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INTERVAL-VALUED FUZZY GENERALIZED BI-IDEALS OF A SEMIGROUP

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Abstract. We introduce the concept of an interval-valued fuzzy generalized bi-ideal of a semigroup, which is an extension of the concept of an interval-valued fuzzy bi-ideal (and of a noninterval-valued fuzzy bi-ideal and a noninterval-valued fuzzy ideal of a semigroup), and characterize regular semigroups, and both intraregular and left quasiregular semigroup in terms of interval-valued fuzzy generalized bi-ideals.

1. Introduction

In 1975, Zadah[12] introduced the concept of an interval-valued fuzzy set as the generalization of a fuzzy set introduced by himself[11]. After then, Biswas[1] applied it to group theory, Garzalczany[6] proposed a method of inference based on interval-valued fuzzy sets, Roy and Biswas[10] studied interval-valued fuzzy relations, and Mondal and Samonta applied it to topology. Recently, Choi et al.[4] investigated interval-valued smooth topological spaces, Hur et al.[7] studied interval-valued fuzzy relations in the sense of lattice theory, and Kang and Hur[7] applied the concept of an interval-valued fuzzy set to algebra. Furthermore, Cheong and Hur[3] invesstigated interval-valued fuzzy ideals and bi-ideals in a semigroup.

In this paper, we will introduce the concept of an interval-valued fuzzy generalized bi-ideal of a semigroup, which is an extension of the notion

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of a noninterval-valued fuzzy generalized bi-ideal (and of a noninterval-valued fuzzy bi-ideal and a noninterval-valued fuzzy ideal), and characterize such semigroups by interval-valued fuzzy generalized bi-ideals.

2. Preliminaries

We will list some concepts needed in the later sections.

Throughout this paper, we will denote the unit interval $[0, 1]$ as I and for an ordinary subset A of a set X, we will denote the characteristic function of A as χ_A.

Definition 2.1. [6, 9, 12] *Let X be a given nonempty set. A mapping $A = [A^L, A^U] : X \to D(I)$ is called an* interval-valued fuzzy set *(briefly, IVFS) in X, where A^L and A^U are fuzzy sets in X satisfying $A^L(x) \leq A^U(x)$ and $A(x) = [A^L(x), A^U(x)]$ for each $x \in X$. In particular,* $\mathbf{0}$ *and* $\mathbf{1}$ *denote the* interval-valued fuzzy empty set *and the* interval-valued fuzzy whole set *in X defined by $\mathbf{0}(x) = [0, 0]$ and $\mathbf{1}(x) = [1, 1]$ for each $x \in X$, respectively.*

We will denote the set of all IVFSs in X as $D(I)^X$.

Notation. Let $X = \{x_1, x_2, \cdots, x_n\}$. Then $A = ([a_1, b_1], [a_2, b_2], \cdots, [a_n, b_n])$ denotes an IVFS in X such that $A^L(x_i) = a_i$ and $A^U(x_i) = b_i$, for all $i = 1, 2, \cdots, n$.

Definition 2.2. [9] *Let X be a nonempty set and let $A, B \in D(I)^X$. Then*

(a) *$A \subset B$ iff $A^L(x) \leq B^L(x)$ and $A^U(x) \leq B^U(x)$ for all $x \in X$.*
(b) *$A = B$ iff $A \subset B$ and $B \subset A$.*
(c) *The complement A^c of A is defined by $A^c(x) = [1 - A^U(x), 1 - A^L(x)]$ for all $x \in X$.*
(d) *If $\{A_i : i \in J\}$ is an arbitrary subset of $D(I)^X$, then*

$$\bigcap A_i(x) = [\bigwedge_{i \in J} A_i^L(x), \bigwedge_{i \in J} A_i^U(x)],$$

$$\bigcup A_i(x) = [\bigvee_{i \in J} A_i^L(x), \bigvee_{i \in J} A_i^U(x)].$$

3. Interval-valued fuzzy ideals of a semigroup

Definition 3.1. [7] *Let $(X, \circ)$ be a groupoid and let $A, B \in D(I)^X$. Then the interval-valued fuzzy product of A and B, $A \circ B$ is defined as follows : For each $x \in X$,*

$$A \circ B(x) = \begin{cases} \left[\bigvee_{x=yz}(A^L(y) \wedge B^L(z)),\ \bigvee_{x=yz}(A^U(y) \wedge B^U(z))\right] & \text{if } x = yz, \\ [0,0] & \text{otherwise.} \end{cases}$$

It is clear that for any $A, B, C \in D(I)^X$, if $B \subset C$, then $A \circ B \subset A \circ C$ and $B \circ A \subset C \circ A$.

