Korean J. Math. **19** (2011), No. 4, pp. 423–436

MULTIPLE SOLUTIONS RESULT FOR THE MIXED TYPE NONLINEAR ELLIPTIC PROBLEM

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Abstract. We obtain a theorem that shows the existence of multiple solutions for the mixed type nonlinear elliptic equation with Dirichlet boundary condition. Here the nonlinear part contain the jumping nonlinearity and the subcritical growth nonlinearity. We first show the existence of a positive solution and next find the second nontrivial solution by applying the variational method and the mountain pass method in the critical point theory. By investigating that the functional I satisfies the mountain pass geometry we show the existence of at least two nontrivial solutions for the equation.

1. Introduction

Let Ω be a bounded subset of R^n with smooth boundary. Let $h(x) \in L^s(\Omega)$ for some $s > n$. In this paper we consider the multiple solutions for the following nonlinear elliptic equation with jumping nonlinearity and subcritical growth nonlinearity and Dirichlet boundary condition

$$\Delta u + bu_+ - pu_-^{p-1} = h(x) \qquad \text{in } \Omega, \tag{1.1}$$
$$u = 0 \qquad \text{on } \partial\Omega,$$

Received October 23, 2011. Revised November 19, 2011. Accepted November 25, 2011.

2000 Mathematics Subject Classification: 35J50, 35J55, 35J20, 35Q72.

Key words and phrases: elliptic equation with jumping nonlinearity and subcritical growth nonlinearity, mountain pass theorem, $(P.S.)$ condition, critical point theory, variational method.

This work(Choi) was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education, Science and Technology (KRF-2011-0003191).

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where $2 < p < 2^*$, $2^* = \frac{2n}{n-2}$, $n \geq 3$, $u_+ = \max\{u, 0\}$, $u_- = -\min\{u, 0\}$, $u(x) \in W_0^{1,2}(\Omega)$ and $h(x) \in L^s(\Omega)$ for some $s > n$. Let us set

$$h(x) = t\phi_1 + \epsilon g(x), \tag{1.2}$$

in such a way that $t \in R$, $\epsilon > 0$ is a small number, $\phi_1(x)$ is a eigenfunction corresponding to the eigenvalue λ_1 of the eigenvalue problem

$$\begin{aligned} -\Delta u = \lambda u \quad & \text{in } \Omega, \\ u = 0 \quad & \text{on } \partial\Omega, \end{aligned} \tag{1.3}$$

and $g \in L^s(\Omega)$ with $\int_\Omega g\phi_1 dx = 0$.

This mixed type nonlinear problem contains the jumping nonlinearity and the subcritical growth nonlinearity. The authors ([1], [2], [4], [5], [9], [10], [11]) considered the jumping nonlinear problem. They investigate the multiplicity results when the constant b of the nonlinear term satisfies $b < \lambda_1$ or $\lambda_k < b < \lambda_{k+1}$, $k \geq 1$. They obtain the multiplicity results by use of the Leray-Schauder degree theory, geometry of the mapping defined on the finite dimensional reduction subspace, mountain pass geometry in the critical point theory, the category theory in the critical point theory. The authors ([3], [6], [7], [8], [11]) also considered the subcritical growth nonlinear problem. They consider the multiplicity results by use of the variational method, the critical point theory and the category theory in the critical point theory.

Let $\lambda_1 < \lambda_2 \leq \ldots \leq \lambda_k \leq \ldots$ be the eigenvalues of the eigenvalue problem (1.3) and ϕ_k the eigenfunction belonging to the eigenvalue λ_k, $k \geq 1$. Let H be a Sobolev space $W_0^{1,2}(\Omega)$ with the norm

$$\|u\|^2 = \int_\Omega |\nabla u(x)|^2 dx.$$

In this paper we are looking for the weak solutions of (1.1) with (1.2) in H, that is, $u \in H$ such that

$$\int_\Omega (\Delta u + bu_+)v dx - p\int_\Omega u_-^{p-1} v - \int_\Omega h(x) v dx = 0 \quad \text{for all} \quad v \in H.$$

