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THE PROPERTIES OF ROUGH APPROXIMATIONS

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Abstract. We investigated the properties of rough approximations induced by two families of preordered sets and closure systems. We study the relations among the lower and upper rough approximations, closure and interior systems, preordered sets.

1. Introduction

Rough set theory was introduced by Pawlak [7] to generalize the classical set theory. Rough approximations are defined by a partition of the universe which is corresponding to the equivalence relation about information. J. Järvinen et.al.[3] define rough approximations on preorder relations that are not necessarily equivalence relations. An information consists of (X, A) where X is a set of objects and A is a set of attributes, a map $a : X \to P(A_a)$ where A_a is the value set of the attribute a. For $B \subset A$,

$$(x, y) \in R \Leftrightarrow (\forall a \in B)(a(x) = a(y)) \quad \text{(Pawlaki's sense)}$$

$$(x, y) \in R \Leftrightarrow (\forall a \in B)(a(x) \subset a(y)) \quad \text{(Järvinen's sense).}$$

It is an important mathematical tool for data analysis and knowledge processing [1-8].

In this paper, we investigated the properties of rough approximations induced by two families of preordered sets and closure systems. We study the relations among the lower and upper rough approximations, closure and interior systems, preordered sets.

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Let X be a set. A relation $e_X \subset X \times X$ is called a preorder if it is reflexive and transitive. We can define a preorder $e_{P(X)} \subset P(X) \times P(X)$ as $(A, B) \in e_{P(X)}$ iff $A \subset B$ for $A, B \in P(X)$. If (X, e_X) is a preordered set and we define a function $(x, y) \in e_X^{-1}$ iff $(y, x) \in e_X$, then (X, e_X^{-1}) is a preordered set.

2. Preliminaries

Definition 2.1 [6]. (1) A family $\mathcal{F} = \{A \in P(X)\}$ is called an *interior system* on X if $\bigcup_{i\in\Gamma} A_i \in \mathcal{F}$ for $\{A_i \mid i \in \Gamma\} \subset \mathcal{F}$. Let $\mathcal{F}_X$ and $\mathcal{F}_Y$ be interior systems on X and Y, respectively. A function $f : (X, \mathcal{F}_X) \to (Y, \mathcal{F}_Y)$ is called an *I-map* if $f^{-1}(A) \in \mathcal{F}_X$ for each $A \in \mathcal{F}_Y$.

(2) A family $\mathcal{G} = \{A \in P(X)\}$ is called a *closure system* on X if $\bigcap_{i\in\Gamma} A_i \in \mathcal{G}$ for $\{A_i \mid i \in \Gamma\} \subset \mathcal{G}$. Let $\mathcal{G}_X$ and $\mathcal{G}_Y$ be closure systems on X and Y, respectively. A function $f : (X, \mathcal{G}_X) \to (Y, \mathcal{G}_Y)$ is called a *C-map* if $f^{-1}(B) \in \mathcal{G}_X$ for each $B \in \mathcal{G}_Y$. Let $\mathcal{F}_1$ (resp. $\mathcal{G}_1$) and $\mathcal{F}_2$ (resp. $\mathcal{G}_2$) be interior (resp. closure) systems on X. $\mathcal{F}_1$ (resp. $\mathcal{G}_1$) is *coarser* than $\mathcal{F}_2$ (resp. $\mathcal{G}_2$) if $\mathcal{F}_1 \subset \mathcal{F}_2$ (resp. $\mathcal{G}_1 \subset \mathcal{G}_2$).

Definition 2.2 [4,6]. Let (X, e_X) be a preodered set. A set $A \in P(X)$ is called an *e_X-upper set* if $(x \in A \ \&\ (x, y) \in e_X) \to y \in A$ for $x, y \in X$.

Theorem 2.3 [4-6]. *Let (X, e_X) be a preordered set. For $A \in P(X)$, we define operations $[e_X], \langle e_X\rangle$ as follows:*

$$[e_X](A) = \{x \in X \mid (\forall z \in X)((x, z) \in e_X \to z \in A)\},$$

$$\langle e_X\rangle(A) = \{x \in X \mid (\exists z \in X)((x, z) \in e_X \ \&\ z \in A)\}.$$

Then the following properties hold.

(1)*If $(e_X)_x = \{z \in X \mid (x, z) \in e_X\}$ and $(e_X)^{-1}_x = \{z \in X \mid (z, x) \in e_X\}$, then $(e_X)_x$ and $((e_X)^{-1}_x)^c$ are e_X-upper sets.*

(2) *A is an e_X-upper set iff $[e_X](A) = A$ iff $[e_X^{-1}](A^c) = A^c$ iff $\langle e_X^{-1}\rangle(A) = A$.*

(3) *If A_i is an e_X-upper set for all $i \in \Gamma$, then $\bigcup_{i\in\Gamma} A_i$ and $\bigcap_{i\in\Gamma} A_i$ are e_X-upper sets.*

(4) *$[e_X](A) = \bigcup_i\{A_i \mid A_i \subset A, A_i : e_X - \text{upper set}\}$.*

(5) *$\langle e_X\rangle(A) = \bigcap_i\{A_i \mid A \subset A_i, A_i : e_X^{-1} - \text{upper set}\}$.*

Definition 2.4 [3,6]. In above theorem, $[e_X](A)$ and $\langle e_X\rangle(A)$ are called *rough lower approximation* and *rough upper approximation*, respectively, for $A \in P(X)$ on a preodered set.

