} \xi_n f_n(z)$, then

$$f(z) = z^p - \sum_{n=p+1}^{\infty} \frac{\mu p(\beta-\alpha)}{[(n-p)+\mu(n\beta-\alpha p)]n(k)} \xi_n z^n = z^p - \sum_{n=p+1}^{\infty} c_n z^n$$

where $c_n = \frac{\mu p(\beta-\alpha)}{[(n-p)+\mu(n\beta-\alpha p)]n(k)}\xi_n$. Thus

$$\sum_{n=p+1}^{\infty} [(n-p)+\mu(n\beta-\alpha p)n(k)]c_n$$
$$= \sum_{n=p+1}^{\infty} [(n-p)+\mu(n\beta-\alpha p)] \frac{n(k)\mu p(\beta-\alpha)\xi_n}{[(n-p)+\mu(n\beta-\alpha p)]n(k)}$$
$$\leq \mu p(\beta-\alpha) \sum_{n=p+1}^{\infty} \xi_n \leq \mu p(\beta-\alpha),$$

since $0 \leq \sum_{n=p+1}^{\infty} \xi_n \leq 1$. Hence $f(z) \in S^{**}(\alpha,\beta)$.

Conversly, suppose that $f(z) = z^p - \sum_{n=p+1}^{\infty} a_n z^n \in S^{**}(\alpha,\beta)$. Therefore we have, for $n = 2,3,4,\cdots$

$$a_n \leq \frac{\mu p(\beta-\alpha)}{[(n-p)+\mu(n\beta-\alpha p)]n(k)}.$$

Setting $\xi_n = \frac{[(n-p)+\mu(n\beta-\alpha p)]n(k)}{\mu p(\beta-\alpha)}a_n$ for $n = 2,3,4,\cdots$ and $\xi_1 = 1 - \sum_{n=p+1}^{\infty} \xi_n$, we find that $\xi_n \geq 0$ for $n = 1,2,3,\cdots$

$$\sum_{n=p+1}^{\infty} \xi_n = \sum_{n=p+1}^{\infty} \frac{[(n-p)+\mu(n\beta-\alpha p)]n(k)}{\mu p(\beta-\alpha)} a_n \leq 1,$$

since $f(z) \in S^{**}(\alpha,\beta)$. And so $\xi_1 = 1 - \sum_{n=1}^{\infty} \xi_n \geq 0$. Thus $\xi_n \geq 0$ for

$n = 1, 2, 3, \cdots$ and $\sum_{n=1}^{\infty} \xi_n = 1$. Now

$$\begin{aligned} f(z) &= z^p - \sum_{n=p+1}^{\infty} a_n z^n \\ &= z^p - \sum_{n=p+1}^{\infty} \frac{\mu p(\beta - \alpha)}{[(n-p) + \mu(n\beta - \alpha p)]n(k)} \xi_n z^n \\ &= \sum_{n=1}^{\infty} \xi_n f_n(z). \end{aligned}$$

□

Hence we complete the proof of theorem. The proof of the Theorem 2.4 follows on lines similar to the proof of theorem on extreme points given in Silverman([8]).

THEOREM 2.5(CLOSURE THEOREM). *Let the function $f_j(z)$ defined by*

$$f_j(z) = z^p - \sum_{n=p+1}^{\infty} a_{n,j} z^n, \quad a_{n,j} \geq 0, \quad j = 1, 2, 3 \cdots m$$

be in the class $S^{**}(\alpha, \beta)$. Then the function

$$h(z) = z^p - \frac{1}{m} \sum_{n=p+1}^{\infty} \left(\sum_{j=1}^{m} a_{n,j} \right) z^n$$

is in the class $S^{**}(\alpha, \beta)$, where $\alpha = \min_{1 \leq j \leq m} \{\alpha_j\}, \quad -1 \leq \alpha_j < 1$.

Proof. Since $f_j(z) \in S^{**}(\alpha, \beta)$ for $j = 1, 2, 3 \cdots m$, by Theorem 2.2

we have

$$\sum_{n=p+1}^{\infty}[(n-p)+\mu(n\beta-\alpha p)]n(k)\left(\frac{1}{m}\sum_{j=1}^{m}a_{n,j}\right)$$
$$=\frac{1}{m}\sum_{j=1}^{m}\left(\sum_{n=p+1}^{\infty}[(n-p)+\mu(n\beta-\alpha p)]n(k)a_{n,j}\right)$$
$$\leq\frac{1}{m}\sum_{j=1}^{m}\mu p(\beta-\alpha j)\leq\mu p(\beta-\alpha).$$

Hence $h(z)\in S^{**}(\alpha,\beta)$. □

Next we prove the theorem for the radius of starlikeness and convexity.