Let S be a semigroup. By a subsemigroup of S we mean a non-empty subset of A of such that $A^2 \subset A$ and by a *left* [resp. *right*] *ideal* of S we mean a non-empty subset A of S such that $SA \subset A$[resp. $AS \subset A$]. By *two-sided ideal* or simply *ideal* we mean a subset A of S which is both a left and a right ideal of S. We well denote the set of all left ideals [resp. right ideals and ideals] of S as LI(S) [resp. RI(S) and I(S)].

Definition 3.2. *Let S be a semigroup and let $A \in D(I)^S$. Then A is called an :*

(1) interval-valued fuzzy subsemigroup *(in short,* IVSG*) of S if*

$$A^L(xy) \geq A^L(x) \wedge A^L(y) \text{ and } A^U(xy) \geq A^U(x) \wedge A^U(y)$$

for any $x, y \in S$,

(2) interval-valued fuzzy left ideal *(in short,* IVLI*) of S if*

$$A^L(xy) \geq A^L(y) \text{ and } A^U(xy) \geq A^U(y)$$

for any $x, y \in S$,

(3) interval-valued fuzzy right ideal *(in short,* IVRI*) of S if*

$$A^L(xy) \geq A^L(x) \text{ and } A^U(xy) \geq A^U(x)$$

for any $x, y \in S$,

(4) interval-valued fuzzy (two-sided) ideal *(in short,* IVI*) of S if it is both an interval-valued fuzzy left and an interval-valued fuzzy right ideal of S.*

We will denote the set of all IVSGs [resp. IVLIs, IVRIs and IVIs] of S as IVSG(S) [resp. IVLI(S), IVRI(S) and IVI(S)]. It is clear that $A \in$ IVI(S) if and only if $A^L(xy) \geq A^L(x) \wedge A^L(y)$ and $A^U(xy) \geq A^U(x) \wedge A^U(y)$ for any $x, y \in S$, and if $A \in$ IVLI(S) [resp. IVRI(S) and

IVI(S)], then $A \in$ IVSG(S).

The following is the immediate result of Definition 3.1 and 3.2.

Theorem 3.3. *Let S be a semigroup and let $\widetilde{0} \neq A \in D(I)^S$. Then $A \in$ IVSG(S) if and only if $A \circ A \subset A$.*

Result 3.A. [3, Theorem 3.2] Let A be a nonempty subset of a semigroup S. Then, A is a subsemigroup of S if and only if $[\chi_A, \chi_A] \in$ IVSG(S).

Result 3.B. [7, Proposition 6.6] Let A be a nonempty subset of a ring R. Then $A \in$ LI(R) [resp. RI(R) and $I(R)$] if and only if $[\chi_A, \chi_A] \in$ IVLI(R) [resp. IVRIR and IVI(R)].

Lemma 3.4. *Let S be a semigroupiod let $A \in D(I)^S$. Then $A \in$ IVLI(S) if and only if $\mathbf{1} \circ A \subset A$.*

Proof. ($\Rightarrow$) : Suppose $A \in$ IVLI(S) and let $a \in S$.

Case (i) : Suppose $(\widetilde{1} \circ A)(a) = [0, 0]$. Then clearly $\widetilde{1} \circ A \subset A$.

Case (ii) : Suppose $(\widetilde{1} \circ A)(a) \neq [0, 0]$. Then there exist $x, y \in S$ with $a = xy$. Thus

$$\begin{aligned}
(\widetilde{1} \circ A)^L(a) &= \bigvee_{a=xy} (\widetilde{1}^L(x) \wedge A^L(y)) \\
&\leq \bigvee_{a=xy} (1 \wedge A^L(xy)) \text{ (Since } A \in \text{IVLI}(S)) \\
&= \bigvee_{a=xy} (1 \wedge A^L(a)) = A^L(a).
\end{aligned}$$

Similarly, we have $(\widetilde{1} \circ A)^U(a) \leq A^U(a)$. Hence, in all, $\widetilde{1} \circ A \subset A$.