If e_X is an equivalence relation, $[e_X](A)$ and $\langle e_X\rangle(A)$ are rough lower approximation and rough upper approximation for $A \in P(X)$ in a Pawlak's sense [7]. Let (X, e_X) and (Y, e_Y) be preordered sets. A function $f : (X, e_X) \to (Y, e_Y)$ is called an order preserving map if $(f(x), f(y)) \in e_Y$ for all $(x, y) \in e_X$.

Theorem 2.5 [6]. *Let $\mathcal{F} = \{B_i \in P(X) \mid i \in \Gamma\}$ be an interior system and $\mathcal{F}^* = \{A_i \in P(X) \mid A_i^c \in \mathcal{F}\}$.*

Then; (1) $\mathcal{F}^*$ is a closure system.

(2) There exists a preorder $e_{\mathcal{F}}$ on X such that $\mathcal{F} \subset \mathcal{F}_{[e_{\mathcal{F}}]}$ with

$$(x, y) \in e_{\mathcal{F}} \text{ iff } (\forall B_i \in \mathcal{F})(x \in B_i \to y \in B_i).$$

(3) There exists a preorder $e_{\mathcal{F}^*}$ on X such that $\mathcal{F}^* \subset \mathcal{F}^*_{\langle e_{\mathcal{F}^*}\rangle}$ with

$$(x, y) \in e_{\mathcal{F}^*} \text{ iff } (\forall A_i \in \mathcal{F}^*)(y \in A_i \to x \in A_i).$$

Theorem 2.6 [6]. *Let (X, e_X) be a preordered set. Define $\mathcal{F}_{[e_X]}, \mathcal{G}_{\langle e_X\rangle}$ as follows:*

$$\mathcal{F}_{[e_X]} = \{A \in P(X) \mid [e_X](A) = A\}$$

$$\mathcal{G}_{\langle e_X\rangle} = \{A \in P(X) \mid \langle e_X\rangle(A) = A\}$$

Then; (1) $[e_X](A) \subset A$ and $[e_X](A) = [e_X]([e_X](A))$, for each $A \in P(X)$ with $x \in [e_X](\{y\}^c)$ iff $(x, y) \notin e_X$.

(2) $A \subset \langle e_X\rangle(A)$ and $\langle e_X\rangle(A) = \langle e_X\rangle(\langle e_X\rangle(A))$, for each $A \in P(X)$ with $x \in \langle e_X\rangle(\{y\})$ iff $(x, y) \in e_X$.

(3) $\mathcal{F}_{[e_X]}$ is an interior and closure system with $I_{\mathcal{F}_{[e_X]}} = [e_X]$ where $I_{\mathcal{F}_{[e_X]}}(A) = \bigcup\{B \mid B \subset A,\ B \in \mathcal{F}_{[e_X]}\}$.

(4) $\mathcal{G}_{\langle e_X\rangle}$ is an interior and closure system with $C_{\mathcal{G}_{\langle e_X\rangle}} = \langle e_X\rangle$ where $C_{\mathcal{G}_{\langle e_X\rangle}}(A) = \bigcap\{B \mid A \subset B,\ B \in \mathcal{G}_{\langle e_X\rangle}\}$.

(5) $e_{\mathcal{F}_{[e_X]}} = e_X$ and $e_{\mathcal{G}_{\langle e_X\rangle}} = e_X$.

3. The properties of rough approximations

THEOREM 3.1. *Let X be a set and $\{(X_i, \mathcal{G}_i)\}_{i\in\Gamma}$ a family of closure systems. Let $f_i : X \to X_i$ be a function. Define*

$$\mathcal{G} = \{\bigcap_{i\in\Gamma} f_i^{-1}(B_i) \mid B_i \in \mathcal{G}_i\}.$$

Then (1) *$\mathcal{G}$ is the coarsest closure system on X for which each f_i is a C-map.*

(2) *$\mathcal{G}^* = \{\bigcup_{i\in\Gamma} f_i^{-1}(A_i) \mid A_i \in \mathcal{G}_i^*\}$. is the coarsest interior system on X for which each f_i is an I-map.*

(3) *A function $f : (Y, \mathcal{G}') \to (X, \mathcal{G})$ is a C-map iff $f_i \circ f : (Y, \mathcal{G}') \to (X_i, \mathcal{G}_i)$ is a C-map, for each $i \in \Gamma$.*

(4) *A function $f : (Y, \mathcal{F}') \to (X, \mathcal{G}^*)$ is an I-map iff $f_i \circ f : (Y, \mathcal{F}) \to (X_i, \mathcal{G}_i^*)$ is an I-map, for each $i \in \Gamma$.*

(5) *$(x, y) \in e_{\mathcal{G}}$ iff $(\forall i \in \Gamma)((f_i(x), f_i(y)) \in e_{\mathcal{G}_{X_i}})$.*

(6) *$(x, y) \in e_{\mathcal{G}^*}$ iff $(\forall i \in \Gamma)((f_i(x), f_i(y)) \in e_{\mathcal{G}^*_{X_i}})$. In particular, $e_{\mathcal{G}^*} = e_{\mathcal{G}}$.*

(7) *$\mathcal{G} \subset \mathcal{G}_{\langle e_{\mathcal{G}}\rangle} = \{\bigcap_{i\in\Gamma} f_i^{-1}(B_i), \bigcup_{i\in\Gamma} f_i^{-1}(B_i) \mid B_i \in \mathcal{G}_i\}$.*

