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THE LIMITING BEHAVIORS OF LINEAR RANDOM FIELDS GENERATED BY LNQD RANDOM VARIABLES ON $\mathbb{Z}^2$

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Abstract. In this paper we establish the central limit theorem and the strong law of large numbers for linear random fields generated by identically distributed linear negative quadrant dependent random variables on $\mathbb{Z}^2$.

1. Introduction

Let Z_+^d, where d is a positive integer, denote the positive integer d-dimensional lattice points. The notation $\mathbf{m} \le \mathbf{n}$, where $\mathbf{m} = (m_1, m_2, \cdots, m_d)$ and $\mathbf{n} = (n_1, n_2, \cdots, n_d)$ in $\mathbb{Z}_+^d$, means that $m_i \le n_i$ for all $1 \le i \le d$.

Two random variables X and Y are said to be negatively quadrant dependent(NQD)[resp. positively quadrant dependent(PQD)] if $P(X \le x, Y \le y) - P(X \le x)P(Y \le y) \le 0$ [resp. ≥ 0] for all $x, y \in \mathbb{R}$. A random field $\{\xi_{\mathbf{k}}, \mathbf{k} \in \mathbb{Z}_+^d\}$ is said to be associated if for any increasing functions f, g and any finite subset $A \subset \mathbb{Z}_+^d$, $Cov(f(\xi_{\mathbf{i}}, \mathbf{i} \in A), g(\xi_{\mathbf{i}}, \mathbf{i} \in A)) \ge 0$ and $\{\xi_{\mathbf{k}}, \mathbf{k} \in \mathbb{Z}_+^d\}$ is said to be negatively associated(NA) if any increasing functions f, g and any disjoint finite subsets $A, B \subset \mathbb{Z}_+^d$, $Cov(f(\xi_{\mathbf{i}}, \mathbf{i} \in A), g(\xi_{\mathbf{j}}, \mathbf{j} \in B)) \le 0$. The definitions of PQD and NQD are given by Lehmann (1966) and the concepts of association and negative association are given by Esary, Proschan and Walkup (1967) and Joag-Dev and Proschan(1983), respectively. Because of their wide applications in multivariate statistical analysis and reliability theory the notions of dependence have received more and more attention recently.

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A random field $\{\xi_{\mathbf{k}}, \mathbf{k} \in \mathbb{Z}_+^d\}$ is said to be linearly negative quadrant dependent(LNQD)[resp.linearly positive quadrant dependent(LPQD)] if for any disjoint finite subsets $A, B \subset \mathbb{Z}_+^d$ and any positive real numbers $r_{\mathbf{i}}$, $r_{\mathbf{j}}$, $\sum_{\mathbf{i}\in A} r_{\mathbf{i}}\xi_{\mathbf{i}}$ and $\sum_{\mathbf{j}\in B} r_{\mathbf{j}}\xi_{\mathbf{j}}$ are NQD[resp. PQD]. This definition is introduced by Newman(1984). Since LNQD is much weaker than NA, studying the limit theorems for LNQD random fields is of interest. Newman(1980) proved the central limit theorem for a stationary associated random field and explained the possibility of the central limit theorem for LPQD random field and Matula(1992) showed the strong law of large numbers for a pairwise NQD random field which is weaker than LNQD random fields.

The following theorem is the well known central limit theorem for LNQD random field obtained by similar method to Newman's(1980) central limit theorem for LPQD random field.

Theorem 1.1(Newman(1980)) Let $\{\xi_{\mathbf{t}}, \mathbf{t} \in \mathbb{Z}^d\}$ be a field of stationary linear negative quadrant dependent random variables with $E\xi_{\mathbf{t}} = 0$ and $E\xi_{\mathbf{t}}^2 < \infty$. Assume that

$$\sigma^2 = \sum_{\mathbf{t}\in\mathbb{Z}^d} Cov(\xi_{\mathbf{0}}, \xi_{\mathbf{t}}) < \infty.$$

Then

$$\frac{1}{\sigma\sqrt{|\mathbf{n}|}} S_{\mathbf{n}} \to N(0,1), \tag{1.1}$$

where $S_{\mathbf{n}} = \sum_{\mathbf{1}\le\mathbf{i}\le\mathbf{n}} \xi_{\mathbf{i}}$.