($\Leftarrow$) : Suppose the necessary condition holds. Let $A \in D(I)^S$ and let $a = xy$ for any $x, y \in S$. Then, by the hypothesis, $\widetilde{1} \circ A \subset A$. Thus

$$\begin{aligned}
A^L(xy) = A^L(a) \geq (\widetilde{1} \circ A)^L(a) &= \bigvee_{a=bc} (\widetilde{1}^L(b) \wedge A^L(c)) \\
&\geq \widetilde{1}^L(x) \wedge A^L(y) \text{ (Since } a = xy) \\
&= 1 \wedge A^L(y) = A^L(y).
\end{aligned}$$

Similarly, we have $A^U(xy) \geq A^U(y)$. Hence $A \in$ IVLI(S). This completes the proof. □

Lemma 3.4′ [The dual of Lemma 3.4] *Let S be a semigroup and let $A \in D(I)^S$. Then $A \in \mathrm{IVRI}(S)$ if and only if $A \circ \widetilde{1} \subset A$.*

The combined effect of these two lemmas is as follows :

Theorem 3.5. *Let S be a semigroup and let $A \in D(I)^S$. Then $A \in \mathrm{IVI}(S)$ if and only if $\widetilde{1} \circ A \subset A$ and $A \circ \widetilde{1} \subset A$.*

4. Interval-valued fuzzy generalized bi-ideals

A subsemigroup A of a semigroup S is called a *bi-ideal* of S if $ASA \subset A$. We will denote the set of all bi-ideals of S as BI (S).

Definition 4.1. [3] *Let S be a semigroup and let $A \in \mathrm{IVSG}(S)$. Then A is called an* interval-valued fuzzy bi-ideal *(in short,* IVBI*) of S if*

$$A^L(xyz) \geq A^L(x) \wedge A^L(z) \text{ and } A^U(xyz) \geq A^U(x) \wedge A^U(z)$$

for any $x, y, z \in S$.

We will denote the set of all IVBIs of S as IVBI(S).

Result 4.A. [3, Theorem 3.7] Let A be a non-empty subset of a semigroup S. Then $A \in \mathrm{BI}(S)$ if and only if $[\chi_A, \chi_A] \in \mathrm{IVBI}(S)$.

Remark 4.2. Let S be a semigroup.

(a) If A is a fuzzy left ideal [resp. right ideal and bi-ideal] of S, then $A = [A, A] \in \mathrm{IVLI}(S)$ [resp. IVRI(S), IVI(S) and IVBI(S)].
(b) If $A \in \mathrm{IVBI}(S)$, then A^L and A^U are fuzzy bi-ideals of S.

A nonempty subset A of a semigroup S is called a *generalized bi-ideal* [8] if $ASA \subset A$. We will denote the set of all generalized bi-ideals of S as GBI(S).

Definition 4.3. *Let S be a semigroup and let $A \in D(I)^S$. Then A is called an* interval-valued fuzzy generalized bi-ideal *(in short,* IVGBI*) of S if for any $x, y, z \in S$, $A^L(xyz) \geq A^L(x) \wedge A^L(z)$ and $A^U(xyz) \geq A^U(x) \wedge A^U(z)$.*

We will denote the set of all IVGBIs of S as IVGBI(S). It is clear that $\mathrm{IVBI}(S) \subset \mathrm{IVGBI}(S)$. But the converse inclusion does not hold in general.

Example 4.4. Let $S = \{a, b, c, d\}$ be the semigroup with the following multiplication table:

	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

We define a mapping $A : S \to D(I)$ as follows :

$$A(a) = [0.4, 0.5],\ A(b) = [0, 0],\ A(c) = [0.2, 0.8],\ A(d) = [0, 0].$$

Then we can easily show that $A \in \mathrm{IVGBI}(S)$ but $A \notin \mathrm{IVBI}(S)$. □

Remark 4.5. Let S be a semigroup.

(a) If A is a fuzzy generalized bi-ideal of S, then $[A, A] \in \mathrm{IVGBI}(S)$.
(b) If $A \in \mathrm{IVGBI}(S)$, then A^L and A^U are fuzzy generalized bi-ideals of S.