Our main result is as follows:

THEOREM 1.1. *Assume that $\lambda_1 < b < \lambda_2$, $t \in R$ and $g(x) \in L^s(\Omega)$ for some $s > n$ with $\int_\Omega g\phi_1 dx = 0$. Then there exists a large number $t_1 > 0$ such that for any t with $t > t_1$, (1.1) with (1.2) has at least two nontrivial solutions.*

The outline of the proof of Theorem 1.1 is as follows: In section 2 we show the existence of a positive solution of (1.1) and the continuity and the *Fréchet* differentiability of the corresponding functional $I(u)$ of (1.1) to approach the variational method. In section 3 we investigate the sub-level sets of the functional F (the functional F is the corresponding functional to find the second solution of (1.1)), the mountain pass geometry of F and find the second nontrivial solution by applying the mountain pass method in the critical point theory, so we prove Theorem 1.1.

2. Existence of a positive solution and the variational approach

LEMMA 2.1. *Let* $u \in H = W_0^{1,2}(\Omega, R)$ *and let* $\|\cdot\|$ *be a Sobolev norm. Then*
(i) $\|u\| \geq C\|u\|_{L^2(\Omega)}$ *for some constant* $C > 0$,
(ii) $\|u\| = 0$ *if and only if* $\|u\|_{L^2(\Omega)} = 0$,
(iii) $-\Delta u \in H$ *implies* $u \in H$,
(iv) Let c *be not an eigenvalue of* $-\Delta$ *and* $f \in W_0^{1,2}(\Omega, R)$. *Then all the solutions of*

$$(-\Delta - c)u = f$$

belong to H.

Proof. (i) and (ii) can be checked easily by the definition of $\|\cdot\|$.
(iii) Let $-\Delta u = f \in W_0^{1,2}(\Omega, R)$. Then f is of the form $f = \sum h_m\phi_m$. Then

$$(-\Delta)^{-1}f = \sum \frac{1}{\lambda_m} h_m\phi_m.$$

We note that for any c, $\{\lambda_m : \lambda_m < |c|\}$ is finite. Thus we have

$$\|(-\Delta)^{-1}f\|^2 = \sum \lambda_m \frac{1}{\lambda_m^2} h_m^2 \leq \sum h_m^2,$$

which means that

$$\|(-\Delta)^{-1}f\| \leq \|f\|_{L^2(\Omega)}.$$

(iv) comes from (iii). □

LEMMA 2.2. *Assume that* $\lambda_1 < b$ *and* b *is not an eigenvalue of* $-\Delta$ *with Dirichlet boundary condition. Then*

$$\Delta u + bu_+ = 0 \quad in \quad H \tag{2.1}$$

has only the trivial solution $u = 0$.

Proof. We note that $u = 0$ is a solution of (2.1). We rewrite (2.1) as

$$(-\Delta - \lambda_1)u = (b - \lambda_1)u_+ + \lambda_1 u_- \quad \text{in} \quad H.$$

We note that $((-\Delta - \lambda_1)u, \phi_1) = 0$. Thus we have

$$\int_\Omega [(b - \lambda_1)u_+ + \lambda_1 u_-]\phi_1 dx = 0. \tag{2.2}$$

Since $\lambda_1 < b$, $(b-\lambda_1)u_+ + \lambda_1 u_-$ is greater than or equal to 0 and strictly greater than zero if u is strictly greater than zero. The only possibility to hold (2.2) is that $u = 0$. That is, $u = 0$ is the only solution of (2.1). □

LEMMA 2.3. *(Existence of a positive solution) Let* $\lambda_1 < b < \lambda_2$ *and* $g(x) \in L^s(\Omega)$, $s > n$ *with* $\int_\Omega g\phi_1 dx = 0$. *Then there exists a large number* $t_1 > 0$ *such that for any* $t > t_1$, *the equation*

$$\Delta u + bu_+ - pu_-^{p-1} = t\phi_1 + \epsilon g(x) \quad in \quad H$$

has a positive solution $u_1(x)$.

