Korean J. Math. **19** (2011), No. 3, pp. 301–308

ALTERNATE SIGNS (A_k) PROPERTY IN BANACH SPACES

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Abstract. In this paper, we define the alternate forms of property (A_k) and study their implications.

1. Introduction

The symbol X denotes a Banach space with closed unit ball B_X and unit sphere S_X. X is said to be reflexive (Rf) if the natural embedding maps X onto X^{**}. $(X, \|\cdot\|)$ is called uniformly convex (UC) if for all $\epsilon > 0$, there exists a $0 < \delta(\epsilon) < 1$ such that for $x,\ y \in B_X$ with $\|x - y\| \geq \epsilon$,

$$\left\| \frac{1}{2}(x + y) \right\| \leq \delta(\epsilon).$$

A Banach space is said to have the Banach-Saks property (weak Banach-Saks property) if any bounded (weakly convergent) sequence in the space admits a subsequence whose arithmetic means converges in norm. A Banach space X is said to have the alternate signs Banach-Saks property (alternate signs weak Banach-Saks property) if any bounded (weakly convergent) sequence $\{x_n\}$ in X, there exist a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ and an alternate signs sequence $\{\epsilon_{n_i}\}$ with $\epsilon_i \in \{\pm 1\}$ such that $\frac{1}{m}\sum_{i=1}^{m} \epsilon_{n_i} x_{n_i}$ is convergent in norm. T. Nishiura and D. Waterman [6] proved that the Banach-Saks property implies reflexivity in Banach spaces and S. Kakutani [4] showed that Uniform convexity implies the Banach-Saks property.

Received July 26, 2011. Revised August 17, 2011. Accepted August 25, 2011.
2000 Mathematics Subject Classification: 46B20.
Key words and phrases: uniformly convexity, Banach-Saks property, alternate signs (A_k) property.
This work was supported by the Inha University Research Grant.
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The natural questions are the followings : For a Banach space X with the Banach-Saks property, is it uniformly convex? And does every reflexive Banach space have the Banach-Saks property? In 1972, A. Baernstein [1] gave an example of a reflexive Banach space which does not have the Banach-Saks property. In 1978, C. J. Seifert [8] showed that the dual of Baernstein space which is not uniformly convex has the Banach-Saks property. We obtain the following strict implication.

$$(UC) \Rightarrow (BS) \Rightarrow (Rf)$$

If E and F are nonempty subsets of $\mathbb{N}$, we write $E < F$ for $\max E < \min F$. If (e_n) is a basis for X, for $x = \sum_{n=1}^{\infty} a_n e_n$, we define the support x, $\text{supp}(x) = \{n : a_n \neq 0\}$. We write $x < y$ for $\text{supp}(x) < \text{supp}(y)$, where $x, y \in X$.

2. Alternate signs (A_k) property

Let k be a natural number and a Banach space X is said to have property (A_k) if it is reflexive and there exists $0 < \alpha < 1$ such that for a weakly null sequence (x_n) in B_X, then there exist $m_1, m_2, \cdots, m_k$ with $\left\| \frac{1}{k} \sum_{i=1}^{k} x_{m_i} \right\| < \alpha$. We say that X has property (A_∞) if it has (A_k) for some k. J. R. Partington show that the following implications hold and they are strict [7]:

$$(UC) \Rightarrow (A_2) \Rightarrow (A_3) \Rightarrow \cdots \Rightarrow (A_\infty) \Rightarrow (BS)$$

We now define the alternating form of property (A_k).

Definition 2.1. A Banach space X is said to have alternate signs (A_k) property (AS-(A_k)) if it is reflexive and there exists $0 < \alpha < 1$ such that for a weakly null sequence (x_n) in B_X, then there exist $m_1, m_2, \cdots, m_k$ and $\epsilon_i \in \{\pm 1\}$ with $\left\| \frac{1}{k} \sum_{i=1}^{k} \epsilon_i x_{m_i} \right\| < \alpha$. We say that X has alternate signs (A_∞) property (AS-(A_∞)) if it has alternate signs (A_k) property (AS-(A_k)) for some k.

It is clear that property (A_k) implies alternate signs (A_k) property. Using the Bessaga-Pelczynski method, we can get the following.

Proposition 2.1. *If a Banach space X has an unconditional basis with unconditional basis constant 1, then property (A_k) is equivalent to alternate signs (A_k) property.*

It is easy to see that alternate signs (A_k) property implies alternate signs (A_{k+1}) property.