THEOREM 2.6.
Let $f(z)\in S^{**}(\alpha,\beta)$. Then
i) $f(z)$ is starlike of order $\delta(0\leq\delta<1)$ in the disc

$$|z|\leq r=\inf_{n\geq p+1}\left\{\frac{[(n-p)+\mu(n\beta-\alpha p)](2-p-\delta)n(k)}{\mu p(\beta-\alpha)(2-n-\delta)}\right\}^{\frac{1}{n-p}},$$

ii) $f(z)$ is convex of order $\delta(0\leq\delta<1)$ in the disc

$$|z|\leq r=\inf_{n\geq p+1}\left\{\frac{p[(n-p)+\mu(n\beta-\alpha p)](2-p-\delta)n(k)}{n\mu p(\beta-\alpha)(2-n-\delta)}\right\}^{\frac{1}{n-p}},$$

where $n=2,3,\cdots,p\in N$. These results are sharp for the function

$$f(z)=z^p-\frac{\mu p(\beta-\alpha)}{[(n-p)+\mu(n\beta-\alpha p)]n(k)}z^n\quad(n=2,3,\cdots).$$

Proof. i) $f(z)$ is starlike of order $\delta(0 \leq \delta < 1)$, we have $Re\left\{\frac{zf'(z)}{f(z)}\right\} > \delta$. That is $\left|\frac{zf'(z)}{f(z)} - 1\right| < 1 - \delta$. Now

$$\begin{aligned}\left|\frac{zf'(z)}{f(z)} - 1\right| &\leq \left|\frac{(p-1)z^p - \sum_{n=p+1}^{\infty}(n-1)a_n z^n}{z^p - \sum_{n=p+1}^{\infty} a_n z^n}\right| \\ &< 1 - \delta \\ &\leq \left|(p-1)z^p - \sum_{n=p+1}^{\infty}(n-1)a_n z^n\right| \\ &< (1-\delta)\left|z^p - \sum_{n=p+1}^{\infty} a_n z^n\right|.\end{aligned}$$

Hence

$$\sum_{n=p+1}^{\infty}\left(\frac{2-n-\delta}{2-p-\delta}\right) a_n |z|^{n-p} < 1. \tag{2.7}$$

We note that $f(z) \in S^{**}(\alpha, \beta)$ if and only if

$$\sum_{n=p+1}^{\infty} \frac{[(n-p) + \mu(n\beta - \alpha p)]n(k)a_n}{\mu p(\beta - \alpha)} < 1 \tag{2.8}$$

Using (2.7) and (2.8) we get

$$\left(\frac{2-n-\delta}{2-p-\delta}\right)|z|^{n-p} < \frac{[(n-p) + \mu(n\beta - \alpha p)]n(k)}{\mu p(\beta - \alpha)}.$$

Thus

$$|z| \leq r = \inf_{n \geq p+1}\left\{\frac{[(n-p) + \mu(n\beta - \alpha p)](2-p-\delta)n(k)}{\mu p(\beta - \alpha)(2-n-\delta)}\right\}^{\frac{1}{n-p}}$$

for $n = 2, 3, 4, \cdots, p \in N$, which proves starlikeness of family.

ii) $f(z)$ is convex of order $\delta (0 \leq \delta < 1)$, we have $Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \delta$, that is

$$\left| \frac{zf''(z)}{f'(z)} \right| < 1 - \delta. \tag{2.9}$$

Thus

$$\left| \frac{p(p-1)z^{p-1} - \sum_{n=p+1}^{\infty} a_n n(n-1) z^{n-1}}{pz^{p-1} - \sum_{n=p+1}^{\infty} a_n n z^{n-1}} \right| < 1 - \delta,$$

that is

$$\sum_{n=p+1}^{\infty} \frac{n(2-n-\delta)}{p(2-p-\delta)} a_n |z|^{n-p} < 1. \tag{2.10}$$

We note that $f(z) \in S^{**}(\alpha, \beta)$ if and only if

$$\sum_{n=p+1}^{\infty} \frac{[(n-p) + \mu(n\beta - \alpha p)]n(k)}{\mu p(\beta - \alpha)} a_n < 1. \tag{2.11}$$

Using (2.10) and (2.11), we get

$$\frac{n}{p} \left(\frac{2-n-\delta}{2-p-\delta} \right) |z|^{n-p} < \frac{[(n-p) + \mu(n\beta - \alpha p)]n(k)}{\mu p(\beta, \alpha)}.$$

Thus

$$|z| \leq r = \inf_{n \geq p+1} \left\{ \frac{p[(n-p) + \mu(n\beta - \alpha p)](2-p-\delta)n(k)}{n\mu p(\beta - \alpha)(2-n-\delta)} \right\}^{\frac{1}{n-p}},$$

for $n = 2, 3, 4, \cdots, p \in N$ which proves convex property of the family.□

3. Partial sums

In this we consider partial sums of functions in the class $S^{**}(\alpha, \beta)$ and obtain sharp lower bounds for the ratios of real part of $f(z)$, $f_k(z)$ and $f'(z)$ to $f'_k(z)$. Silverman([8]) and Silvia([9]) have studied the partial sums of analytic functions.