Proof. We first consider the following equation

$$\Delta u + bu = t\phi_1 \quad \text{in} \quad H. \tag{2.3}$$

For $t > 0$, the linear equation (2.3) has a unique positive solution $u_*(x) > 0$, which is of the form

$$u_*(x) = \frac{t\phi_1(x)}{b - \lambda_1}.$$

We also consider the equation

$$\Delta u + bu = \epsilon g(x) \quad \text{in} \quad H. \tag{2.4}$$

The linear equation (2.4) has a unique solution $u_{\epsilon g}(x)$. For the given $g(x) \in L^s(\Omega)$, $s > n$ with $\int_\Omega g\phi_1 dx = 0$, we can choose $t_0 > 0$ such that for any $t > t_0$,

$$u_* + u_{\epsilon g} = \frac{t\phi_1}{b - \lambda_1} + u_{\epsilon g} > 0.$$

Next we consider the equation

$$\Delta u - pu_-^{p-1} = 0 \quad \text{in} \quad H. \tag{2.5}$$

Since the operator $-\Delta$ is a positive operator and $-pu_-^{p-1}$ is negative, the solution $\breve{u}$ of (2.5) is $\breve{u} \leq 0$. Thus there exists a large number $t_1 > 0$ such that $u_1 = u_* + u_{\epsilon g} + \breve{u} > 0$ is a positive solution of (1.1) with (1.2) for any t with $t > t_1$. We proved the lemma. □

Now we will try to find the second nontrivial weak solutions of (1.1). By the following Proposition 2.1, the weak solutions of (1.1) coincide with the critical points of the corresponding functional

$$I \in C^1(H, R),$$

$$I(u) = \int_\Omega [\frac{1}{2}|\nabla u|^2 - \frac{b}{2}|u_+|^2 - u_-^p + h(x)u]dx. \tag{2.6}$$

PROPOSITION 2.1. *Assume that $\lambda_1 < b$, b is not an eigenvalue and $h(x) \in L^s$, $s > n$. Then the functional $I(u)$ is continuous, Fréchet differentiable in H with Fréchet derivative*

$$\nabla I(u)v = \int_\Omega [(-\Delta u)\cdot v - bu_+ \cdot v + pu_-^{p-1}\cdot v + h(x)\cdot v]dx.$$

Moreover $DI \in C$. That is, $I \in C^1$.

Proof. First we shall prove that $I(u)$ is continuous at u. For u, $v \in H$,

$$\begin{aligned} |I(u+v) - I(u)| &= |\frac{1}{2}\int_\Omega (-\Delta u - \Delta v)\cdot(u+v)dx \\ &\quad - \int_\Omega [\frac{b}{2}|(u+v)_+|^2 + (u+v)_-^p - h(x)(u+v)]dx \\ &\quad -\frac{1}{2}\int_\Omega (-\Delta u)\cdot udx + \int_\Omega [\frac{b}{2}|u_+|^2 + u_-^p - h(x)u]dx| \\ &= |\frac{1}{2}\int_\Omega (-\Delta u\cdot v - \Delta v\cdot u - \Delta v\cdot v)dx \\ &\quad - \int_\Omega (\frac{b}{2}|(u+v)_+|^2 + (u+v)_-^p - h(x)v \\ &\quad -\frac{b}{2}|u_+|^2 - u_-^p)dx|. \end{aligned}$$

Let $u = \sum h_n\phi_n$, $v = \sum k_n\phi_n$. Then we have

$$|\int_\Omega (-\Delta u)\cdot vdx| = |\sum \lambda_n h_n k_n| \le \|u\|\cdot\|v\|,$$

$$|\int_\Omega (-\Delta v)\cdot udx| = |\sum \lambda_n k_n h_n| \le \|u\|\cdot\|v\|,$$

$$|\int_\Omega (-\Delta v)\cdot vdx| = |\sum \lambda_n k_n k_n| \le \|v\|^2,$$

from which we have

$$(2.7) \qquad |\frac{1}{2}\int_\Omega(-\Delta u\cdot v-\Delta v\cdot u-\Delta v\cdot v)dx|\le\|u\|\cdot\|v\|+\|v\|^2.$$