PROPOSITION 2.2. *If X has alternate signs (A_k) property then it has alternate signs (A_{k+1}) property.*

Proof. Suppose that X has alternate signs (A_k) property. Then X reflexive and there exists $0 < \alpha < 1$ such that for a weakly null sequence (x_n) in B_X there exist $n_1 < n_2 < \cdots < n_k$ and $\epsilon_i \in \{\pm 1\}$ with

$$\left\| \frac{1}{k} \sum_{i=1}^{k} \epsilon_i x_{n_i} \right\| < \alpha.$$

Let $n_{k+1} = n_k + 1$ and $\epsilon_{k+1} = 1$. Then

$$\begin{aligned} \left\| \frac{1}{k+1} \sum_{i=1}^{k+1} \epsilon_i x_{n_i} \right\| &\leq \frac{k}{k+1} \left\| \frac{1}{k} \sum_{i=1}^{k} \epsilon_i x_{n_i} \right\| + \frac{1}{k+1} \|x_{n_{k+1}}\| \\ &\leq \frac{k\alpha}{k+1} + \frac{1}{k+1} < 1. \end{aligned}$$

Letting $\beta = \frac{1}{k+1}(k\alpha + 1)$, we get the result. □

Since Uniformly convexity implies property (A_2), we get the following proposition.

PROPOSITION 2.3. *If X is uniformly convex then it has alternate signs (A_2) property.*

We consider the converse of Proposition 2.3. The implication of Proposition 2.3 is strict. There exists a non-uniformly convex Banach space with alternate signs (A_2) property.

EXAMPLE 2.2. Consider $(\mathbb{R}^2, \|\cdot\|_\infty)$. Let $x = (1, 1)$ and $y = (1, 0)$. Then $\|x\|_\infty = \|y\|_\infty = 1$ and $\|x - y\|_\infty = 1$. But $\frac{1}{2}\|x + y\|_\infty = 1$. This means that $(\mathbb{R}^2, \|\cdot\|_\infty)$ is not uniformly convex. Since weakly convergence is equivalent to norm convergence in finite dimensional space, it is easy to see that $(\mathbb{R}^2, \|\cdot\|_\infty)$ has alternate signs (A_2) property.

We need the following Proposition 2.4 which is found in [2].

PROPOSITION 2.4. *A Banach space has weak Banach-Saks property if and only if it has alternate signs weak Banach-Saks property.*

Banach spaces with alternate signs (A_k) property have alternate signs weak Banach-Saks property.

THEOREM 2.3. *If X has alternate signs (A_k) property, it has alternate signs weak Banach-Saks property.*

Proof. Suppose that X has alternate signs (A_k) property. Then there exists $0 < \alpha < 1$ such that for all weakly null sequence (x_n) in B_X, there exist $n_1 < n_2 < \cdots < n_k$ and $\epsilon_i \in \{\pm 1\}$ with

$$\left\| \frac{1}{k} \sum_{i=1}^{k} \epsilon_i x_{n_i} \right\| < \alpha.$$

Suppose that (x_n) is a weakly null sequence in X. Without loss of generality, we may assume that $\|x_n\| \leq 1$. Then there exist $n_1 < n_2 < \cdots < n_k$ and $\epsilon_i \in \{\pm 1\}$ with

$$\left\| \frac{1}{k} \sum_{i=1}^{k} \epsilon_i x_{n_i} \right\| < \alpha.$$

Since $(x_n)_{n>n_k}$ is weakly null and $\|x\| \leq 1$ for $n > n_k$, there exist $(n_k <) < n_{k+1} < n_{k+2} < \cdots < n_{2k}$ and $\epsilon_i \in \{\pm 1\}$ such that

$$\left\| \frac{1}{k} \sum_{i=k+1}^{2k} \epsilon_i x_{n_i} \right\| < \alpha.$$

Continue this process, we obtain a subsequence (x_{n_m}) and $\epsilon_i \in \{\pm 1\}$ which given any $k \in \mathbb{N}$

$$\left\| \frac{1}{k} \sum_{i=jk+1}^{(j+1)k} \epsilon_i x_{n_i} \right\| < \alpha,$$

for all $j \in \mathbb{N}$. Thus we have a block of x_i and $\epsilon_i \in \{\pm 1\}$, i.e, $\{x_{n_{jk+1}}, \cdots, x_{(j+1)k}\}$, for $j = 0, 1, \cdots$ such that

$$\left\| \frac{1}{k} \sum_{i=1}^{k} \epsilon_i x_{n_{jk+i}} \right\| < \alpha,$$

for all $k \in \mathbb{N}$. Now by applying Kakutani's method (see [4] and [5]), we obtain a subsequence (x'_n) of (x_n) such that

$$\left\| \frac{1}{n} \sum_{i=1}^{n} \epsilon_i x'_n \right\| \to 0 \quad \text{as } n \to \infty.$$

This completes our proof. □

Alternate signs (A_k) property implies weak Banach-Saks property, by Proposition 2.4 and Theorem 2.3. Since Banach-Saks property is equivalent to weak Banach-Saks property in reflexive Banach space, we get the following corollary.