THEOREM 3.1. *Let $f(z) \in S^{**}(\alpha, \beta)$ be given by (1.1) and define the partial sums of $f_1(z)$ to $f_k(z)$ by $f_1(z) = z^p$ and for $k = 2, 3, \cdots$*

$$f_k(z) = z^p + \sum_{n=p+1}^{k} a_n z^n. \tag{3.1}$$

If $\sum_{n=p+1}^{\infty} t_n |a_n| \leq 1$ *and*

$$t_n = \frac{[(n-p) + \mu(n\beta - \alpha p)]}{\mu p(\beta - \alpha)} n(k), \tag{3.2}$$

then $f(z) \in S^{**}(\alpha, \beta)$.

Furthermore,

$$Re\left\{\frac{f(z)}{f_k(z)}\right\} > 1 - \frac{1}{t_{k+1}} \tag{3.3}$$

and

$$Re\left\{\frac{f_k(z)}{f(z)}\right\} > \frac{t_{k+1}}{1 + t_{k+1}}, \tag{3.4}$$

where $z \in U, k \in N$.

Proof. For the coefficient t_n given by (3.2), it is not difficult to verify that,

$$t_{n+1} > t_n > 1. \tag{3.5}$$

Therefore we have

$$\sum_{n=p+1}^{\infty} |a_n| + t_{k+1} \sum_{n=k+1}^{\infty} |a_n| \leq \sum_{n=p+1}^{\infty} t_n |a_n| \leq 1. \tag{3.6}$$

By using (3.2) and by setting

$$\begin{aligned} g_1(z) &= t_{k+1}\left\{\frac{f(z)}{f_k(z)} - \left(1 - \frac{1}{t_{k+1}}\right)\right\} \\ &= 1 + \frac{t_{k+1}\sum_{n=k+1}^{\infty} a_n z^n}{z^p + \sum_{n=p+1}^{k} a_n z^n} \end{aligned} \tag{3.7}$$

and using (3.6), we find that for $z \in U$,

(3.8)

$$\left|\frac{g_1(z)-1}{g_1(z)+1}\right| \leq \frac{t_{k+1}\sum_{n=k+1}^{\infty} a_n z^n}{2z^p + 2\sum_{n=p+1}^{k} a_n z^n + t_{k+1}\sum_{n=k+1}^{\infty} a_n z^n}$$
$$\leq \frac{t_{k+1}\sum_{n=k+1}^{\infty} |a_n|}{2 + 2\sum_{n=p+1}^{k} |a_n| + t_{k+1}\sum_{n=k+1}^{\infty} |a_n|}$$
$$\leq 1$$

which really gives the assertion (3.3) of Theorem 3.1. □

In order to see that $f(z) = z^p + \frac{z^{k+1}}{t^{k+1}}$ gives sharp result, we observe that for $z = re^{i\pi/k}$, $\frac{f(z)}{f_k(z)} = 1 + \frac{z^k}{t_{k+1}} \to 1 - \frac{1}{t_{k+1}}$ as z^-. Similarly if we take

(3.10) $$g_2(z) = (1 + t_{k+1})\left\{\frac{f_k(z)}{f(z)} - \frac{t_{k+1}}{1 + t_{k+1}}\right\}$$
$$= 1 + \frac{(1 + t_{k+1})\sum_{n=p+1}^{k} a_n z^n}{z^p + \sum_{n=p+1}^{\infty} a_n z^n}$$

and making use of (3.6), we can deduce that

(3.11)

$$\left|\frac{g_2(z)-1}{g_2(z)+1}\right| \leq \frac{(1 + t_{k+1})\sum_{n=p+1}^{k} a_n z^n}{2z^p + 2\sum_{n=p+1}^{\infty} a_n z^n + (1 + t_{k+1})\sum_{n=p+1}^{k} a_n z^n}$$
$$\leq \frac{(1 + t_{k+1})\sum_{n=p+1}^{k} |a_n|}{2 + 2\sum_{n=p+1}^{\infty} |a_n| + (1 + t_{k+1})\sum_{n=p+1}^{k} |a_n|}$$

which leads us to the assertion (3.4) of Theorem 3.1. The bound in (3.4) is sharp for each $k \in N$ with the extremal function $f(z)$ given.