On the other hand

$$||(u+v)_+|^2-|u_+|^2|\le 2u_+|v|+|v|^2,$$

$$||(u+v)_-|^p-|u_-|^p|\le C_1|u_-^{p-1}||v|+R_2(|u_-|,|v_-|)$$

and hence we have

$$\begin{aligned}(2.8)|\int_\Omega(|(u+v)_+|^2-|u_+|^2)dx| &\le 2\|u_+\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)}+\|v\|^2_{L^2(\Omega)}\\ &\le 2\|u\|\cdot\|v\|+\|v\|^2,\end{aligned}$$

$$\begin{aligned}(2.9) \qquad &|\int_\Omega(|(u+v)_-|^p-|u_-|^p)dx|\\ &\le C_1\|u_-^{p-1}\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)}+R_2(\|u\|_{L^2(\Omega)},\|v\|_{L^2(\Omega)})\\ &\le C_2\|u_-^{p-1}\|\|v\|+R_2(\|u\|,\|v\|)=O(\|v\|).\end{aligned}$$

Combining (2.7) with (2.8) and (2.9), we have

$$|I(u+v)-I(u)|=O(\|v\|)$$

from which we can conclude that $I(u)$ is continuous at u. Next we shall prove that $I(u)$ is *Fréchet* differentiable in H. For $u,v\in H$,

$$\begin{aligned}&|I(u+v)-I(u)-\nabla I(u)v|\\ &=|\frac{1}{2}\int_\Omega(-\Delta u-\Delta v)\cdot(u+v)dx\\ &\quad-\int_\Omega[\frac{b}{2}|(u+v)_+|^2+(u+v)_-^p-h(x)(u+v)]dx\\ &\quad-\frac{1}{2}\int_\Omega(-\Delta u)\cdot udx+\int_\Omega[\frac{b}{2}|u_+|^2+u_-^p-h(x)u]dx\\ &\quad-\int_\Omega(-\Delta u-bu_++pu_-^{p-1}+h(x))\cdot vdx|\\ &=|\int_\Omega[\frac{1}{2}(-\Delta v)\cdot v-\frac{b}{2}|(u+v)_+|^2-(u+v)_-^p\\ &\quad+\frac{b}{2}|u_+|^2+u_-^p+bu_+v-pu_-^{p-1}v]dx|.\end{aligned}$$

Combining (2.7) with (2.8) and (2.9), we have that

$$|I(u+v) - I(u) - \nabla I(u)v| = O(\|v\|).$$

Thus $I(u)$ is *Fréchet* differentiable in H. Similarly, it is easily checked that $I \in C^1$. □

3. Existence of the second solution and proof of theorem 1.1

From now on we shall show the existence of the second nontrivial solution of (1.1) with (1.2) by using the mountain pass geometry in the critical point theory. We notice that the nontrivial weak solution u of (1.1) with (1.2) is of the form $u = u_1 + \bar{u}$, where $u_1 = u_* + u_{\epsilon g} + \check{u}$ is a positive solution of (1.1) with (1.2) and $\bar{u}$ is a nontrivial solution of the equation

$$\Delta u + b(u_1 + u)_+ - bu_1 - p(u_1 + u)_-^{p-1} = 0 \quad \text{in} \quad H. \tag{3.1}$$

We note that the weak solutions of (3.1) coincide with the critical points of the corresponding functional

$$F : H \to R \in C^1,$$

$$F(u) = \int_\Omega [\frac{1}{2}|\nabla u|^2 - \frac{b}{2}|(u_1 + u)_+|^2 + bu_1 u - (u_1 + u)_-^p]dx. \tag{3.2}$$

Thus it suffices to find the nontrivial critical points for F. Now we shall show that F satisfies the mountain pass geometry in the critical point theory. Assume that $\lambda_1 < b < \lambda_2$. Let us set

$$X = \text{span}\{\phi_1(x)\}, \quad Y = X^\perp.$$

Then X is one dimensional subspace and

$$H = X \oplus Y.$$

We have the following inequalities:

