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ASYMPTOTICALLY LINEAR BEAM EQUATION AND REDUCTION METHOD

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Abstract. We prove a theorem which shows the existence of at least three $\pi-$periodic solutions of the wave equation with asymptotical linearity. We obtain this result by the finite dimensional reduction method which reduces the critical point results of the infinite dimensional space to those of the finite dimensional subspace. We also use the critical point theory and the variational method.

1. Introduction

Let g be a C^1 function defined on R with $g(0) = 0$. Let

$$g'(0) = \lim_{|u|\to 0} \frac{g(u)}{u}, \qquad g'(\infty) = \lim_{|u|\to\infty} \frac{g(u)}{u}.$$

In this paper we investigate the number of the π periodic weak solutions of the following asymptotically linear wave equation with Dirichlet boundary condition and periodic condition

$$\begin{aligned} u_{tt} - u_{xx} &= g(u), \\ u(-\frac{\pi}{2}, t) &= u(\frac{\pi}{2}, t) = 0, \\ u(x,t) &= u(-x,t) = u(x,-t) = u(-x, t+\pi). \end{aligned} \tag{1.1}$$

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As the physical model for this problem we can find a string with travelling wave, which is suspended by the cable under a load. Choi and Jung investigate in [2], [3] the existence and multiplicity of the solutions of the nonlinear wave equation with Dirichlet boundary condition. In [1], [4] the authors investigate the existence and multiplicity of the solutions of the nonlinear wave equation. We assume that $g \in C^1(R, R)$ and satisfies the following conditions:

$(g1)$ $g(u) = o(u)$ as $u \to 0$, $g(0) = 0$ and $G(0) = 0$, where

$$G(\psi) = \int_0^\psi g(s)ds.$$

$(g2)$ There exist constants α, β such that $-7 < \alpha < -3 < \beta < 1$ and

$$\alpha \leq g'(u) \leq \beta, \qquad \forall u \in R.$$

$(g3)$ $g'(0)$ and $g'(\infty)$ exist and satisfy

$$-7 < \alpha < g'(0) < -3 < g'(\infty) < \beta < 1.$$

$(g4)$ g is a π periodic function with respect to t.

The eigenvalue problem

$$\begin{aligned} &u_{tt} - u_{xx} = \lambda u \qquad \text{in } (-\frac{\pi}{2}, \frac{\pi}{2}) \times R, \\ &u(\pm\frac{\pi}{2}, t) = 0, \\ &u(x,t) = u(-x,t) = u(x,-t) = u(-x, t+\pi) \end{aligned} \tag{1.2}$$

has infinitely many eigenvalues

$$\lambda_{mn} = (2n+1)^2 - 4m^2, \qquad (m, n = 0, 1, 2, \ldots)$$

and corresponding normalized eigenfunctions $\phi_{mn}(x,t)$, m, $n > 0$, given by

$$\begin{aligned} \phi_{0n} &= \frac{\sqrt{2}}{\pi}\cos(2n+1)x \qquad \text{for } n \geq 0, \\ \phi_{mn} &= \frac{2}{\pi}\cos 2mt\cos(2n+1)x \qquad \text{for } m > 0, n \geq 0. \end{aligned}$$

We note that $\{\lambda_{mn} |\ m, n = 0, 1, 2, \ldots\}$ is unbounded from above and from below and has no finite accumulation point. The only eigenvalues in the interval $(-15, 9)$ are given by

$$\lambda_{32} = -11 < \lambda_{21} = -7 < \lambda_{10} = -3 < \lambda_{00} = 1 < \lambda_{11} = 5.$$

The main result is as follows:

THEOREM 1.1. *Assume that g satisfies the conditions $(g1) - (g4)$. Then (1.1) has at least three $\pi-$periodic solutions.*

Theorem 1.1 will be proved in Section 3 via the finite dimensional reduction method, the critical point theory and the variational method. The finite dimensional reduction method combined with the critical point theory and the variational methods reduce the critical point result on the infinite dimensional space to that on the finite dimensional subspace. So we obtain the critical points result of the functional on the infinite space E from the critical points result of the correponding functional $\tilde{I}(v)$ on the finite dimensional reduction subspace. The outline of this paper is as follows: In section 2 we introduce the Hilbert normed space E and show that the corresponding functional $I(u)$ of (1.1) is in $C^1(E, R)$, *Fréchet* differentiable and satisfies the Palais-Smale condition. In section 3, we prove Theorem 1.1.