THEOREM 3.2. *If $f(z)$ of the form (1.1) satisfies the condition (2.1) then,*

(3.12) $$Re\left\{\frac{f'(z)}{f_k'(z)}\right\} \geq 1 - \frac{k+1}{t_{k+1}}.$$

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$$g(z) = t_{k+1}\left\{\frac{f'(z)}{f_k'(z)} - \left(1 - \frac{k+1}{t_{k+1}}\right)\right\}$$
$$= 1 + \frac{\frac{t_{k+1}}{k+1}\sum_{n=k+1}^{\infty} a_n n z^{n-1}}{p z^{p-1} + \sum_{n=p+1}^{k} a_n n z^{n-1}}.$$

Now

$$\left|\frac{g(z)-1}{g(z)+1}\right| \leq \frac{\frac{t_{k+1}}{k+1}\sum_{n=k+1}^{\infty} n|a_n|}{-2p - 2\sum_{n=p+1}^{k} n|a_n| - \frac{t_{k+1}}{k+1}\sum_{n=k+1}^{\infty}|a_n|}. \tag{3.13}$$

If

$$\frac{\frac{t_{k+1}}{k+1}\sum_{n=k+1}^{\infty} n|a_n|}{-2p - 2\sum_{n=p+1}^{k} n|a_n| - \frac{t_{k+1}}{k+1}\sum_{n=k+1}^{\infty} n|a_n|} \leq 1,$$

then

$$\frac{t_{k+1}}{k+1}\sum_{n=k+1}^{\infty} n|a_n| \leq -2p - 2\sum_{n=p+1}^{k} n|a_n| - \frac{t_{k+1}}{k+1}\sum_{n=p+1}^{\infty} n|a_n|,$$

that is

$$-\frac{1}{p}\frac{t_{k+1}}{k+1}\sum_{n=k+1}^{\infty} n|a_n| - \frac{1}{p}\sum_{n=p+1}^{k} n|a_n| \leq 1. \tag{3.14}$$

Since the left hand size of (3.14) is bounded above by $\sum_{n=p+1}^{k} t_n|a_n|$, if

$$-\frac{1}{p}\frac{t_{k+1}}{k+1}\sum_{n=k+1}^{\infty} n|a_n| - \frac{1}{p}\sum_{n=p+1}^{k} n|a_n| \leq \sum_{n=p+1}^{k} t_n|a_n|$$

then

$$\sum_{n=p+1}^{k}\left(t_n + \frac{n}{p}\right)|a_n| + \sum_{n=k+1}^{\infty}\left[t_n + \frac{n t_{k+1}}{p(k+1)}\right]|a_n| \geq 0. \tag{3.15}$$

The result is sharp for the extremal function $f(z)$. □

THEOREM 3.3. *If $f(z)$ of the form (1.1) satisfies the condition (2.1) then,*

$$Re\left\{\frac{f_k'(z)}{f'(z)}\right\} \geq \frac{t_{k+1}}{k+1+t_{k+1}}.$$

Proof. By setting

$$\begin{aligned} g(z) &= (k+1+t_{k+1})\left\{\frac{f_k'(z)}{f'(z)} - \frac{t_{k+1}}{k+1+t_{k+1}}\right\} \\ &= 1 - \frac{\frac{t_{k+1}}{k+1}\sum_{n=k+1}^{\infty} a_n n z^{n-1}}{pz^{p-1} + \sum_{n=p+1}^{\infty} a_n n z^{n-1}}. \end{aligned}$$

Making use of (3.13), we deduce that

$$\begin{aligned} \left|\frac{g(z)-1}{g(z)+1}\right| &\leq \left|\frac{\frac{t_{k+1}}{k+1}\sum_{n=k+1}^{\infty} a_n n z^{n-1}}{2pz^{p-1} + 2\sum_{n=p+1}^{\infty} a_n n z^{n-1} - \frac{t_{k+1}}{k+1}\sum_{n=p+1}^{\infty} a_n n z^{n-1}}\right| \\ &\leq \frac{\frac{t_{k+1}}{k+1}\sum_{n=k+1}^{\infty} n\,|a_n|}{2p + 2\sum_{n=p+1}^{\infty} n\,|a_n| - \frac{t_{k+1}}{k+1}\sum_{n=k+1}^{\infty} n\,|a_n|} \\ &\leq 1. \end{aligned}$$

The result is sharp for the function $f(z)$. □

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