LEMMA 3.1. *Assume that* $\lambda_1 < b < \lambda_2$. *Then there exist* $\rho > 0$ *and a small ball* B_ρ *with radius* ρ *such that* $B_\rho \cap Y \neq \emptyset$,

$$\inf_{u \in \partial B_\rho \cap Y} F(u) > 0 \quad \textit{and} \quad \inf_{u \in B_\rho \cap Y} F(u) > -\infty.$$

Proof. We note that

$$\text{if } \ u \in Y, \text{ then } \int_\Omega [-\Delta u \cdot u - \frac{b}{2}u^2]dx \geq \frac{1-\frac{b}{\lambda_2}}{2}\|u\|^2 > 0.$$

Let $u \in Y$. Then we have

$$F(u) = \int_\Omega [\frac{1}{2}|\nabla u|^2 - \frac{b}{2}u^2 + \frac{b}{2}|(u_1+u)_-|^2 - (u_1+u)_-^p]dx$$

$$\geq \frac{1-\frac{b}{\lambda_2}}{2}\|u\|^2 - \int_\Omega (u_1+u)_-^p dx.$$

Let us define

$$C_p(\Omega) = \inf_{u \in H\setminus\{0\}} \frac{\int_\Omega |\nabla(u+u_1)|^2}{(\int_\Omega (u+u_1)_-^p)^{\frac{2}{p}}}.$$

Then we have

$$F(u) \geq \frac{1-\frac{b}{\lambda_2}}{2}\|u\|^2 - (C_p(\Omega))^{-\frac{p}{2}}\|u+u_1\|^p.$$

Since $\lambda_2 - b > 0$ and $p > 2$, there exist $\rho > 0$ and a ball B_ρ with radius ρ such that $\inf_{u \in \partial B_\rho \cap Y} F(u) > 0$ and $\inf_{u \in B_\rho \cap Y} F(u) > -(C_p(\Omega))^{-\frac{p}{2}}\|u + u_1\|^p > -\infty$. □

LEMMA 3.2. *Assume that $\lambda_1 < b < \lambda_2$. Then there exist $e \in \partial B_1 \cap Y$ and $Q \equiv (\bar{B}_R \cap X) \oplus \{re|\ 0 < r < R\}$ such that*

$$\sup_{u \in \partial Q} F(u) < 0 \ \ \textit{and} \ \ \sup_{u \in Q} F(u) < \infty.$$

Proof. Let $u \in X \oplus \{re|\ r > 0\}$, $u = v + re$, $v \in X$, $e \in \partial B_1 \cap Y$. We note that

$$\text{if } \ u \in X, \text{ then } \int_\Omega [-\Delta u \cdot u - \frac{b}{2}u^2]dx \leq \frac{1-\frac{b}{\lambda_1}}{2}\|u\|^2 < 0.$$

For $s > 0$ we have

$$\begin{aligned} F(su) &= s^2(\int_\Omega [\frac{1}{2}|\nabla(v+re)|^2 - \frac{b}{2}(v+re)^2 + \frac{b}{2}|(v+re+\frac{u_1}{s})_-|^2]dx \\ &\qquad -s^p \int_\Omega (v+re+\frac{u_1}{s})_-^p]dx \\ &\le \frac{s^2(1-\frac{b}{\lambda_1})}{2}\|v\|^2 + \frac{s^2(1-\frac{b}{\lambda_n})}{2}r^2 + \frac{s^2 b}{2}\int_\Omega |(v+re+\frac{u_1}{s})_-|^2 dx \\ &\qquad -s^p \int_\Omega (v+re+\frac{u_1}{s})_-^p]dx \end{aligned}$$

for some $\lambda_n \geq \lambda_2$. Since $p > 2$, $F(su) = F(s(v+re)) \to -\infty$ as $s \to \infty$. Thus there exist $R > 0$, a ball B_R and $Q \equiv (\bar{B}_R \cap X) \oplus \{re|\ 0 < r < R\}$ such that if $u \in \partial Q$, then $\sup F(u) < 0$. Moreover if $u \in Q$ then $\sup F(u) < \frac{s^2(1-\frac{b}{\lambda_n})}{2}r^2 + \frac{s^2 b}{2}\int_\Omega |(v+re+\frac{u_1}{s})_-|^2 dx < \infty$. Thus we prove the lemma. □