(8) *$\mathcal{G}^* \subset \mathcal{F}_{[e_{\mathcal{G}^*}]} = \{\bigcap_{i\in\Gamma} f_i^{-1}(A_i), \bigcup_{i\in\Gamma} f_i^{-1}(A_i) \mid A_i \in \mathcal{G}_i^*\} = \{A \mid A^c \in \mathcal{G}_{\langle e_{\mathcal{G}}\rangle}\}$.*

Proof. (1) For each $k \in K$ with index set K, $A_k = \bigcap_{i\in\Gamma} f_i^{-1}(B_{ik}) \in \mathcal{G}$, then $\bigcap_{k\in K} A_k = \bigcap_{i\in\Gamma} f_i^{-1}(\bigcap_{k\in K} B_{ik}) \in \mathcal{G}$. Hence $\mathcal{G}$ is a closure system on X. For $B_i \in \mathcal{G}_i$, since $X_j \in \mathcal{G}_j$ because it is the intersection of empty family, then $f_i^{-1}(B_i) = f_i^{-1}(B_i) \cap (\bigcap_{j\in\Gamma-\{i\}} f_j^{-1}(X_j)) \in \mathcal{G}$. Hence f_i is a C-map. Let $f_i : (X, \mathcal{G}') \to (X_i, \mathcal{G}_i)$ be a C-map. For $B = \bigcap_{i\in\Gamma} f_i^{-1}(B_i) \in \mathcal{G}$ with $B_i \in \mathcal{G}_i$, then $f_i^{-1}(B_i) \in \mathcal{G}'$. Since $\mathcal{G}'$ is a closure system on X, $B \in \mathcal{G}'$, that is, $\mathcal{G} \subset \mathcal{G}'$. Hence $\mathcal{G}$ is the coarsest closure system on X for which each f_i is a C-map.

(3) ($\Rightarrow$) It is trivial because the composition of C-maps is a C-map.

($\Leftarrow$) For $B = \bigcap_{i\in\Gamma} f_i^{-1}(B_i) \in \mathcal{G}$ with $B_i \in \mathcal{G}_i$, then $f^{-1}(f_i^{-1}(B_i)) \in \mathcal{G}'$. It implies $f^{-1}(B) = \bigcap_{i\in\Gamma} f^{-1}(f_i^{-1}(B_i)) \in \mathcal{G}'$. Thus, f is a C-map.

(2) and (4) are similarly proved as in (1) and (3), respectively.

(5) Suppose there exist $x, y \in X$ such that $(x, y) \in e_{\mathcal{G}}$, but $\not\vdash (\forall i \in \Gamma)((f_i(x), f_i(y)) \in e_{\mathcal{G}_{X_i}})$;i.e.

$$\not\vdash (\forall i \in \Gamma)(\forall B_i \in \mathcal{G}_i)(f_i(y) \in B_i \to f_i(x) \in B_i).$$

Then there exists $i_0 \in \Gamma$ such that $(x,y) \in e_{\mathcal{G}}$,

$$\nvdash (\forall B \in \mathcal{G}_{i_0})(f_{i_0}(y) \in B \to f_{i_0}(x) \in B).$$

Then there exists $B_{i_0} \in \mathcal{G}_{i_0}$ such that $(x,y) \in e_{\mathcal{G}}$,

$$f_{i_0}(y) \in B_{i_0} \to f_{i_0}(x) \notin B_{i_0}.$$

Put $B = f_{i_0}^{-1}(B_{i_0}) \cap (\bigcap_{i \neq i_0} f_i^{-1}(X_i))$. Then

$$\vdash (x,y) \in e_{\mathcal{G}} \to \Big(y \in B \to x \in B\Big).$$

Since $\vdash (x,y) \in e_{\mathcal{G}}$, by Modus Ponens, $\vdash (f_{i_0}(y) \in B_{i_0} \to f_{i_0}(x) \in B_{i_0})$. It is a contradiction. Hence, if $(x,y) \in e_{\mathcal{G}}$, then $(\forall i \in \Gamma)((f_i(x), f_i(y)) \in e_{\mathcal{G}_{X_i}})$.

Suppose there exist $x, y \in X$ such that

$$(x,y) \notin e_{\mathcal{G}}, \vdash (\forall i \in \Gamma)((f_i(x), f_i(y)) \in e_{\mathcal{G}_{X_i}}).$$

By the definition of $e_{\mathcal{G}}$, there exists $B = \bigcap_{i \in \Gamma} f_i^{-1}(B_i) \in \mathcal{G}$ such that

$$\nvdash \Big(y \in \bigcap_{i \in \Gamma} f_i^{-1}(B_i) \to x \in \bigcap_{i \in \Gamma} f_i^{-1}(B_i)\Big)$$

$$\vdash (\forall i \in \Gamma)((f_i(x), f_i(y)) \in e_{\mathcal{G}_{X_i}}).$$

Since

$$\vdash \bigcap_{i \in \Gamma} \Big(y \in f_i^{-1}(B_i) \to x \in f_i^{-1}(B_i)\Big) \to$$

$$\Big(y \in \bigcap_{i \in \Gamma} f_i^{-1}(B_i) \to x \in \bigcap_{i \in \Gamma} f_i^{-1}(B_i)\Big)$$

and $\vdash (\forall i \in \Gamma)((f_i(x), f_i(y)) \in e_{\mathcal{G}_{X_i}})$, by Modus Ponens,

$$\vdash \Big(y \in \bigcap_{i \in \Gamma} f_i^{-1}(B_i) \to x \in \bigcap_{i \in \Gamma} f_i^{-1}(B_i)\Big)$$

It is a contradiction. Hence, if $(\forall i \in \Gamma)((f_i(x), f_i(y)) \in e_{\mathcal{G}_{X_i}})$, then $(x,y) \in e_{\mathcal{G}}$.