2. Finite dimensional reduction method

Let Ω be the square $[-\frac{\pi}{2}, \frac{\pi}{2}] \times [-\frac{\pi}{2}, \frac{\pi}{2}]$ and E' the Hilbert space defined by

$$E' = \{v \in L^2(\Omega) |\ v \text{ is even in } x \text{ and } t\}.$$

The set of functions $\{\phi_{mn}\}$ is an orthonormal basis in E'. Let us denote an element v in E', as

$$v = \sum h_{mn}\phi_{mn}$$

and we define a subspace E of E' as

$$E = \{v \in E' |\ \sum |\lambda_{mn}| h_{mn}^2 < \infty\}.$$

This is a completely normed space with a norm

$$\|v\| = [\sum |\lambda_{mn}| h_{mn}^2]^{\frac{1}{2}}.$$

Since $\lambda_{mn} \to +\infty$ and c is fixed, we have
(i) $u_{tt} - u_{xx} \in E$ implies $u \in E$,
(ii) $\|u\| \geq C\|u\|_{L^2(\Omega)}$, for some $C > 0$,

(iii) $\|u\|_{L^2(\Omega)} = 0$ if and only if $\|u\| = 0$.
From the conditions $(g1) - (g4)$, we have the following lemma:

LEMMA 2.1. *Assume that g is $\pi-$periodic in t and satisfies the conditions $(g1)$-$(g4)$. Then the solutions in $L^2(\Omega)$ of*

$$u_{tt} - u_{xx} = g(u) \qquad \text{in } L^2(\Omega)$$

belong to E.

Proof. Let $g(u) = \sum h_{mn}\phi_{mn} \in L^2(\Omega)$. Then

$$(D_{tt} - D_{xx})^{-1}(g(u)) = \sum \frac{1}{\lambda_{mn}} h_{mn}\phi_{mn}.$$

Hence we have

$$\|(D_{tt} - D_{xx})^{-1} g(u)\|^2 = \sum |\lambda_{mn}| \frac{1}{\lambda_{mn}^2} h_{mn}^2 \le C \sum h_{mn}^2$$

for some $C > 0$, which means that

$$\|(D_{tt} - D_{xx})^{-1} g(u)\| \le C_1 \|u\|_{L^2(\Omega)}.$$

□

With the aid of Lemma 2.1 it is enough that we investigate the existence of solutions of (1.1) in the subspace E of $L^2(\Omega)$. We consider the following functional associated with (1.1),

$$I(u) = \frac{1}{2}\int_\Omega [-|u_t|^2 + |u_x|^2] dx\, dt - \int_\Omega G(u) dx\, dt, \tag{2.1}$$

where

$$G(u) = \int_0^u g(s) ds.$$

Then I is well defined. By $(g1) - (g4)$, $I(u) \in C(E, R)$, *Fréchet* differentiable in E, so the solutions of (1.1) coincide with the critical points of $I(u)$.

LEMMA 2.2. *Assume that $g(u)$ is $\pi-$periodic in t and satisfies the conditions $(g1)$-$(g4)$. Then $I(u)$ is continuous and Fréchet differentiable in E and*

$$DI(u)(h) = \int_\Omega [-u_t \cdot h_t + u_x \cdot h_x - g(u)h] dx\, dt \tag{2.2}$$

for $h \in H$. Moreover if we set

$$F(u) = \int_\Omega G(u) dx\, dt,$$

then $F'(u)$ is continuous with respect to weak convergence, $F'(u)$ is compact, and

$$F'(u)h = \int_\Omega g(u)h dx\, dt \qquad \text{for all } \ h \in E.$$

This implies that $I \in C^1(E,R)$ and $F(u)$ is weakly continuous.