Theorem 1.2(Matula(1992)) Let $\{\xi_{\mathbf{t}}, \mathbf{t} \in \mathbb{Z}^d\}$ be a field of centered and identically distributed NQD random variables. Then, $E|\xi_{\mathbf{1}}|(\log^+|\xi_{\mathbf{1}}|)^{d-1} < \infty$ implies $|\mathbf{n}|^{-1}\sum_{\mathbf{1}\le\mathbf{t}\le\mathbf{n}} \xi_{\mathbf{t}} \to 0$ *a.s. as* $\mathbf{n} \to \infty$, where $\log^+ x = \max\{1, \log x\}$.

Define a linear random field

$$\begin{aligned} X(\mathbf{t}) &= \sum_{\mathbf{k}\ge\mathbf{0}} a(\mathbf{k})\xi(\mathbf{t}-\mathbf{k}) \\ &= \sum_{k_1=0}^{\infty}\cdots\sum_{k_d=0}^{\infty} a(k_1,\cdots,k_d)\xi(t_1-k_1,\cdots,t_d-k_d), \end{aligned} \tag{1.2}$$

where the coefficients $\{a(\mathbf{k}), \mathbf{k} \in \mathbb{Z}^d\}$ and the random variables $\{\xi(\mathbf{t}), \mathbf{t} \in \mathbb{Z}^d\}$ are such that the linear random field $\{X(\mathbf{t}), \mathbf{t} \in \mathbb{Z}^d\}$ is well defined and stationary.

Marinucci and Poghosyan(2001) proved the invariance principle and the strong convergence for linear random fields generated by independent and identically distributed random fields and Kim et al.(2008) investigated the invariance principle for the linear random field with associated random field. Paulauskas (2010) showed that an analogue of the Beveridge-Nelson decomposition can be applied to limit theorems for sums of linear random fields and Banys, Davydov and Paulauskas(2010) proved a strong law of large numbers for linear random field generated by a strictly stationary centered ergodic random field. Ko(2011) also proved a strong law of large numbers for linear random field generated by NA random field.

In this paper we prove the central limit theorem and the strong law of large numbers for the linear random field generated by centered and identically distributed LNQD random fields on $\mathbb{Z}^2$ by using the so-called Beveridge-Nelson decomposition. As an example we also give a Doubly Geometric Spatial Autoregressive Model.

2. Decomposition of bivariate polynomials

Define a linear random field (two parameter stochastic process) on $\mathbb{Z}^2$ by

$$X(t_1,t_2)=\sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}a(i_1,i_2)\xi(t_1-i_1,t_2-i_2),(t_1,t_2)\in\mathbb{Z}^2, \tag{2.1}$$

where $\{\xi(t_1,t_2)\}$ is a 2-parameter array of identically distributed random variables with $E\xi(t_1,t_2)=0$ and $E(\xi(t_1,t_2))^2<\infty$ and $\{a(i_1,i_2)\}$ is an array of real numbers such that

$$a(i_1,i_2)\geq 0 \text{ for all } (i_1,i_2),\ i_1,i_2\in\mathbb{N}\cup\{0\}. \tag{2.2}$$

To consider the decomposition of bivariate polynomials (see Marinucci and Poghsyan (2001)) put

$$A(x_1,x_2)=\sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}a(i_1,i_2)x_1^{i_1}x_2^{i_2},\ (x_1,x_2)\in\mathbb{R}^2, \tag{2.3}$$

where $|x_i|\leq 1,\ i=1,2$, and

$$\sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\sum_{k_1=i_1+1}^{\infty}\sum_{k_2=i_2+1}^{\infty}a(k_1,k_2)<\infty. \tag{2.4}$$

Note that (2.4) implies

$$A(1,1)=\sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}a(i_1,i_2)<\infty.$$

The following lemma extends a result known for $d=1$ as the Beveridge-Nelson decomposition(cf.Phillips and Solo(1992)) to the case $d=2$.