(6) Suppose there exist $x, y \in X$ such that

$$(x,y) \in e_{\mathcal{G}^*},\ \not\vdash (\forall i \in \Gamma)((f_i(x), f_i(y)) \in e_{\mathcal{G}^*_{X_i}}.$$

Then there exists $i_0 \in \Gamma$ such that

$$(x,y) \in e_{\mathcal{G}^*},\ \ (f_{i_0}(x), f_{i_0}(y)) \notin e_{\mathcal{G}^*_{X_{i_0}}}.$$

Then there exists $A_{i_0} \in \mathcal{G}^*_{i_0}$ such that

$$(x,y) \in e_{\mathcal{G}^*},\ \not\vdash (f_{i_0}(x) \in A_{i_0} \to f_{i_0}(y) \in A_{i_0}).$$

Put $A = f_{i_0}^{-1}(A_{i_0}) \cup (\bigcup_{i \neq i_0} f_i^{-1}(\emptyset))$. Then $\vdash ((x,y) \in e_{\mathcal{G}^*} \to (f_{i_0}(x) \in A_{i_0} \to f_{i_0}(y) \in A_{i_0}))$. By Modus Ponens, $\vdash (f_{i_0}(x) \in A_{i_0} \to f_{i_0}(y) \in A_{i_0}))$. It is a contradiction.
Hence, if $(x,y) \in e_{\mathcal{G}^*}$, then $(\forall i \in \Gamma)((f_i(x), f_i(y)) \in e_{\mathcal{G}^*_{X_i}})$.

Suppose there exist $x, y \in X$ such that

$$(x,y) \notin e_{\mathcal{G}^*}, \vdash (\forall i \in \Gamma)((f_i(x), f_i(y)) \in e_{\mathcal{G}^*_{X_i}}).$$

Then there exists $A = \bigcup_{i\in\Gamma} f_i^{-1}(A_i) \in \mathcal{G}^*$ such that

$$\not\vdash \Big(x \in \bigcup_{i\in\Gamma} f_i^{-1}(A_i) \to y \in \bigcup_{i\in\Gamma} f_i^{-1}(A_i)\Big)$$

$$\vdash (\forall i \in \Gamma)((f_i(x), f_i(y)) \in e_{\mathcal{G}^*_{X_i}}).$$

Since

$$\vdash \bigcap_{i\in\Gamma} \Big(x \in f_i^{-1}(A_i) \to y \in f_i^{-1}(A_i)\Big) \to$$

$$\Big(x \in \bigcup_{i\in\Gamma} f_i^{-1}(A_i) \to y \in \bigcup_{i\in\Gamma} f_i^{-1}(A_i)\Big)$$

By Modus Ponens,

$$\vdash \Big(x \in \bigcup_{i\in\Gamma} f_i^{-1}(A_i) \to y \in \bigcup_{i\in\Gamma} f_i^{-1}(A_i)\Big).$$

It is a contradiction. Hence, if $(\forall i \in \Gamma)((f_i(x), f_i(y)) \in e_{\mathcal{G}^*_{X_i}})$, then $(x, y) \in e_{\mathcal{G}^*}$.

We have $(x, y) \in e_{\mathcal{G}^*}$ iff $\vdash (\forall B \in \mathcal{G}^*)(x \in B \to y \in B)$ iff $\vdash (\forall B^* \in \mathcal{G})(y \in B^* \to x \in B^*)$ iff $(x, y) \in e_{\mathcal{G}}$.

(7) Let $B \in \mathcal{G}$. Then $B \subset \{x \in X \mid (\exists y \in X)((x, y) \in e_{\mathcal{G}} \ \& \ y \in B)\}$ because

$$\vdash ((x, x) \in e_{\mathcal{G}} \ \& \ x \in B) \to (\exists y \ \in X)((x, y) \in e_{\mathcal{G}} \ \& \ y \in B).$$

Since $B \in \mathcal{G}$, then

$$\vdash ((x, y) \in e_{\mathcal{G}} \ \& \ y \in B) \to (y \in B \to x \in B) \ \& \ y \in B$$

$$\vdash (y \in B \to x \in B) \ \& \ y \in B \to x \in B$$

By Modus Ponens, $\vdash ((x, y) \in e_{\mathcal{G}} \ \& \ y \in B) \to x \in B$. Thus, $\{x \in X \mid (\exists y \in X)((x, y) \in e_{\mathcal{G}} \ \& \ y \in B)\} \subset B$. Hence $B = \langle e_{\mathcal{G}} \rangle(B)$. Thus, $B \in \mathcal{G}_{\langle e_{\mathcal{G}} \rangle}$. So, $\mathcal{G} \subset \mathcal{G}_{\langle e_{\mathcal{G}} \rangle}$.

For $B = \bigcup_{i \in \Gamma} f_i^{-1}(B_i)$ with $B_i \in \mathcal{G}_i$, $(y \in \bigcup_{i \in \Gamma} f_i^{-1}(B_i)) \ \& \ (x, y) \in e_{\mathcal{G}}$ implies $(\exists i \in \Gamma)(f_i(y) \in B_i \ \& \ (f_i(x), f_i(y)) \in e_{\mathcal{G}_i}$ implies $(\exists i \in \Gamma)(f_i(y) \in B_i \ \& \ (f_i(y) \in B_i \to f_i(x) \in B_i)$ implies $x \in \bigcup_{i \in \Gamma} f_i^{-1}(B_i)$. Thus B is an e_X^{-1}-upper set. By Theorem 2.3(2), $B = \langle e_{\mathcal{G}} \rangle(B)$. Hence $\bigcup_{i \in \Gamma} f_i^{-1}(B_i) \in \mathcal{G}_{\langle e_{\mathcal{G}} \rangle}$.