Proof. Let $u \in E$. First we will prove that $I(u)$ is continuous. We consider

$$I(u+v)-I(u) = \int_\Omega [u\cdot(v_{tt}-v_{xx})+\frac{1}{2}v\cdot(v_{tt}-v_{xx})-G(u+v)+G(u)]dxdt.$$

Let $u = \sum h_m n\phi_m n$, $v = \sum \tilde{h}_{mn}\phi_{mn}$. Then we have

$$\begin{aligned} |\int_\Omega u\cdot(v_{tt}-v_{xx})dxdt| &= |\sum\int_\Omega \lambda_{mn}h_{mn}\tilde{h}_{mn}| \le \|u\|\|v\|, \\ |\int_\Omega v\cdot(v_{tt}-v_{xx})dxdt| &= |\sum\lambda_{mn}\tilde{h}^2_{mn}| \le \|v\|^2. \end{aligned}$$

On the other hand, by mean value theorem and $(g2)$, we have

$$\begin{aligned} G(u+v)-G(u) &= \int_0^{u+v} g(s)ds - \int_0^u g(s)ds \\ &= \frac{1}{2}g'(t)(u+v)^2 - \frac{1}{2}g'(t')u^2 \\ &\le \max\{|\alpha|,|\beta|\}|v|(|u|+|v|) \\ &\le C\max\{|\alpha|,|\beta|\}\|v\|(\|u\|+\|v\|). \end{aligned}$$

With the above results, we see that $I(u)$ is continuous at u. To prove $I(u)$ is *Fréchet* differentiable at $u \in E$, we consider

$$\begin{aligned} &|I(u+v)-I(u)-DI(u)v| \\ &\quad = |\int_\Omega \frac{1}{2}v(v_{tt}-v_{xx}) - G(u+v)+G(u)-g(u)v| \\ &\quad \le \frac{1}{2}\|v\|^2 + C\gamma\|v\|(\|u\|+\|v\|)+M\|v\| \\ &\quad \le C'\|v\|(\|v\|+\|u\|+\|v\|+1). \end{aligned}$$

□

Let V be the 1-dimensional subspace of E spanned by ϕ_{10} whose eigenvalue is $\lambda_{10} = -3$. Let W be the orthogonal complement of V in E. Let $P : E \to V$ be the orthogonal projection of E onto V and

$I-P : E \to W$ denote that of E onto W. Then every element $u \in L^2(\Omega)$ is expressed by $u = v+z$, $v \in Pu$, $z = (I-P)u$. Then (1.1) is equivalent to the two systems in the two unknowns v and z:

$$
\begin{aligned}
v_{tt} - v_{xx} &= P(g(v+z)) \qquad \text{in } \Omega, \\
z_{tt} - z_{xx} &= (I-P)(g(v+z)) \qquad \text{in } \Omega, \\
v(-\frac{\pi}{2}, t) &= v(\frac{\pi}{2}, t) = 0, \\
z(-\frac{\pi}{2}, t) &= z(\frac{\pi}{2}, t) = 0, \\
v(x,t) = v(-x,t) &= v(x,-t) = v(-x, t+\pi), \\
z(x,t) = z(-x,t) &= z(x,-t) = z(-x, t+\pi).
\end{aligned} \tag{2.3}
$$

Let W_1 be a subspace of W spanned by eigenfunctions corresponding to the eigenvalues $\lambda_{mn} \le -7$ and let W_2 be a subspace of W spanned by eigenfunctions corresponding to the eigenvalues $\lambda_{mn} \ge 1$. Let $v \in V$ be fixed and consider the function $h : W_1 \times W_2 \to R$ defined by

$$h(w_1, w_2) = I(v + w_1 + w_2).$$

The function h has continuous partial *Fréchet* derivatives $D_1 h$ and $D_2 h$ with respect to its first and second variables given by

$$D_i h(w_1, w_2)(y_i) = DI(v + w_1 + w_2)(y_i) \tag{2.4}$$

for $y_i \in W_i$, $i = 1, 2$. By Lemma 2.2, I is a function of class C^1.

By the following Lemma 2.3, we can get the critical points of the functional $I(u)$ on the infinite dimensional space E from that of the functional on the finite dimensional subspace V.