Lemma 2.1(Marinucci and Poghosyan(2001)) Let Γ be the class of all subsets γ of $\{1,2\}$. Let $y_j=x_j$ if $j\in\gamma$ and $y_j=1$ if $j\notin\gamma$. Then we have

$$A(x_1,x_2)=\sum_{\gamma\in\Gamma}\{\Pi_{j\in\gamma}(x_j-1)\}A_\gamma(y_1,y_2),$$

where $\Pi_{j\in\phi}=1$, and

$$A_\gamma(y_1,y_2)=\sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}a_\gamma(i_1,i_2)y_1^{i_1}y_2^{i_2}, \tag{2.5}$$

$$a_\gamma(i_1,i_2)=\sum_{s_1=i_1+1}^{\infty}\sum_{s_2=i_2+1}^{\infty}a(s_1,s_2), \tag{2.6}$$

where the sum is taken over (s_1,s_2) such that $s_j\geq i_j+1$, if $j\in\gamma$ and $s_j=i_j$ otherwise.

It follows from (2.3), (2.5) and (2.6) that $A_\varnothing(1,1)=A(1,1)$.

Let $A_{\{1\}}=A_1,\ A_{\{2\}}=A_2$, and $A_{\{1,2\}}=A_{12}$.

In other words, we have

$$\begin{aligned}A(x_1,x_2)&=A(1,x_2)+(x_1-1)A_1(x_1,x_2),\\ A(1,x_2)&=A(1,1)+(x_2-1)A_2(1,x_2),\\ A_1(x_1,x_2)&=A_1(x_1,1)+(x_2-1)A_{12}(x_1,x_2),\end{aligned}$$

where

$$A_1(x_1,x_2)=\sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\sum_{k_1=i_1+1}^{\infty}a(k_1,i_2)x_1^{i_1}x_2^{i_2},$$

$$A_{12}(x_1,x_2)=\sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\sum_{k_1=i_1+1}^{\infty}\sum_{k_2=i_2+1}^{\infty}a(k_1,k_2)x_1^{i_1}x_2^{i_2},$$

hence

$$\begin{aligned}A(x_1,x_2)&=A(1,1)+(x_1-1)A_1(x_1,1)+(x_2-1)A_2(1,x_2)\\ &\quad+(x_1-1)(x_2-1)A_{12}(x_1,x_2).\end{aligned}$$

As in Marinucci and Poghosyan(2001) we also consider the partial backshift operator satisfying

$$B_1\xi(t_1,t_2)=\xi(t_1-1,t_2) \text{ and } B_2\xi(t_1,t_2)=\xi(t_1,t_2-1), \tag{2.7}$$

which enables us to write (2.1) more compactly as

$$\begin{aligned} X(t_1,t_2) &= \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty} a(i_1,i_2)B_1^{i_1}B_2^{i_2}\xi(t_1,t_2) \\ &= A(B_1,B_2)\xi(t_1,t_2), \end{aligned} \tag{2.8}$$

where

$$A(B_1,B_2)=\sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty} a(i_1,i_2)B_1^{i_1}B_2^{i_2}.$$

The above ideas shall be exploited to establish the limit theorems(strong law of large numbers, central limit theorem) for the linear random field on $\mathbb{Z}^2$. To this aim, we write

$$\xi_\gamma(t_1,t_2)=A_\gamma(L_1,L_2)\xi(t_1,t_2), \tag{2.9}$$

where for $i=1,2$ the operator L_i is defined as $L_i=B_i$ for $i\in\gamma$, $L_i=1$ otherwise; that is

$$\begin{aligned} \xi_1(t_1,t_2) &= A_1(B_1,1)\xi(t_1,t_2), \\ \xi_2(t_1,t_2) &= A_2(1,B_2)\xi(t_1,t_2), \\ \xi_{12}(t_1,t_2) &= A_{12}(B_1,B_2)\xi(t_1,t_2). \end{aligned}$$

3. Results

Lemma 3.1(Zhang(2000)) Let $\{\xi_{\mathbf{t}},\mathbf{t}\in\mathbb{Z}^d\}$ be a field of stationary LNQD random variables with $E\xi_{\mathbf{t}}=0$. Then,

(i) there exists a positive constant D_p such that

$$E|\sum_{\mathbf{1}\le\mathbf{t}\le\mathbf{n}}\xi_{\mathbf{t}}|^p\le D_p|\mathbf{n}|^{\frac{p}{2}}E|\xi_{\mathbf{t}}|^p \tag{3.1}$$

for any $p\ge 2$ and for any $\mathbf{n}\in\mathbb{Z}_+^d$,

(ii) there exists a positive constant D_q such that

$$E\max_{\mathbf{1}\le\mathbf{m}\le\mathbf{n}}|\sum_{\mathbf{j}\le\mathbf{m}}\xi_{\mathbf{j}}|^q\le D_q|\mathbf{n}|^{\frac{q}{2}}E|\xi_{\mathbf{j}}|^q \tag{3.2}$$

for any $q>2$ and for any $\mathbf{n}\in\mathbb{Z}_+^d$.