COROLLARY 2.4. *Alternate signs (A_k) property implies Banach-Saks property.*

By Proposition 2.2, Proposition 2.3 and Corollary 2.4, we get the following implications.

$$(UC) \Rightarrow AS\text{-}(A_2) \Rightarrow AS\text{-}(A_3) \Rightarrow \cdots \Rightarrow AS\text{-}(A_\infty) \Rightarrow (BS).$$

We now show that the implications are not reversible. The following can be found in [7].

EXAMPLE 2.5. For $x = (a_n) \in l_2$, we define a norm $\|x\|_{(k)}$ by

$$\|x\|_{(k)} = \left[\sup_{n_1 < n_2 < \cdots < n_k} \left(\sum_{i=1}^{s} |a_{n_i}| \right)^2 + \sum_{n \neq n_1, n_2 \cdots, n_k} |a_n|^2 \right]^{\frac{1}{2}}.$$

Then $\|x\|_2 \leq \|x\|_{(k)} \leq \sqrt{k}\|x\|_2$. Let $X_k = (l_2, \|\cdot\|_{(s)})$.

The following can be found in [3].

LEMMA 2.6. *If X is a Banach space with basis (e_n) and (x_n) is a weakly null sequence in X, then for all $\epsilon > 0$ there exists a subsequence (x_{n_i}) of (x_n) and block sequence (u_i) of (e_n) such that $\|x_{n_i} - u_i\| < \frac{\epsilon}{2^{i+1}}$.*

We need the following lemma.

LEMMA 2.7. *Let X_k be the space defined by in Example 2.5. If $x_1,\ x_2,\ \cdots,\ x_k,\ x_{k+1} \in B_{X_k}$ with $x_1 < x_2 < \cdots < x_k < x_{k+1}$ and $\epsilon_i \in \{\pm 1\}$ then*

$$\left\| \sum_{i=1}^{k+1} \epsilon_i x_i \right\|_{(k)} \leq \sqrt{k^2 + 1}.$$

Proof. This is proved by straightforward computation using the following inequality

$$(n-1)\sum_{i=1}^{n} a_i^2 \geq 2 \sum_{1\leq i<j\leq n} a_i a_j,$$

where (a_i) is a real sequence. For simplicity, we give the proof in case $k=2$. Suppose that $\epsilon_1 x = (a_n)$, $\epsilon_2 y = (b_n)$, $\epsilon_3 z = (c_n) \in B_{X_2}$ for $\epsilon_i \in \{\pm 1\}$ and $x < y < z$. Without loss of generality, it suffices to consider the following two cases.
Case 1:

$$\begin{aligned}
&\|\epsilon_1 x + \epsilon_2 y + \epsilon_3 z\|_{(2)}^2 \\
&\quad = \sup_{n_1,n_2} (|a_{n_1}| + |a_{n_2}|)^2 + \sum_{n\neq n_1,n_2} |a_n|^2 + \sum_n |b_n|^2 + \sum_n |c_n|^2.
\end{aligned}$$

$$\begin{aligned}
&\|\epsilon_1 x + \epsilon_2 y + \epsilon_3 z\|_{(2)}^2 \\
&\quad = \sup_{n_1,n_2} (|a_{n_1}| + |a_{n_2}|)^2 + \sum_{n\neq n_1,n_2} |a_n|^2 + \sum_n |b_n|^2 + \sum_n |c_n|^2 \\
&\quad \leq \|x\|_{(2)}^2 + \|y\|_2^2 + \|z\|_2^2 \leq \|x\|_{(2)}^2 + \|y\|_{(2)}^2 + \|z\|_{(2)}^2 = 3.
\end{aligned}$$