The following two lemmas are easily seen.

Lemma 4.6. *Let A be a nonempty subset of a semigroup S. Then $A \in \mathrm{GBI}(S)$ if and only if $[\chi_A, \chi_A] \in \mathrm{IVGBI}(S)$.*

Lemma 4.7. *Let S be a semigroup and let $A \in D(I)^S$. Then $A \in \mathrm{IVGBI}(S)$ if and only if $A \circ \widetilde{1} \circ A \subset A$.*

5. Regular semigroups

A semigroup S is said to be *regular* if for each $a \in S$, there exists an $x \in S$ such that $a = axa$.

Proposition 5.1. *Let S be a regular semigroup. Then* $\mathrm{IVGBI}(S) \subset \mathrm{IVBI}(S)$.

Proof. Let $A \in \mathrm{IVGBI}(S)$ and let $a, b \in S$. Since S is regular, there exists an $x \in S$ such that $b = bxb$. Then $A^L(ab) = A^L(a(bxb)) = A^L(a(bx)b) \geq A^L(a) \wedge A^L(b)$. Similarly, we have $A^U(ab) \geq A^U(a) \wedge A^U(b)$. Thus $A \in \mathrm{IVSG}(S)$. So $A \in \mathrm{IVBI}(S)$. Hence $\mathrm{IVGBI}(S) \subset \mathrm{IVBI}(S)$. □

Theorem 5.2. *Let S be a semigroup. Then S is regular if and only if $A = A \circ \widetilde{1} \circ A$ for each $A \in \mathrm{IVGBI}(S)$.*

Proof. ($\Rightarrow$) : Suppose S is regular. Let $A \in$ IVGBI(S) and let $a \in S$. Since S is regular, there exists an $x \in S$ such that $a = axa$. Then

$$\begin{aligned}(A \circ \widetilde{1} \circ A)^L(a) &= \bigvee_{a=yz} ((A \circ \widetilde{1})^L(y) \wedge A^L(z)) \\ &\geq (A \circ \widetilde{1})^L(ax) \wedge A^L(a) \text{ (Since } a = axa) \\ &= \left(\bigvee_{ax=pq} A^L(p) \wedge \widetilde{1}^L(q) \right) \wedge A^L(a) \\ &\geq A^L(a) \wedge \widetilde{1}^L(x) \wedge A^L(a) \\ &= A^L(a) \wedge 1 \wedge A^L(a) = A^L(a).\end{aligned}$$

Similarly, we have $(A \circ \widetilde{1} \circ A)^U(a) \geq A^U(a)$. Thus $A \subset A \circ \widetilde{1} \circ A$. Since $A \in IVGBI(S)$, by Lemma 4.7, $A \circ \widetilde{1} \circ A \subset A$. Hence $A = A \circ \widetilde{1} \circ A$.

($\Leftarrow$) : Suppose the necessary condition holds. Let $A \in$ GBI(S). There, by Lemma 4.6, $[\chi_A, \chi_A] \in$ IVGBI(S). Thus, by the hypothesis, $[\chi_A, \chi_A] \circ \widetilde{1} \circ [\chi_A, \chi_A] = [\chi_A, \chi_A]$. Let $a \in S$. Then

$$([\chi_A, \chi_A] \circ \widetilde{1} \circ [\chi_A, \chi_A])^L(a) = \bigvee_{a=yz} (([\chi_A, \chi_A] \circ \widetilde{1})^L(y) \wedge \chi_A(z)) = \chi_A(a) = 1.$$

Similarly, we have $([\chi_A, \chi_A] \circ \widetilde{1} \circ [\chi_A, \chi_A])^U(a) = 1$. Thus there exist $b, c \in S$ with $a = bc$ such that $([\chi_A, \chi_A] \circ \widetilde{1})^L(b) = \chi_A(c) = 1$ and $([\chi_A, \chi_A] \circ \widetilde{1})^U(b) = \chi_A(c) = 1$. So $\bigvee_{b=pq}(\chi_A(p) \wedge \widetilde{1}^L(q)) = 1$ and $\bigvee_{b=pq}(\chi_A(p) \wedge \widetilde{1}^U(q)) = 1$. Then there exist $d, e \in S$ with $b = de$ such that $\chi_A(d) = \widetilde{1}^L(e) = 1$ and $\chi_A(d) = \widetilde{1}^U(e) = 1$. Thus $d \in A, e \in S, c \in S$ and $a = bc = (de)c \in ASA$. So $A \subset ASA$. Since $A \in$ GBI(S), it is clear that $ASA \subset A$. Hence $A = ASA$. Therefore A is regular. This completes the proof. □

The following result is due to Lemma 4.7 and Theorem 5.2.