Lemma 3.2 Let $\{\xi(t_1,t_2)\}$ be a field of identically distributed LNQD random varkables with $E\xi(t_1,t_2)=0$ and $E|\xi(t_1,t_2)|^q<\infty$ for $q>2$. Assume that (2.2) and (2.4) hold. Then,

$$E|\xi_\gamma(t_1,t_2)|^q<\infty \text{ for } \gamma\in\Gamma \tag{3.3}$$

Proof It follows from (2.2), (2.4) and (2.6) that

$$0\le\sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}a_\gamma(i_1,i_2)<\infty.$$

Hence,

$$\xi_\gamma(t_1,t_2)=\sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}a_\gamma(i_1,i_2)\xi(t_1-i_1,t_2-i_2) \tag{3.4}$$

by (2.5), (2.7) and (2.9). From (3.4) we have

$$\begin{aligned}\xi_\gamma(0,0) &= \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}a_\gamma(i_1,i_2)\xi(-i_1,-i_2)\\ &= \sum_{i=0}^{\infty}a_\gamma(\phi(i))\xi(-\phi(i))\end{aligned}$$

where $\phi:\mathbb{Z}\to\mathbb{Z}^2$ and $\{\xi(-\phi(i))\}$ is a sequence of identically distributed LNQD random variables. Hence, $q>2$

$$\begin{aligned}E|\xi_\gamma(t_1,t_2)|^q &= E|\xi_\gamma(0,0)|^q\\ &= [\{E|\sum_{i=0}^{\infty}a_\gamma(\phi(i))\xi(-\phi(i))|^q\}^{\frac{1}{q}}]^q\\ &\le [\sum_{i=0}^{\infty}a_\gamma(\phi(i))(E|\xi(-\phi(i))|^q)^{\frac{1}{q}}]^q\\ &\le C[\sum_{i=0}^{\infty}a_\gamma(\phi(i))]^q<\infty,\end{aligned}$$

where the first bound follows from Minkowski's inequality and the second bound from condition (2.4).

Theorem 3.3 Let $\{X(t_1,t_2)\}$ be defined as in (2.1) and $\{\xi(t_1,t_2),(t_1,t_2)\in\mathbb{Z}^2\}$ a field of identically distributed LNQD random variables with $E\xi(t_1,t_2)$

$= 0$ and $E|\xi(t_1,t_2)|^q < \infty$ for $q > 2$. Assume that (2.2) and (2.4) hold. Then,

$$\sigma^{-1}|\mathbf{n}|^{-\frac{1}{2}} \sum_{t_1=1}^{n_1} \sum_{t_2=1}^{n_2} X(t_1,t_2) \to^{\mathcal{D}} A(1,1)N(0,1), \tag{3.5}$$

where $\mathbf{n} = (n_1,n_2)$ and $\sigma^2 = \sum_{(t_1,t_2)\in\mathbb{Z}^2} Cov(\xi(0,0),\xi(t_1,t_2)) < \infty$.

Corollary 3.4 Let $X(t_1,t_2)$ satisfy model (2.1) and $\{\xi(t_1,t_2)\}$ a 2-parameter array of identically distributed LNQD random variables with $E\xi(t_1,t_2) = 0$, $E|\xi(t_1,t_2)|^q < \infty$ for $q > 2$. If $a(i_1,i_2) = 1$ for $i_1 = i_2 = 0$, $a(i_1,i_2) = 0$ otherwise, then for $d = 2$, (1.1) holds.