(8) Let $A \in \mathcal{G}^*$. $\vdash (\forall y \in X)(((x, y) \in e_{\mathcal{G}^*} \to y \in A) \to x \in A)$. Conversely, since $\vdash (\forall y \in X)((x \in A) \ \& \ (x \in A \to y \in A) \to y \in A)$ and $\vdash (\forall x, y \in X)((x, y) \in e_{\mathcal{G}^*} \to (x \in A \to y \in A)$, we have $\vdash (\forall y \in X)(x \in A \ \& \ (x, y) \in e_{\mathcal{G}^*} \to y \in A$ iff $\vdash (x \in A) \to ((x, y) \in e_{\mathcal{G}^*} \to y \in A)$. Hence $[e_{\mathcal{G}^*}](A) = A$. Thus, $A \in \mathcal{F}_{e_{\mathcal{G}^*}}$. So, $\mathcal{G}^* \subset \mathcal{F}_{[e_{\mathcal{G}^*}]}$.

For $A = \bigcap_{i \in \Gamma} f_i^{-1}(A_i)$ with $A_i \in \mathcal{G}_i^*$, $(x \in \bigcup_{i \in \Gamma} f_i^{-1}(A_i)) \ \& \ (x, y) \in e_{\mathcal{G}^*}$ implies $(\exists i \in \Gamma)(f_i(x) \in A_i \ \& \ (f_i(x), f_i(y)) \in e_{\mathcal{G}_i^*}$ implies $(\exists i \in \Gamma)(f_i(x) \in A_i \ \& \ (f_i(x) \in A_i \to f_i(y) \in A_i)$ implies $y \in \bigcup_{i \in \Gamma} f_i^{-1}(A_i)$. Thus A is e_X-upper set. By Theorem 2.3(2), $A = [e_{\mathcal{G}^*}](A)$. Hence $A = \bigcap_{i \in \Gamma} f_i^{-1}(A_i) \in \mathcal{F}_{[e_{\mathcal{G}^*}]}$. □

EXAMPLE 3.2. Let $X = \{a, b, c, d\}$ be a set, $\mathcal{G}_1 = \{\{a, b\}, X\}$ and $\mathcal{G}_2 = \{\{b, c\}, X\}$. Then $\mathcal{G} = \{\{a, b\}, \{b, c\}, \{b\}, X\}$ be the coarsest closure system which is finer than $\mathcal{G}_i$ for $i = 1, 2$. We obtain

$$\begin{aligned} e_{\mathcal{G}_1} &= \{(a,a)(a,b),(a,c),(a,d),(b,a)(b,b),(b,c),(b,d),\\ &\quad (c,c),(c,d),(d,c),(d,d)\},\\ e_{\mathcal{G}_2} &= \{(a,a)(a,d),(b,a)(b,b),(b,c),(b,d),\\ &\quad (c,a),(c,b)(c,c),(c,d),(d,a),(d,d)\},\\ e_{\mathcal{G}} &= e_{\mathcal{G}_1} \cap e_{\mathcal{G}_1} = \{(a,a)(a,d),(b,a)(b,b),(b,c),(b,d),\\ &\quad (c,c),(c,d),(d,d)\},\\ \mathcal{G}_1^* &= \{\emptyset,\{c,d\}\},\ \ \mathcal{G}_2^* = \{\emptyset,\{a,d\}\}. \end{aligned}$$

Then $\mathcal{G}^* = \{\emptyset, \{c,d\}, \{a,d\}, \{d\}\}$ is the coarsest interior system which is finer than $\mathcal{G}_i^*$ for $i = 1, 2$. In particular, $e_{\mathcal{G}_1} = e_{\mathcal{G}_1^*}, e_{\mathcal{G}_2} = e_{\mathcal{G}_2^*}, e_{\mathcal{G}} = e_{\mathcal{G}^*}$ and

$$\mathcal{G}_{\langle e_{\mathcal{G}} \rangle} = \{\emptyset, \{a,b\}, \{b,c\}, \{b\}, \{a,b,c\}, X\}$$

$$\mathcal{F}_{[e_{\mathcal{G}^*}]} = \{\emptyset, \{c,d\}, \{a,d\}, \{a,c,d\}, \{d\}, X\}.$$

THEOREM 3.3. *Let X be a set and $\{(X_i, e_{X_i})\}_{i\in\Gamma}$ a family of preorder sets. Let $f_i : X \to X_i$ be a function for each $i \in \Gamma$. We define the relation $e_X \subset X \times X$ by*

$$(x,y) \in e_X \text{ iff } (\forall i \in \Gamma)((f_i(x), f_i(y)) \in e_{X_i}).$$

Then we have the following statements:

(1) *e_X is the coarsest preorder on X for which each f_i is an order preserving map.*

(2) *A function $f : (Y, e'_X) \to (X, e_X)$ is an order preserving map iff $f_i \circ f : (Y, e'_X) \to (X_i, e_{X_i})$ is an order preserving map, for each $i \in \Gamma$.*

(3) *$e_X = e_{\mathcal{G}_{\langle e_X \rangle}} = e_{\mathcal{G}}$ where $\mathcal{G} = \{\bigcap_{i\in\Gamma} f_i^{-1}(B_i) \mid B_i \in \mathcal{G}_{\langle e_{X_i} \rangle}\}$.*