Theorem 5.3. *A semigroup S is regular if and only if* IVGBI(S) *is a regular semigroup.*

Theorem 5.4. *A semigroup S is regular if and only if for each $A \in$* IVGBI(S) *and each $B \in IVI(S)$, $A \cap B = A \circ B \circ A$.*

Proof. ($\Rightarrow$) : Suppose S is regular. Let $A \in$ IVGBI(S) and let $B \in$ IVI(S). Then, by Lemma 4.7, $A \circ B \circ A \subset A \circ \widetilde{1} \circ A \subset A$. Also, $A \circ B \circ A \subset \widetilde{1} \circ B \circ \widetilde{1} \subset \widetilde{1} \circ B \subset B$. So $A \circ B \circ A \subset A \cap B$. Now let $a \in S$.

Since S is regular, there exists an $x \in S$ such that $a = axa(= axaxa)$. Since $B \in \mathrm{IVI}(S)$,

$$B^L(xax) \geq B^L(ax) \geq B^L(a) \text{ and } B^U(xax) \geq B^U(ax) \geq B^U(a).$$

Then

$$\begin{aligned}(A \circ B \circ A)^L(a) &= \bigvee_{a=yz} (A^L(y) \wedge (B \circ A)^L(z)) \\ &\geq A^L(a) \wedge (B \circ A)^L(xaxa) \text{ (Since } a = axaxa) \\ &= A^L(a) \wedge \left(\bigvee_{xaxa=pq} \left(B^L(p) \wedge A^L(p)\right) \right) \\ &\geq A^L(a) \wedge B^L(xax) \wedge A^L(a) \\ &\geq A^L(a) \wedge B^L(a) \wedge A^L(a) \\ &= A^L(a) \wedge B^L(a) = (A \cap B)^L(a).\end{aligned}$$

By the similar arguments, we have that $(A \circ B \circ A)^U(a) \geq (A \cap B)^U(a)$. So $A \cap B \subset A \circ B \circ A$. Hence $A \circ B \circ A = A \cap B$.

($\Leftarrow$) : Suppose the necessary condition holds. It is clear that $\mathbf{1} \in \mathrm{IVI}(S)$. Let $A \in \mathrm{IVGBI}(S)$. Then, by the hypothesis, $A = A \cap \widetilde{1} = A \circ \widetilde{1} \circ A$. Hence, by Theorem 5.2, S is regular. This completes the proof. □

Result 5.A. [9, Theorems 1 and 4] Let S be a semigroup. Then the following are equivalent:

(a) S is regular.
(b) $A \cap L \subset AL$ for each $A \in \mathrm{GBI}(S)$ and each $L \in \mathrm{LI}(S)$.
(c) $R \cap A \cap L \subset RAL$ for each $A \in \mathrm{GBI}(S)$, each $L \in \mathrm{LI}(S)$ and each $R \in \mathrm{RI}(S)$.

Now we give a characterization of a regular semigroup in terms of interval-valued fuzzy generalized bi-ideals and interval-valued fuzzy bi-ideals.

Theorem 5.5. *Let S be a semigroup. Then the following are equivalent:*

(a) *S is regular.*
(b) *$A \cap B \subset A \circ B$ for each $A \in \mathrm{IVBI}(S)$ and each $B \in \mathrm{IVLI}(S)$.*
(c) *$A \cap B \subset A \circ B$ for each $A \in IVGBI(S)$ and each $B \in \mathrm{IVLI}(S)$.*
(d) *$C \cap A \cap B \subset C \circ A \circ B$ for each $A \in \mathrm{IVBI}(S)$, each $B \in \mathrm{IVLI}(S)$ and each $C \in \mathrm{IVRI}(S)$.*

(e) $C \cap A \cap B \subset C \circ A \circ B$ *for each* $A \in \text{IVGBI}(S)$ *and each* $B \in \text{IVRI}(S)$.