Lemma 3.3. *Assume that $\lambda_1 < b < \lambda_2$. Then F satisfies the $(P.S.)_c$ condition for every real number $c \in R$.*

Proof. Let $c \in R$ and $(u_n)_n$ be a sequence such that

$$u_n \in H,\ \forall n,\ F(u_n) \to c,\ \nabla F(u_n) \to 0.$$

We claim that $(u_n)_n$ is bounded. By contradiction we suppose that $\|u_n\| \to +\infty$ and set $\hat{u_n} = \frac{u_n}{\|u_n\|}$. Then we have

$$\begin{aligned} \langle \nabla F(u_n), \hat{u_n} \rangle &= \frac{2F(u_n)}{\|u_n\|} - \frac{\int_\Omega [bu_n + b(u_n+u_1)_- - p(u_n+u_1)_-^{p-1}] \cdot u_n dx}{\|u_n\|} \\ &\qquad + \frac{2\int_\Omega [\frac{b}{2}u^2 - \frac{b}{2}|(u_n+u_1)_-|^2 + (u_n+u_1)_-^p]dx}{\|u_n\|} \\ &\longrightarrow 0. \end{aligned}$$

Hence

$$\begin{aligned} &\frac{\int_\Omega [bu_n + b(u_n+u_1)_- - p(u_n+u_1)_-^{p-1}] \cdot u_n dx}{\|u_n\|} \\ &\qquad - \frac{2\int_\Omega [\frac{b}{2}u_n^2 - \frac{b}{2}|(u_n+u_1)_-|^2 + (u_n+u_1)_-^p]dx}{\|u_n\|} \longrightarrow 0. \end{aligned}$$

Thus

$$\frac{1}{\|u_n\|}[\int_\Omega [b(u_n+u_1)_- - p(u_n+u_1)_-^{p-1}]\cdot u_n dx + \int_\Omega [b|(u_n+u_1)_-|^2 - 2(u_n+u_1)_-^p]dx] \longrightarrow 0.$$

We claim that

$$\lim_{n\to\infty}\int_\Omega \frac{b(u_n+u_1)_-\cdot u_n dx + b|(u_n+u_1)_-|^2}{\|u_n\|}dx = 0.$$

In fact,

$$\begin{aligned}
&\lim_{n\to\infty}\int_\Omega \frac{b(u_n+u_1)_-\cdot u_n dx + b|(u_n+u_1)_-|^2}{\|u_n\|}dx \\
&\le \lim_{n\to\infty}\int_\Omega \frac{b(u_n+u_1)_-\cdot (u_n+u_1) dx + b|(u_n+u_1)_-|^2}{\|u_n\|}dx \\
&= \lim_{n\to\infty}\int_\Omega \frac{-b(u_n+u_1)_-\cdot (u_n+u_1)_- dx + b|(u_n+u_1)_-|^2}{\|u_n\|}dx = 0.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\lim_{n\to\infty}\int_\Omega \frac{b(u_n+u_1)_-\cdot u_n dx + b|(u_n+u_1)_-|^2}{\|u_n\|}dx \\
&= \lim_{n\to\infty}\int_\Omega b(u_n+u_1)_-\cdot \hat{u_n} dx + b(u_n+u_1)_-\cdot (\hat{u_n}+\frac{u_1}{\|u_n\|})_- dx \\
&= \lim_{n\to\infty}\int_\Omega b(u_n+u_1)_-\cdot \hat{u_n} dx + b((u_n+u_1)_-)\cdot (\hat{u_n})_- dx \\
&= \lim_{n\to\infty}\int_\Omega b(u_n+u_1)_-\cdot (\hat{u_n})_+ dx \ge 0.
\end{aligned}$$