(4) *$\langle e_X \rangle(A) \subset \bigcap_{i\in\Gamma} f_i^{-1}(\langle e_{X_i} \rangle(f_i(A)))$. If $\Gamma = \{i\}$, the equality holds.*

(5) *$[e_X](A) \supset \bigcup_{i\in\Gamma} f_i^{-1}([e_{X_i}]((f_i(A^c)^c)))$. If $\Gamma = \{i\}$, the equality holds.*

(6) *$\bigcap_{i\in\Gamma} f_i^{-1}(\mathcal{F}_{[e_{X_i}]}) \subset \mathcal{F}_{[e_X]}$. If $\Gamma = \{i\}$, the equality holds.*

(7) *$\bigcup_{i\in\Gamma} f_i^{-1}(\mathcal{G}_{\langle e_{X_i} \rangle}) \subset \mathcal{G}_{\langle e_X \rangle}$. If $\Gamma = \{i\}$, the equality holds.*

Proof. (1) We easily show that e_X is the coarsest preorder on X for which each f_i is an order preserving map.

(2) ($\Rightarrow$) It is trivial because the composition of order preserving maps is an order preserving map.

($\Leftarrow$) Let $f_i \circ f : (Y, e'_Y) \to (X_i, e_{X_i})$ be an order preserving map for each $i \in \Gamma$. For each $x, y \in Y$,

$$\begin{aligned} &\vdash (x,y) \in e'_Y \to (\forall i \in \Gamma)((f_i(f(x)), f_i(f(y))) \in e_{X_i}) \\ \text{iff} \quad &\vdash (x,y) \in e'_Y \to (f(x), f(y)) \in e_X. \end{aligned}$$

Hence $f : (Y, e'_Y) \to (X, e_X)$ is an order preserving map.

(3) First, we show that $(\forall i \in \Gamma, B_i \in \mathcal{G}_{\langle e_{X_i}\rangle})(y \in \bigcap_{i\in\Gamma} f_i^{-1}(B_i) \to x \in \bigcap_{i\in\Gamma} f_i^{-1}(B_i))$ iff $(\forall i \in \Gamma)(B_i \in \mathcal{G}_{\langle e_{X_i}\rangle})(y \in f_i^{-1}(B_i) \to x \in f_i^{-1}(B_i))$. Since $f_i^{-1}(B_i) = f_i^{-1}(B_i) \cap (\bigcap_{j\in\Gamma-\{i\}} f_j^{-1}(X_j))$, $(\forall i \in \Gamma, B_i \in \mathcal{G}_{\langle e_{X_i}\rangle})(y \in \bigcap_{i\in\Gamma} f_i^{-1}(B_i) \to x \in \bigcap_{i\in\Gamma} f_i^{-1}(B_i))$ implies $(\forall i \in \Gamma)(B_i \in \mathcal{G}_{\langle e_{X_i}\rangle})(y \in f_i^{-1}(B_i) \to x \in f_i^{-1}(B_i))$. Conversely, it follows from

$$\begin{aligned} &\vdash \bigcap_{i\in\Gamma} \Big(y \in f_i^{-1}(B_i) \to x \in f_i^{-1}(B_i)\Big) \\ &\to \Big(y \in \bigcap_{i\in\Gamma} f_i^{-1}(B_i) \to x \in \bigcap_{i\in\Gamma} f_i^{-1}(B_i)\Big) \end{aligned}$$

Thus,

$$\begin{aligned} (x,y) \in e_{\mathcal{G}} \ &\text{iff} (\forall B \in \mathcal{G})(y \in B \to x \in B) \\ &\text{iff} (\forall i \in \Gamma, B_i \in \mathcal{G}_{\langle e_{X_i}\rangle})(y \in \bigcap_{i\in\Gamma} f_i^{-1}(B_i) \\ &\quad \to x \in \bigcap_{i\in\Gamma} f_i^{-1}(B_i)) \\ &\text{iff} (\forall i \in \Gamma)(B_i \in \mathcal{G}_{\langle e_{X_i}\rangle})(y \in f_i^{-1}(B_i) \to x \in f_i^{-1}(B_i)) \\ &\text{iff} (\forall i \in \Gamma)((f_i(x), f_i(y)) \in e_{\mathcal{G}_{\langle e_X\rangle}}) \\ &\text{iff} (x,y) \in e_{\mathcal{G}_{\langle e_X\rangle}} = e_X \text{ (by Theorem 2.6(5)).} \end{aligned}$$

(4) Let $y \in f_i(\langle e_X\rangle(A))$ with $x \in \langle e_X\rangle(A)$ and $y = f_i(x)$. Then $(x,z) \in e_X \;\&\; z \in A$ implies $(f_i(x), f_i(z)) \in e_{X_i} \;\&\; f_i(z) \in f_i(A)$. Then

$y = f_i(x) \in \langle e_{X_i}\rangle(f_i(A))$. Then $f_i(\langle e_X\rangle(A)) \subset \langle e_{X_i}\rangle(f_i(A))$. Hence f_i is C-map for each $i \in \Gamma$. Thus, $\langle e_X\rangle(A) \subset \bigcap_{i\in\Gamma} f_i^{-1}(\langle e_{X_i}\rangle(f_i(A)))$.