Proof. (a) ⇒ (b): Suppose S is regular. Let $A \in \text{IVBI}(S)$, let $B \in \text{IVLI}(S)$ and let $a \in S$. Since S is regular, there exists an $x \in S$ such that $a = axa$. Then $(A \circ B)(a) \neq [0, 0]$. Thus

$$\begin{aligned}(A \circ B)^L(a) &= \bigvee_{a=yz} (A^L(y) \wedge B^L(z)) \\ &\geq A^L(a) \wedge B^L(xa) \text{ (Since } a = axa) \\ &\geq A^L(a) \wedge B^L(a) \text{ (Since } B \in \text{IVFLI}(S)) \\ &= (A \cap B)^L(a).\end{aligned}$$

Similarly, we have $(A \circ B)^U(a) \geq (A \cap B)^U(a)$. Hence $A \cap B \subset A \circ B$.

(b ⇒ (c): It is clear.

(c) ⇒ (a): Suppose the condition (c) holds. Let $A \in \text{GBI}(S)$, let $L \in \text{LI}(S)$ and let $a \in A \cap L$. Then $a \in A$ and $a \in L$. Since $A \in \text{GBI}(S)$, by Lemma 4.6, $[\chi_A, \chi_A] \in \text{IVGBI}(S)$. By Result 3.B, $[\chi_L, \chi_L] \in \text{IVLI}(S)$. Thus, by the hypothesis, $[\chi_A, \chi_A] \cap [\chi_L, \chi_L] \subset [\chi_A, \chi_A] \circ [\chi_L, \chi_L]$. So

$$([\chi_A, \chi_A] \circ [\chi_L, \chi_L])^L(a) \geq ([\chi_A, \chi_A] \cap [\chi_L, \chi_L])^L(a) = \chi_A(a) \wedge \chi_L(a) = 1.$$

Similarly, we have that $([\chi_A, \chi_A] \circ [\chi_L, \chi_L])^U(a) \geq 1$. Then

$$([\chi_A, \chi_A] \circ [\chi_L, \chi_L])(a) \neq [0, 0].$$

Thus

$$\bigvee_{a=yz} (\chi_A(y) \wedge \chi_L(z)) = 1 \text{ and } \bigvee_{a=yz} (\chi_A(y) \wedge \chi_L(z)) = 1.$$

So there exist $b, c \in S$ with $a = bc$ such that $\chi_A(b) = 1$ and $\chi_L(c) = 1$. Thus $b \in A$ and $c \in L$, i.e., $a = bc \in AL$. So $A \cap L \subset AL$. Hence, by Result 5.A, S is regular.

(a) ⇒ (d): Suppose S is regular. Let $A \in \text{IVBI}(S)$, let $B \in \text{IVLI}(S)$ and let $C \in \text{IVRI}(S)$. Since S is regular, there exists an $x \in S$ such that

$a = axa$. Then

$$
\begin{aligned}
(C \circ A \circ B)^L(a) &= \bigvee_{a=yz} (C^L(y) \wedge (A \circ B)^L(z)) \\
&\geq C^L(ax) \wedge (A \circ B)^L(a) \text{ (Since } a = axa) \\
&\geq C^L(a) \wedge \left(\bigvee_{a=pq} \left(A^L(p) \wedge B^L(q)\right) \right) \text{ (Since } C \in \mathrm{IVRI}(S)) \\
&\geq C^L(a) \wedge A^L(a) \wedge B^L(xa) \text{ (Since } a = axa) \\
&\geq C^L(a) \wedge A^L(a) \wedge B^L(a) \text{ (Since } C \in \mathrm{IVRI}(S)) \\
&= (C \cap A \cap B)^L(a).
\end{aligned}
$$

Similarly, we have that $(C \circ A \circ B)^U(a) \geq (C \cap A \cap B)^U(a)$. Hence $C \cap A \cap B \subset C \circ A \circ B$.

(d) $\Rightarrow$ (e): It is clear.