LEMMA 2.3. *(REDUCTION METHOD) Assume that g is $\pi-$periodic and satisfies the conditions (g1)-(g4).Then*
(i) there exists a unique solution $z \in W$ of the equation

$$
\begin{aligned}
z_{tt} - z_{xx} &= (I-P)(g(v+z)) \qquad \textit{in } \Omega, \\
z(-\frac{\pi}{2}, t) &= z(\frac{\pi}{2}, t) = 0, \\
z(x,t) &= z(-x,t) = z(x,-t) = z(-x, t+\pi).
\end{aligned} \tag{2.5}
$$

If we put $z = \theta(v)$, then θ is continuous on V and satisfies a uniform Lipschitz condition in v with respect to the L^2 norm(also norm $\|\cdot\|$). Moreover

$$DI(v + \theta(v))(w) = 0 \qquad \textit{for all } w \in W.$$

(ii) There exists $m_1 < 0$ such that if w_1 and y_1 are in W_1 and $w_2 \in W_2$, then

$$(D_1h(w_1, w_2) - D_1h(y_1, w_2))(w_1 - y_1) \leq m_1\|w_1 - y_1\|^2.$$

(iii) There exists $m_2 > 0$ such that if w_2 and y_2 are in W_2 and $w_1 \in W_1$, then

$$(D_2h(w_1, w_2) - D_2h(w_1, y_2))(w_2 - y_2) \geq m_2\|w_2 - y_2\|^2.$$

(iv) If $\tilde{I} : V \to R$ is defined by $\tilde{I}(v) = I(v+\theta(v))$, then $\tilde{I}$ has a continuous Fréchet derivative $D\tilde{I}$ with respect to v, and

$$D\tilde{I}(v)(h) = DI(v + \theta(v))(h) \qquad \text{for all } v, h \in V. \tag{2.6}$$

(v) $v_0 \in V$ is a critical point of $\tilde{I}$ if and only if $v_0 + \theta(v_0)$ is a critical point of I.

Proof. (i) Let $\delta = \frac{\alpha+\beta}{2}$. If $g_1(\xi) = g(\xi) - \delta\xi$, the equation (2.5) is equivalent to

$$z = (D_{tt} - D_{xx} - \delta)^{-1}(I - P)(g_1(v + z)) \tag{2.7}$$

The operator $(D_{tt} - D_{xx} - \delta)^{-1}(I - P)$ is self adjoint, compact and linear map from $(I-P)L^2(\Omega)$ into itself and its L_2 norm is $(\min\{|\lambda_{21} - \delta|, |\lambda_{00} - \delta|\}^{-1} = (\min\{| - 7 - \delta|, |1 - \delta|\}^{-1}$. Since $|g_1(\xi_2) - g_1(\xi_1)| \leq \max\{|\alpha-\delta|, |\beta-\delta|\}|\xi_2-\xi_1| = \frac{|\alpha+\beta|}{2}|\xi_2-\xi_1|$, it follows that the right-hand side of (2.7) defines, for fixed $v \in V$, a Lipschitz mapping of $(I-P)L^2(\Omega)$ into itself with Lipschitz constant $r < 1$. Therefore, by the contraction mapping principle, for given $v \in V$, there exists a unique $z = (I - P)L^2(\Omega)$ which satisfies (2.7). If $\theta(v)$ denote the unique $z \in (I-P)L^2(\Omega)$ which solves (2.5), then θ is continuous and satisfies a uniform Lipschitz condition in v with respect to the L^2 norm(also norm $\|\cdot\|$). In fact, if $z_1 = \theta(v_1)$ and $z_2 = \theta(v_2)$, then

$$\begin{aligned}
&\|z_1 - z_2\|_{L^2(\Omega)} \\
&\quad = \|(D_{tt} - D_{xx} - \delta)^{-1}(I - P)(g_1(v_1 + z_1) - g_1(v_2 + z_2))\|_{L^2(\Omega)} \\
&\quad \leq r\|(v_1 + z_1) - (v_2 + z_2)\|_{L^2(\Omega)} \\
&\quad \leq r(\|v_1 - v_2\|_{L^2(\Omega)} + \|z_1 - z_2\|_{L^2(\Omega)}) \leq r\|v_1 - v_2\| + r\|z_1 - z_2\|.
\end{aligned}$$