If $\Gamma = \{i\}$, suppose that there exists $A \in P(X)$ with

$$\langle e_X\rangle(A) \not\supset f_i^{-1}(\langle e_{X_i}\rangle(f_i(A))).$$

Then there exists $x \in X$ such that

$$x \in f_i^{-1}(\langle e_{X_i}\rangle(f_i(A))), \ \ x \notin \langle e_X\rangle(A).$$

Since $x \in f_i^{-1}(\langle e_{X_i}\rangle(f_i(A)))$, then $f_i(x) \in \langle e_{X_i}\rangle(f_i(A))$ iff $(\exists z \in X)((f_i(x), f_i(z)) \in e_{X_i}$ & $z \in A)$ iff $(x, z) \in e_X$ & $z \in A$. Then $x \in \langle e_X\rangle(A)$. It is a contradiction. Hence $\langle e_X\rangle(A) \supset f_i^{-1}(\langle e_{X_i}\rangle(f_i(A)))$.

(5) Let $y \in f_i^{-1}([e_{X_i}](B))$. Then $f_i(y) \in [e_{X_i}](B)$. Then $(\forall z \in X_i)((f_i(x), z) \in e_{X_i} \to z \in B)$ implies $(\forall y \in X)((f_i(x), f_i(y)) \in e_{X_i} \to f_i(y) \in B)$ implies $(\forall y \in X)((x, y) \in e_X \to y \in f_i^{-1}(B))$. Hence $f_i^{-1}([e_{X_i}](B)) \subset [e_X](f_i^{-1}(B))$. So,

$$[e_X](A) \supset \bigcup_{i\in\Gamma} f_i^{-1}([e_{X_i}]((f_i(A^c)^c))).$$

If $\Gamma = \{i\}$, suppose that there exists $A \in P(X)$ with

$$[e_X](A) \not\subset f_i^{-1}([e_{X_i}]((f_i(A^c)^c))).$$

Then there exists $x \in X$ such that

$$x \notin f_i^{-1}([e_{X_i}]((f_i(A^c)^c))), \ \ x \in [e_X](A).$$

Since $x \notin f_i^{-1}([e_{X_i}]((f_i(A^c)^c)))$, $f_i(x) \in ([e_{X_i}]((f_i(A^c)^c)))^c$ iff $f_i(x) \in (\langle e_{X_i}\rangle(f_i(A^c)))$ iff $(\exists y \in X_i)((f_i(x), y) \in e_{X_i}$ & $y \in f_i(A^c))$ iff $(\exists z \in X)((x, z) \in e_X$ & $z \in A^c)$, then $x \notin [e_X](A)$. It is a contradiction. Hence $[e_X](A) \subset f_i^{-1}([e_{X_i}]((f_i(A^c)^c)))$.

(6) Let $A = \bigcup f_i^{-1}(A_i) \in \bigcup_{i\in\Gamma} f_i^{-1}(\mathcal{F}_{[e_{X_i}]})$ with e_{X_i}-upper set A_i. Since $x \in \bigcup f_i^{-1}(A_i)$ & $(f_i(x), f_i(y)) \in \bigcap_{i\in\Gamma} e_{X_i}$ implies $(\exists i \in \Gamma)\Big(x \in f_i^{-1}(A_i)$ & $(f_i(x), f_i(y)) \in e_{X_i} \to x \in f_i^{-1}(A_i)$, then $A = \bigcup f_i^{-1}(A_i)$ is e_X-upper set;i.e. $e_X(A) = A$. Hence $A \in \mathcal{F}_{[e_X]}$.

If $\Gamma = \{i\}$, let $A \in \mathcal{F}_{[e_X]}$. By (5), Then

$$A = [e_X](A) = f_i^{-1}([e_{X_i}]((f_i(A^c)^c))).$$

Since $[e_{X_i}]((f_i(A^c)^c)) = [e_{X_i}]([e_{X_i}]((f_i(A^c)^c)))$, $A \in f_i^{-1}(\mathcal{F}_{[e_{X_i}]})$.

(7) Let $B = \bigcap f_i^{-1}(B_i) \in \bigcap_{i\in\Gamma} f_i^{-1}(\mathcal{G}_{\langle e_{X_i}\rangle})$ with $e_{X_i}^{-1}$-upper set B_i. Since $y \in \bigcap f_i^{-1}(B_i)$ & $(f_i(x), f_i(y)) \in \bigcap_{i\in\Gamma} e_{X_i}$ implies $(\forall i \in \Gamma)\Big(x \in f_i^{-1}(B_i)$ & $(f_i(x), f_i(y)) \in e_{X_i}\Big) \to x \in \bigcap f_i^{-1}(B_i)$, then $B \in \mathcal{G}_{\langle e_X\rangle}$.

If $\Gamma = \{i\}$, let $A \in \mathcal{G}_{\langle e_X\rangle}$. By (4), Then

$$A = \langle e_X\rangle(A) = f_i^{-1}(\langle e_{X_i}\rangle(f_i(A))).$$

Since $\langle e_{X_i}\rangle(f_i(A)) = \langle e_{X_i}\rangle(\langle e_{X_i}\rangle(f_i(A)))$, $A \in f_i^{-1}(\mathcal{G}_{\langle e_{X_i}\rangle})$.