(e) $\Rightarrow$ (a): Suppose the condition (e) holds. Let $A \in \mathrm{GBI}(S)$, let $B \in \mathrm{LI}(S)$ and let $R \in \mathrm{RI}(S)$. Let $a \in R \cap A \cap L$. Then $a \in R$, $a \in A$ and $a \in L$. Since $A \in \mathrm{GBI}(S)$, by Lemma 4.6, $[\chi_A, \chi_A] \in \mathrm{IVGBI}(S)$. By Result 3.B, $[\chi_R, \chi_R] \in \mathrm{IVRI}(S)$ and $[\chi_L, \chi_L] \in \mathrm{IVLI}(S)$. By the hypothesis,

$$[\chi_R, \chi_R] \cap [\chi_A, \chi_A] \cap [\chi_L, \chi_L] \subset [\chi_R, \chi_R] \circ [\chi_A, \chi_A] \circ [\chi_L, \chi_L].$$

Then

$$
\begin{aligned}
([\chi_R, \chi_R] \circ [\chi_A, \chi_A] \circ [\chi_L, \chi_L])^L(a) &\geq ([\chi_R, \chi_R] \cap [\chi_A, \chi_A] \cap [\chi_L, \chi_L])^L(a) \\
&= \chi_R(a) \wedge \chi_A(a) \wedge \chi_L(a) = 1.
\end{aligned}
$$

Similarly, we have that

$$((\chi_R, \chi_R] \circ [\chi_A, \chi_A] \circ [\chi_L, \chi_L])^U(a) \geq 1.$$

Thus $[\chi_R, \chi_R] \circ [\chi_A, \chi_A] \circ [\chi_L, \chi_L] \neq [0, 0]$. So

$$\bigvee_{a=yz} (([\chi_R, \chi_R] \circ [\chi_A, \chi_A])^L(y) \wedge \chi_L(z)) = 1.$$

Similarly, we have that

$$\bigvee_{a=yz} (([\chi_R, \chi_R] \circ [\chi_A, \chi_A])^U(y) \wedge \chi_L(z)) = 1.$$

Then there exist $b, c \in S$ with $a = bc$ such that

$$
\begin{aligned}
&([\chi_R, \chi_R] \circ [\chi_A, \chi_A])^L(b) = 1, \\
&([\chi_R, \chi_R] \circ [\chi_A, \chi_A])^U(b) = 1
\end{aligned}
$$

and

$$\chi_L(c) = 1. \tag{5.1}$$

Thus $([\chi_R, \chi_R] \circ [\chi_A, \chi_A])(b) \neq [0,0]$. So

$$\bigvee_{b=pq} (\chi_R(p) \wedge \chi_A(q)) = 1 \text{ and } \bigvee_{b=pq} (\chi_R(p) \wedge \chi_A(q)) = 1.$$

Then there exist $d, e \in S$ with $b = de$ such that

$$\chi_R(d) = 1 \text{ and } \chi_A(e) = 1. \tag{5.2}$$

By (5.1) and (5.2), $d \in R$, $e \in A$ and $c \in L$. Thus $a = bc = dec \in RAL$. So $R \cap A \cap L \subset RAL$. Hence, by Result 4.A, S is regular. This completes the proof. □

6. Left quasiregular semigroups

A semigroup S is said to be *left quasiregular* if every left ideal of S is globally idempotent.

Result 6.A. [2, Proposition 1.1] A semigroup S is left quasiregular if and only if for each $a \in S$, there exist $x, y \in S$ such that $a = xaya$.

The following result can be easily proved.

Lemma 6.1. *Let S be a semigroup. If S is left quasiregular, then* IVGBI(S) = IVBI(S), *i.e.,* IVGBI(S) ⊂ IVBI(S).

Lemma 6.2. *Let S be a semigroup. Then S is left quasiregular if and only if $A \circ A = A$ for each $A \in$ IVLI(S).*

Proof. (⇒): Suppose S is left quasiregular, and let $A \in$ IVLI(S). Then, by Theorem 3.3, $A \circ A \subset A$. Let $a \in S$. Then, by Result 6.A, there exist $x, y \in S$ such that $a = xaya$. Thus

$$\begin{aligned} (A \circ A)^L(a) &= \bigvee_{a=pq} (A^L(p) \wedge A^L(q)) \\ &\geq A^L(xa) \wedge A^L(ya) \text{ (Since } a = xaya) \\ &\geq A^L(a) \wedge A^L(a) \text{ (Since } A \in \text{IVFLI}(S)) \\ &= A^L(a). \end{aligned}$$

Similarly, we have that $(A \circ A)^U(a) \geq A^U(a)$. So $A \subset A \circ A$. Hence $A \circ A = A$.