Thus we prove the claim. Therefore we have

$$
\begin{aligned}
0 \quad \longleftarrow \quad & \frac{1}{\|u_n\|}[\int_\Omega [b(u_n+u_1)_- - p(u_n+u_1)_-^{p-1}] \cdot u_n dx \\
& \qquad + \int_\Omega [b|(u_n+u_1)_-|^2 - 2(u_n+u_1)_-^p]dx] \\
= \quad & \frac{1}{\|u_n\|}[\int_\Omega (-p(u_n+u_1)_-^{p-1}) \cdot u_n dx - 2\int_\Omega (u_n+u_1)_-^p dx] \\
\geq \quad & \frac{1}{\|u_n\|}[\int_\Omega (-p(u_n+u_1)_-^{p-1}) \cdot (u_n+u_1) dx - 2\int_\Omega (u_n+u_1)_-^p dx] \\
= \quad & \frac{1}{\|u_n\|}[\int_\Omega (p(u_n+u_1)_-^{p-1}) \cdot (u_n+u_1)_- dx - 2\int_\Omega (u_n+u_1)_-^p dx] \\
= \quad & (p-2)\frac{\int_\Omega (u_n+u_1)_-^p dx}{\|u_n\|} = (p-2)\frac{\|(u_n+u_1)_-\|_{L^p(\Omega)}^p}{\|u_n\|}.
\end{aligned}
$$

Since $p > 2$,

$$
\frac{\|(u_n+u_1)_-\|_{L^p(\Omega)}^p}{\|u_n\|} \quad \text{converges to } 0.
$$

On the other hand

$$
\|p(u_n+u_1)_-^{p-1}\| \leq C_1\|(u_n+u_1)_-^{p-1}\|_{L^{2^{*'}}(\Omega)} \tag{3.3}
$$

for suitable constant C_1. Then we have

$$
\|\frac{bu_n + b(u_n+u_1)_- - p(u_n+u_1)_-^{p-1}}{\|u_n\|}\| \leq 2b + C_1\|\frac{(u_n+u_1)_-^{p-1}}{\|u_n\|}\|_{L^{2^{*'}}(\Omega)}.
$$

If $p \geq 2^{*'}(p-1)$, then by the $H\ddot{o}lder's$ inequality, it is easily checked that $\|\frac{(u_n+u_1)_-^{p-1}}{\|u_n\|}\|_{L^{2^{*'}}(\Omega)}$ can be estimated in terms of $\frac{\|(u_n+u_1)_-\|_{L^p(\Omega)}^p}{\|u_n\|}$. If $p \leq 2^{*'}(p-1)$, then by the standard interpolation inequalities,

$$
\begin{aligned}
&\|\frac{(u_n+u_1)_-^{p-1}}{\|u_n\|}\|_{L^{2^{*'}}(\Omega)} \\
&\qquad \leq C_2(\frac{\|(u_n+u_1)_-\|_{L^p(\Omega)}^p}{\|u_n\|})^{\frac{(p-1)\alpha}{p}}\|(u_n+u_1)_-\|^\beta
\end{aligned}
$$

for some constant C_2, where $\alpha > 0$ is such that $\frac{\alpha}{p} + \frac{1-\alpha}{2^*} = \frac{1}{2^{*'}}$ and $\beta = (1-\alpha)(p-1) - 1 - \frac{(p-1)\alpha}{p}$. Since $p-1 \leq 2^* - 1 - (2^*-p)(1-\frac{2^{*'}}{2^*})$,

$\beta < 0$. Thus we have

$$\|\frac{p(u_n+u_1)_-^{p-1}}{\|u_n\|}\| \le C_2(\frac{\|(u_n+u_1)_-\|_{L^p(\Omega)}^p}{\|u_n\|})^{\frac{(p-1)\alpha}{p}}\|(u_n+u_1)_-\|^\beta \quad \text{and}$$

$$\|\frac{bu_n+b(u_n+u_1)_- - p(u_n+u_1)_-^{p-1}}{\|u_n\|}\|$$
$$\le 2b + C_2(\frac{\|(u_n+u_1)_-\|_{L^p(\Omega)}^p}{\|u_n\|})^{\frac{(p-1)\alpha}{p}}\|(u_n+u_1)_-\|^\beta$$

for a constant C_2. Since $\frac{\|(u_n+u_1)_-\|^p}{\|u_n\|}$ converges to 0 and $\beta < 0$,

$$\frac{p(u_n+u_1)_-^{p-1}}{\|u_n\|} \text{ converges to } 0 \text{ and} \tag{3.4}$$

$$\frac{bu_n+b(u_n+u_1)_- - p(u_n+u_1)_-^{p-1}}{\|u_n\|} \text{ converges.} \tag{3.5}$$