Hence

$$\|z_1 - z_2\| \leq C\|v_1 - v_2\|, \qquad C = \frac{r}{1 - r}. \tag{2.8}$$

Let $u = v + z$, $v \in V$ and $z = \theta(v)$. If $w \in (I-P)L^2(\Omega) \cap E$,

$$DI(v+\theta(v))(w) = \int_\Omega [-(v+z)_t \cdot w_t + (v+z)_x \cdot w_x - Pg(v+z)w - (I-P)g(v+w)w]dxdt.$$

From (2.5) we see that

$$\int_\Omega [-z_t \cdot w_t + z_x \cdot w_x - (I-P)(g(v+z)w)]dxdt = 0.$$

Since

$$\int_\Omega z_t \cdot w_t = 0 \qquad \text{and} \int_\Omega z_x \cdot w_x = 0,$$

we have

$$DI(v+\theta(v))(w) = 0. \tag{2.9}$$

(ii) If w_1 and y_1 are in W_1 and $w_2 \in W_2$, then

$$\begin{aligned}&(D_1h(w_1,w_2) - D_1h(y_1,w_2))(w_1-y_1)\\ &= \int_\Omega [-|(w_1-y_1)_t|^2 + |(w_1-y_1)_x|^2\\ &\quad -(g(v+w_1+w_2) - g(v+y_1+w_2))(w_1-y_1)]dtdx.\end{aligned}$$

Since $\int_\Omega -|(w_1-y_1)_t|^2 + |(w_1-y_1)_x|^2 dtdx = -\|w_1-y_1\|^2$ and $(g(\xi_2) - g(\xi_1))(\xi_2-\xi_1) > \alpha(\xi_2-\xi_1)^2$, we see that if w_1 and y_1 are in W_1 and $w_2 \in W_2$, then $\|w_1-y_1\|_{L^2(\Omega)} \le \frac{1}{7}\|w_1-y_1\|^2$ and

$$g(v+w_1+w_2) - g(v+y_1+w_2) \le \frac{-\alpha}{7}\|w_1-y_1\|^2$$

and

$$(D_1h(w_1,w_2) - D_1h(y_1,w_2))(w_1-y_1) \le (-1-\frac{\alpha}{7})\|w_1-y_1\|^2$$

where $(-1-\frac{\alpha}{7}) < 0$.

(iii) Similarly, using the fact that $\int_\Omega -|(w_2-y_2)_t|^2 + |(w_2-y_2)_x|^2 dtdx = \|w_2-y_2\|^2$ and $(g(\xi_2)-g(\xi_1))(\xi_2-\xi_1) \le \beta(\xi_2-\xi_1)^2$ we see that if w_2 and y_2 are in W_2 and $w_1 \in W_1$, then

$$(D_2h(w_1,w_2) - D_2h(w_1,y_2))(w_2-y_2) \ge (1-\beta)\|w_2-y_2\|^2$$

where $(1-\beta) > 0$.

(iv) Since the functional I has a continuous *Fréchet* derivative DI, $\tilde{I}$ has a continuous *Fréchet* derivative $D\tilde{I}$ with respect to v.

(v) Suppose that there exists $v_0 \in V$ such that $D\tilde{I}(v_0) = 0$. From $D\tilde{I}(v)(h) = DI(v+\theta(v))(h)$ for all $v, h \in V$, $DI(v_0+\theta(v_0))(h) =$

$D\tilde{I}(v_0)(h) = 0$ for all $h \in V$. Since $DI(v + \theta(v))(w) = 0$ for all $w \in W$ and E is the direct sum of V and W, it follows that $DI(v_0 + \theta(v_0)) = 0$. Thus $v_0 + \theta(v_0)$ is a solution of (1.1). Conversely if u is a solution of (1.1) and $v = Pu$, then $D\tilde{I}(v) = 0$.

□

REMARK. *We note that* $\theta(v) = 0$.