Case 2:

$$\begin{aligned}
&\|\epsilon_1 x + \epsilon_2 y + \epsilon_3 z\|_{(2)}^2 \\
&\quad = \sup_{n_1,n_2} (|a_{n_1}| + |b_{n_2}|)^2 + \sum_{n\neq n_1} |a_n|^2 + \sum_{n\neq n_2} |b_n|^2 + \sum_n |c_n|^2.
\end{aligned}$$

$$\begin{aligned}
&\|\epsilon_1 x + \epsilon_2 y + \epsilon_3 z\|_{(2)}^2 \\
&\quad = \sup_{n_1,n_2} (|a_{n_1}| + |b_{n_2}|)^2 + \sum_{n\neq n_1} |a_n|^2 + \sum_{n\neq n_2} |b_n|^2 + \sum_n |c_n|^2 \\
&\quad \leq 2\sup_{n_1,n_2} \left(|a_{n_1}|^2 + |b_{n_2}|^2\right) + \sum_{n\neq n_1} |a_n|^2 + \sum_{n\neq n_2} |b_n|^2 + \sum_n |c_n|^2 \\
&\quad \leq \sup_{n_1,n_2} \left(|a_{n_1}|^2 + |b_{n_2}|^2\right) + \sum_n |a_n|^2 + \sum_n |b_n|^2 + \sum_n |c_n|^2 \\
&\quad \leq \|x\|_2^2 + \|y\|_2^2 + \|x\|_2^2 + \|y\|_2^2 + \|z\|_2^2 = 5.
\end{aligned}$$

This implies that $\|x - y + z\|_{(2)} \leq \sqrt{5}$. □

By the above lemmas, we get the following.

PROPOSITION 2.5. *Alternate signs* (A_{k+1}) *property does not imply Alternate signs* (A_k) *property.*

Proof. Since the space X_k is isomorphic to l_2, unit vector basis (e_n) is weakly null in X_k. But

$$\left\|\sum_{i=1}^{k}\epsilon_i e_{n_i}\right\|_{(k)} = k$$

for all choice of n_i and $\epsilon_i \in \{\pm 1\}$. This means that X_k does not have alternate signs (A_k) property.

Let (x_n) be a weak null sequence in B_{X_k}. By Lemma 2.6, for all $\epsilon > 0$ there exists a subsequence (x_{n_i}) of (x_n) and block sequence (u_i) of (e_n) such that $\|x_{n_i} - u_i\| < \frac{\epsilon}{2^{i+1}}$. We note that for all $\epsilon_i \in \{\pm 1\}$,

$$\|\sum_{i=1}^{k+1}\epsilon_i u_i\|_{(k)} \le \sqrt{k^2+1},$$

by Lemma 2.7. For some large $i_1 < i_2 < \cdots < i_k < i_{k+1}$,

$$\|x_{n_{i_j}} - u_{i_j}\| < \frac{1}{k+1}\left(\sqrt{k^2+2} - \sqrt{k^2+1}\right),$$

where $j = 1, 2, \cdots, k+1$. Then we have

$$\begin{aligned}\left\|\sum_{j=1}^{k+1}\epsilon_i x_{n_{i_j}}\right\| &\le \sum_{j=1}^{k+1}\left\|x_{n_{i_j}} - u_{i_j}\right\| + \left\|\sum_{j=1}^{k+1}\epsilon_i u_{n_j}\right\| \\ &\le \sqrt{k^2+2}.\end{aligned}$$

Let $\alpha = \frac{\sqrt{k^2+2}}{k+1}$. Then $0 < \alpha < 1$ and this leads that the space X_k has alternate signs (A_{k+1}) property. □

PROPOSITION 2.6. *Banach-Saks property does not imply alternate signs* (A_∞) *property.*

Proof. Consider $\left(\prod_{s\ge 2} X_s\right)_{l_2}$. Then $\left(\prod_{s\ge 2} X_s\right)_{l_2}$ has Banach-Saks property [7].

Let $k \in \mathbb{N}$. If $x^{(n)} = (0, 0, \cdots, 0, e_n, 0, \cdots)$ where usual unit vector e_n in k-th coordinate is only nonzero element of $x^{(n)}$, then $x^{(n)} \in$

$\left(\prod_{s\geq 2} X_s\right)_{l_2}$ and $\|x^{(n)}\|_{\left(\prod_{s\geq 2} X_s\right)_{l_2}} = 1$. We note that $x^{(n)}$ is weakly null in $\left(\prod_{s\geq 2} X_s\right)_{l_2}$. But for all $\epsilon_i \in \{\pm 1\}$,

$$\left\|\sum_{j=1}^{k} \epsilon_i x^{(n_i)}\right\|_{\left(\prod_{s\geq 2} X_s\right)_{l_2}} = \left\|\sum_{j=1}^{k} \epsilon_i e_{n_i}\right\|_{(k)} = k.$$

This means that $\left(\prod_{s\geq 2} X_s\right)_{l_2}$ has no alternate signs (A_∞) property . □

Acknowledgements. We are grateful to the referee for very useful comments and providing us with Proposition 2.1.

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