By (3.4),

$$\int_\Omega \frac{(u_n+u_1)_-^{p-1}}{\|u_n\|}dx = \int_\Omega (\hat{u_n} + \frac{u_1}{\|u_n\|})_-^{p-1}\|u_n\|^p dx \longrightarrow 0.$$

Thus $\hat{u_n} \rightharpoonup 0$. We get

$$\frac{\nabla F(u_n)}{\|u_n\|} = -\Delta\hat{u_n} - \frac{bu_n+b(u_n+u_1)_- - p(u_n+u_1)_-^{p-1}}{\|u_n\|} \longrightarrow 0.$$

By (3.5), $-\Delta\hat{u_n}$ converges. Since $(\hat{u_n})_n$ is bounded and the inverse operator of $-\Delta$ is a compact mapping, up to subsequence, $(\hat{u_n})_n$ has a limit. Since $\hat{u_n} \rightharpoonup 0$, we get $\hat{u_n} \to 0$, which is a contradiction to the fact that $\|\hat{u_n}\| = 1$. Thus $(u_n)_n$ is bounded. We can now suppose that $u_n \rightharpoonup u$ for some $u \in H$. We claim that $u_n \to u$ strongly. We have that

$$\langle \nabla F(u_n), u_n\rangle$$
$$= (\|u_n\|^2 - \int_\Omega [b(u_n+u_1)_+u_n - bu_1 - p(u_n+u_1)_-^{p-1}u_n]dx) \longrightarrow 0.$$

Since $\int_\Omega [b(u_n+u_1)_+u_n - bu_1 - p(u_n+u_1)_-^{p-1}u_n]dx \longrightarrow \int_\Omega [b(u+u_1)_+u - bu_1 - p(u+u_1)_-^{p-1}u]dx$, $\|u_n\|^2$ converge. Thus $(u_n)_n$ converges to some u strongly with $\nabla F(u) = \lim \nabla F(u_n) = 0$. Thus we prove the lemma. □

[Proof of Theorem 1.1]
Let $X = \text{span}\{\phi_1(x)\}$ and $Y = X^{\perp}$. Then X is a one dimensional subspace and $H = X \oplus Y$. From Proposition 2.1 we can deduce that the functional F belong to $C^1(H, R^1)$. By Lemma 3.1 and Lemma 3.2, there exist $\rho > 0$, a small ball B_ρ with radius ρ, $e \in \partial B_1 \cap Y$ and $Q \equiv (\bar{B}_R \cap X) \oplus \{re|\ 0 < r < R\}$ such that $B_\rho \cap Y \neq \emptyset$,

$$\sup_{u\in\partial Q} F(u) < \inf_{u\in\partial B_\rho \cap Y} F(u)$$

and

$$\sup_{u\in Q} F(u) < \infty \quad \text{and} \quad -\infty < \inf_{u\in B_\rho \cap Y} F(u).$$

By Lemma 3.3, the functional $F(u)$ satisfies the $(P.S.)_c$ condition for any $c \in R$. Let us set

$$\Gamma = \{\gamma \in C(\bar{Q}, H)|\ \gamma = id \text{ on } \partial Q\}.$$

Then by the mountain pass theorem, F possesses a critical value $c \geq 0$ which can be characterized as

$$c = \inf_{\gamma\in\Gamma} \max_{u\in Q} F(\gamma(u)).$$

Thus (3.1) has a nontrivial solution $\bar{u}$, so (1.1) has at least two nontrivial solutions, one of which is a positive solution u_1 and the other solution is of the form $u = u_1 + \bar{u}$.

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