3. Proof of theorem 1.1

LEMMA 3.1. *Assume that g is $\pi-$periodic and satisfies the conditions $(g1)$-$(g4)$. Then $v = 0$ is a strict local point of maximum of $\tilde{I}(v)$.*

Proof.

$$
\begin{aligned}
&\tilde{I}(v) = I(v + \theta(v)) \\
&= \int_\Omega [-\frac{1}{2}|vt + \theta(v)_t|^2 + \frac{1}{2}|v_x + \theta(v)_x|^2]dtdx - \int_\Omega G(v + \theta(v))dtdx \\
&= \int_\Omega [-\frac{1}{2}|v_t|^2 + \frac{1}{2}|v_x|^2]dtdx + C,
\end{aligned}
$$

where

$$
\begin{aligned}
C &= \int_\Omega [-\frac{1}{2}|\theta(v)_t|^2 + \frac{1}{2}|\theta(v)_x|^2]dxdt - \int_\Omega G(\theta(v))dxdt \\
&\qquad - \int_\Omega [G(v + \theta(v)) - G(\theta(v))]dxdt \\
&= I(\theta(v)) - \int_\Omega [G(v + \theta(v)) - G(\theta(v))]dxdt \\
&= \tilde{I}(0) - \int_\Omega [G(v + \theta(v)) - G(\theta(v))]dxdt.
\end{aligned}
$$

Thus we have

$$\begin{aligned}
&\lim_{|v|\to 0} \tilde{I}(v) - \tilde{I}(0)\\
&= \lim_{|v|\to 0}\{\int_\Omega [-\frac{1}{2}|v_t|^2 + \frac{1}{2}|v_x|^2]dxdt - \int_\Omega [G(v+\theta(v)) - G(\theta(v))]dxdt\}\\
&= \lim_{|v|\to 0}\{-\int_\Omega \int_0^1 g(\theta(v)+sv)ds + \int_\Omega [-\frac{1}{2}|v_t|^2 + \frac{1}{2}|v_x|^2]dxdt\}\\
&= \lim_{|v|\to 0}\{\int_\Omega \int_0^1 g'(\theta(v)+sv)vvsds - \int_\Omega [-\frac{1}{2}|v_t|^2 + \frac{1}{2}|v_x|^2]dxdt\}\\
&= \frac{1}{2}(g'(0)) - (-3)) \lim_{|v|\to 0}\int_\Omega v^2 < 0.
\end{aligned}$$

Thus $v = 0$ is a strictly local point of maximum of $\tilde{I}(v)$ whose critical value is 0. □

We shall show that $-\tilde{I}(v)$ is bounded from below and $-\tilde{I}(v)$ satisfies the (P.S.) condition.

LEMMA 3.2. *Assume that g is $\pi-$periodic and satisfies the conditions $(g1)$-$(g4)$. Then $-\tilde{I}(v)$ is bounded from below and $\tilde{I}(v)$ satisfies the Palsis-Smale condition.*

Proof. Let us set $u(v) = v + \theta_1(v) + \theta_2(v)$, $v \in V$, $\theta_1(v) \in W_1$, $\theta_2(v) \in W_2$. Then we have

$$\begin{aligned}
\tilde{I}(v) \;\; = \;\; & \int_\Omega [-\frac{1}{2}|v_t + \theta_1(v)_t + \theta_2(v)_t|^2 + \frac{1}{2}|v_x + \theta_1(v)_x + \theta_2(v)_x|^2]dtdx\\
& - \int_\Omega G(v + \theta_1(v) + \theta_2(v))dtdx.
\end{aligned}$$

Moreover we have

$$
\begin{aligned}
\tilde{I}(v) &= I(v+\theta_1(v)+\theta_2(v)) = I(u(v)) \\
&= \int_\Omega [-\frac{1}{2}|u(v)_t|^2 + \frac{1}{2}|u(v)_x|^2 - \int_\Omega G(u(v))dxdt \\
&= \int_\Omega [-\frac{1}{2}|(v+\theta_1(v))_t|^2 + \frac{1}{2}|(v+\theta_1(v))_x|^2]dxdt - \int_\Omega G(v+\theta_1(v))dxdt \\
&\quad +\{\int_\Omega [-\frac{1}{2}|u(v)_t|^2 + \frac{1}{2}|u(v)_x|^2] + \frac{1}{2}|(v+\theta_1(v))_t|^2 - \frac{1}{2}|(v+\theta_1(v))_x|^2 \\
&\quad - \int_\Omega [G(u(v)) - G(v+\theta_1(v))]dxdt\}.
\end{aligned}
$$