($\Leftarrow$): Suppose the necessary condition holds. Let $L \in \mathrm{LI}(S)$ and let $a \in L$. By Result 3.B, $[\chi_L, \chi_L] \in \mathrm{IVLI}(S)$. Then, by the hypothesis, $[\chi_L, \chi_L] \circ [\chi_L, \chi_L] = [\chi_L, \chi_L]$. Thus

$$([(\chi_L, \chi_L] \circ [\chi_L, \chi_L])^L(a) = \chi_L(a) = 1 \text{ and } ([\chi_L, \chi_L] \circ [\chi_L, \chi_L])^U(a) = \chi_L(a) = 1.$$

So $([\chi_L, \chi_L] \circ [\chi_L, \chi_L])(a) \neq [0, 0]$. Then $\bigvee_{a=pq}(\chi_L(p) \wedge \chi_L(q)) = 1$. Thus there exist $b, c \in S$ with $a = bc$ such that $\chi_L(b) = 1$, $\chi_L(c) = 1$. So $b \in L$, i.e., $a = bc \in LL$. Then $L \subset LL$. It is clear that $LL \subset L$. Thus $L = LL$. Hence S is left quasiregular. This completes the proof. □

A semigroup S is said to be *intraregular* if for each $a \in S$, there exist $x, y \in S$ such that $a = xa^2y$.

Result 6.B. [9, Theorem 6] Let S be a semigroup. Then S is both intraregular and left quasiregualr if and only if for each $B \in \mathrm{GBI}(S)$, each $L \in \mathrm{LI}(S)$ and each $R \in \mathrm{RI}(S)$, $L \cap R \cap B \subset LRB$.

We give a characterization of a semigroup that is both intraregular and left quasiregular in terms of interval-valued fuzzy sets.

Theorem 6.3. *Let S be a semigroup. Then the following are equivalent:*

(a) *S is both intraregular and left quasiregular.*
(b) *$B \cap C \cap A \subset B \circ C \circ A$ for each $A \in \mathrm{IVBI}(S)$, each $B \in \mathrm{IVLI}(S)$ and each $C \in \mathrm{IVRI}(S)$.*
(c) *$B \cap C \cap A \subset B \circ C \circ A$ for each $A \in \mathrm{IVGBI}(S)$, each $B \in \mathrm{IVLI}(S)$ and each $C \in \mathrm{IVRI}(S)$.*

Proof. (b) $\Rightarrow$(c): It is clear.

(c) $\Rightarrow$(a): It can be seen as in the proof of Theorem 5.5[(e) implies (a)].

(a) $\Rightarrow$(b): Suppose the condition (a) holds. Let $A \in \mathrm{IVBI}(S)$, let $B \in \mathrm{IVLI}(S)$ and let $C \in \mathrm{IVRI}(S)$. Let $a \in S$. Since S is left quasiregular, by Result 6.A, there exist $u, v \in S$ such that $a = uava$. Then

$$a = uava = u(xa^2y)va = ((ux)a)((a(yv))a).$$

Thus

$$
\begin{aligned}
(B \circ C \circ A)^L(a) &= \bigvee_{a=pq} (B^L(p) \wedge (C \circ A)^L(q)) \\
&\geq B^L((ux)a) \wedge (C \circ A)^L((avy)a) \\
&\geq B^L(a) \wedge \left(\bigvee_{ayva=pq} \left(C^L \wedge A^L(q)\right) \right) \text{ (Since } B \in \text{IVLI}(S)) \\
&\geq B^L(a) \wedge C^L(a(yv)) \wedge A^L(a) \\
&\geq B^L(a) \wedge C^L(a) \wedge A^L(a) \text{ (Since } C \in \text{IVRI}(S)) \\
&= (B \cap C \cap A)^L(a).
\end{aligned}
$$

Similarly, we have $(B \circ C \circ A)^U(a) \geq (B \cap C \cap A)^U(a)$. Hence $B \cap C \cap A \subset B \circ C \circ A$. This completes the proof. □

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