□

Example 3.4. Let $X = \{a, b, c, d\}$, $Y = \{x, y, z\}$ and $Z = \{u, v, w\}$ be sets. Let e_Y and e_z be preodered set as follows

$$e_Y = \{(x, x), (y, y), (z, z), (x, z), (z, y)(x, y)\}$$

$$e_Z = \{(u, u), (v, v), (w, w), (u, v), (v, w)(u, w)\}$$

Define $f_1 : X \to Y$ and $f_2 : X \to Z$ as

$$f_1(a) = x,\ f_1(b) = f_1(c) = y,\ f_1(d) = z,$$

$$f_2(a) = f_2(b) = u,\ f_1(c) = v,\ f_2(d) = w.$$

We obtain

$$e_X = \{(a, a), (a, b), (a, c), (a, d), (b, b), (b, c), (c, c),\ (c, d), (d, d)\},$$

$$[e_Y](A) = \begin{cases} \{y\} & \text{if } \{y\} \subset A, \{y, z\} \not\subset A, \\ \{y, z\} & \text{if } A = \{y, z\} \\ Y & \text{if } A = Y, \\ \emptyset & \quad\text{otherwise,} \end{cases}$$

$$[e_Z](B) = \begin{cases} \{w\} & \text{if } \{w\} \subset A, \{v,w\} \not\subset A, \\ \{v,w\} & \text{if } A = \{v,w\} \\ Z & \text{if } A = Y, \\ \emptyset & \quad \text{otherwise,} \end{cases}$$

$$[e_X](A) = \begin{cases} \{d\} & \text{if } \{d\} \subset A, \{c\} \not\subset A, \\ \{c\} & \text{if } \{c\} \subset A, \{d\} \not\subset A, \\ \{b,c\} & \text{if } \{b,c\} \subset A, \{d\} \not\subset A, \\ \{c,d\} & \text{if } \{c,d\} \subset A, \{b\} \not\subset A, \\ \{b,c,d\} & \text{if } A = \{b,c,d\}, \\ X & \text{if } A = X, \\ \emptyset & \quad \text{otherwise,} \end{cases}$$

Then $[e_X](A) \supset \bigcup_{i\in\Gamma} f_i^{-1}([e_{X_i}]((f_i(A^c)^c)))$. In general,

$$\{c\} = [e_X](\{c\}) \neq \bigcup_{i\in\Gamma} f_i^{-1}([e_{X_i}]((f_i(\{c\}^c)^c))) = \emptyset.$$

Moreover,

$$\mathcal{F}_{[e_Y]} = \{\emptyset, \{y\}, \{y,z\}, Y\}, \ \ \mathcal{F}_{[e_Z]} = \{\emptyset, \{w\}, \{v,w\}, Z\},$$
$$\mathcal{F}_{[e_X]} = \{\emptyset, \{c\}, \{d\}, \{b,c\}, \{c,d\}, \{b,c,d\}, X\}.$$

$$\langle e_Y\rangle(A) = \begin{cases} Y & \text{if } \{y\} \subset A, \\ \{x,z\} & \text{if } \{z\} \subset A, \{y\} \not\subset A, \\ \{x\} & \text{if } \{x\} \subset A, \{z\} \not\subset A, \\ \emptyset & \ \text{if } A = \emptyset, \end{cases}$$

$$\langle e_Z\rangle(B) = \begin{cases} Z & \text{if } \{w\} \subset A, \\ \{u,v\} & \text{if } \{v\} \subset A, \{u\} \not\subset A, \\ \{u\} & \text{if } \{u\} \subset A, \{v\} \not\subset A, \\ \emptyset & \ \text{if } A = \emptyset, \end{cases}$$

$$\langle e_X\rangle(A)=\begin{cases}\{a\} & \text{if } \{a\}\subset A, \{b\}\not\subset A, \{d\}\not\subset A\\ \{a,b\} & \text{if } \{b\}\subset A, \{d\}\not\subset A,\\ \{a,d\} & \text{if } \{d\}\subset A, \{b\}\not\subset A,\\ \{a,b,c\} & \text{if } \{c\}\subset A, \{d\}\not\subset A,\\ \{a,b,d\} & \text{if } A=\{a,b,d\},\\ \emptyset & \text{if } A=\emptyset,\\ X & \text{otherwise,}\end{cases}$$

Then $\langle e_X\rangle(A)\subset\bigcap_{i\in\Gamma} f_i^{-1}(\langle e_{X_i}\rangle((f_i(A))))$. In general,

$$\begin{aligned}\{a,b,d\}&=\langle e_X\rangle(\{a,b,d\})\\ &\neq\bigcap_{i\in\Gamma} f_i^{-1}(\langle e_{X_i}\rangle(f_i(\{a,b,d\})))=X.\end{aligned}$$

Moreover,

$$\begin{aligned}&\mathcal{G}_{\langle e_Y\rangle}=\{\emptyset,\{x\},\{x,z\},Y\},\ \mathcal{G}_{\langle e_Y\rangle}=\{\emptyset,\{u\},\{u,v\},Z\},\\ &\mathcal{G}_{\langle e_X\rangle}=\{\emptyset,\{a\},\{a,d\},\{a,b\},\{a,b,d\},\{a,b,c\},X\}.\end{aligned}$$

Using Theorem 3.3, we can define subspaces and products in the obvious way.

Definition 3.4. Let (X,e_X) be a preordered space and A a subset of X. The pair (A,e_A) is said to be a *subspace* of (X,e_X) if e_A is the coarsest preorder on A which the inclusion function $i:A\to X$ is an ordered preserving map.

Definition 3.5. Let $\{(X_i,e_{X_i})\mid i\in\Gamma\}$ be a family of preordered spaces. Let $X=\prod_{i\in\Gamma}X_i$ be a product set. The coarsest preorder $e_X=\otimes e_{X_i}$ on X with respect to $(X,\pi_i,(X_i,e_{X_i}))$ where $\pi_i:X\to X_i$ is projection map is called the *product preorder* of $\{e_{X_i}\mid i\in\Gamma\}$.

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