The terms in the bracket are equal to

$$
\begin{aligned}
&-\int_\Omega \int_0^1 [g(s\theta_2(v)+v+\theta_1(v))\theta_2(v)ds]dxdt] \\
&\qquad +\frac{1}{2}\int_\Omega (u(v)_{tt} - u(v)_{xx})\theta_2(v)dxdt \\
&= \int_\Omega [\int_0^1 g'(s\theta_2(v)+v+\theta_1(v))\theta_2(v)\theta_2(v)sds]dxdt \\
&\qquad -\frac{1}{2}\int_\Omega (\theta_2(v)_{tt} - \theta_2(v)_{xx})\theta_2(v)dxdt \le 0.
\end{aligned}
$$

We also have that

$$
\begin{aligned}
&\lim_{|v|\to\infty} \int_\Omega G(v+\theta_1(v))dxdt \\
&\quad = \lim_{|v|\to\infty} \{\int_\Omega [\int_0^1 g(sv+s\theta_1(v))(v+\theta_1(v))ds]dxdt\} \\
&\quad = \lim_{|v|\to\infty} \int_\Omega [\int_0^1 g'(sv+s\theta_1(v))(v+\theta_1(v))(v+\theta_1(v))sds]dxdt \\
&\quad = \frac{1}{2} g'(\infty) \lim_{|v|\to\infty} \int_\Omega (v+\theta_1(v))^2 dxdt.
\end{aligned}
$$

Thus we have

$$\begin{aligned}
&\lim_{|v|\to\infty} \tilde{I}(v) \\
&\quad \le \lim_{|v|\to\infty} \{\int_\Omega [-\frac{1}{2}|(v+\theta_1(v))_t|^2 + \frac{1}{2}|(v+\theta_1(v))_x|^2]dxdt \\
&\qquad\qquad - \int_\Omega G(v+\theta_1(v))dxdt\} \\
&\quad \le \frac{1}{2}(-3-g'(\infty)) \lim_{|v|\to\infty} \int_\Omega (v+\theta_1(v))^2 dxdt \\
&\quad = \frac{1}{2}(-3-g'(\infty)) \lim_{|v|\to\infty} \|v+\theta_1(v)\|^2_{L^2(\Omega)} \longrightarrow -\infty.
\end{aligned}$$

Thus $-\tilde{I}(v)$ is bounded from below and, so satisfies the (P.S.) condition. □

[Proof of Theorem 1.1]
By Lemma 2.2, $\tilde{I}(v)$ is continuous and *Fréchet* differentiable in V. By Lemma 3.2, $\tilde{I}(v)$ is bounded above, satisfies the (P.S.) condition and $\tilde{I}(v) \to -\infty$ as $\|v\| \to \infty$. By Lemma 3.1, $v = 0$ is a strictly local point of maximum of $\tilde{I}(v)$ with critical value $\tilde{I}(0) = 0$. We note that $\max_{v\in V}\tilde{I}(v) > 0$ is another critical value of $\tilde{I}$. By the shape of the graph of the functional $\tilde{I}$ on the 1-dimensional subspace V, there exist the third critical point of $\tilde{I}(v)$. Thus (1.1) has at least three solutions, one of which is trivial solution $u = v + \theta(v) = 0 + 0 = 0$.

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References

[1] V. Benci and P.H. Rabinowitz, *Critical point theorems for indefinite functionals*, Invent. Math. **52** (1979), 241–273.
[2] Q.H. Choi and T. Jung, *An application of a variational reduction method to a nonlinear wave equation*, J. Differential Equations **117** (2) (1995), 390–410.
[3] Q.H. Choi and T. Jung, *Multiple periodic solutions of a semilinear wave equation at double external resonances*, Commun. Appl. Anal. **3** (1) (1999), 73–84.
[4] P.H. Rabinowitz, *Free vibrations for a semilinear wave equation*, Comm. Pure Appl. Math. **31** (1968), 31–68.

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