Example 3.5 Let $A(x_1,x_2) = 1 + x_1 + x_1x_2 + x_2^2$ and let

$$\begin{aligned} X(t_1,t_2) &= \xi(t_1,t_2) + \xi(t_1-1,t_2) + \xi(t_1-1,t_2-1) + \xi(t_1,t_2-1) \\ &= A(B_1,B_2)\xi(t_1,t_2) \end{aligned}$$

for $A(B_1,B_2) = 1 + B_1 + B_1B_2 + B_2^2$. Then Theorem 3.3 implies, as $\mathbf{n} \to \infty$,

$$(\sigma^2|\mathbf{n}|)^{-\frac{1}{2}} \sum_{t_1=1}^{n_1} \sum_{t_2=1}^{n_2} X(t_1,t_2) \to^{\mathcal{D}} 4N(0,1),\ \mathbf{n} = (n_1,n_2).$$

From Corollary in Matula(1992) we obtain the following lemma.

Lemma 3.6 Let $\{\xi_n, n \geq 1\}$ be a sequence of identically distributed LNQD random variables with $E\xi_1 = 0$ and $E\xi_1^2 < \infty$. Then

$$\sum_{i=1}^{n} \xi_i/n \to 0 \ a.s. \ as \ n \to \infty.$$

Theorem 3.7 Let $\{X(t_1,t_2)\}$ be defined as in (2.1), where $\{\xi(t_1,t_2), (t_1,t_2) \in \mathbb{Z}^2\}$ is a field of the identically distributed LNQD random variables with $E\xi(t_1,t_2) = 0$, $E|\xi(t_1,t_2)|^q < \infty$ for $q > 2$ and $\{a(k_1,k_2)\}$ is a collection of real numbers such that $a(k_1,k_2) \geq 0$ for all (k_1,k_2), $k_1,k_2 \in N \cup \{0\}$. Then $E|\xi_{\mathbf{1}}|(\log^+|\xi_{\mathbf{1}}|)^{d-1} < \infty$ implies

$$|\mathbf{n}|^{-1} \sum_{\mathbf{1}\leq\mathbf{t}\leq\mathbf{n}} X(t_1,t_2) \to 0 \ a.s. \ as \ \mathbf{n} \to \infty, \tag{3.6}$$

where $\mathbf{t} = (t_1,t_2) \in \mathbb{Z}^2$ and $\log^+ x = \max\{1, \log x\}$.

Finally, we give a simple example satisfying Theorems 3.3 and 3.7.

Example 3.8 Let

(3.7)
$$X(t_1,t_2)=\alpha X(t_1-1,t_2)+\beta X(t_1,t_2-1)-\alpha\beta X(t_1-1,t_2-1)+\xi(t_1,t_2),$$

where $0<\alpha,\beta<1$. By using the partial back shifts B_1 and B_2 defined as (2.7), the model (3.7) can be written as

$$(1-\alpha B_1)(1-\beta B_2)X(t_1,t_2)=\xi(t_1,t_2). \tag{3.8}$$

Therefore

$$\begin{aligned}X(t_1,t_2) &= \frac{1}{(1-\alpha B_1)(1-\beta B_2)}\xi(t_1,t_2)\\ &= (\sum_{i_1=0}^{\infty}\alpha^{i_1}B_1^{i_1})(\sum_{i_2=0}^{\infty}\beta^{i_2}B_2^{i_2}\xi(t_1,t_2))\\ &= \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\alpha^{i_1}\beta^{i_2}B_1^{i_1}B_2^{i_2}\xi(t_1,t_2)\\ &= \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\alpha^{i_1}\beta^{i_2}\xi(t_1-i_1,t_2-i_2)\end{aligned}$$

where $a(i_1,i_2)=\alpha^{i_1}\beta^{i_2}$ and $A(B_1,B_2)=\sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\alpha^{i_1}\beta^{i_2}B_1^{i_1}B_2^{i_2}$.

The representation (3.8) elucidates the meaning of a "Doubly Geometric Spatial Autoregressive Model".

If $\{\xi(t_1,t_2)\in\mathbb{Z}^2\}$ is a field of identically distributed LNQD random variables with mean zero and finite variance, then under conditions of Theorems 3.3 and 3.7, the random field $X(t_1,t_2)$ satisfying (3.7) provides a simple example that ensures (3.5) and (3.6).

4. Proofs

Proof of Theorem 3.3: From Theorem 1.1 we have

$$\frac{1}{\sigma\sqrt{|\mathbf{n}|}}\sum_{t_1=1}^{n_1}\sum_{t_2=1}^{n_2}\xi(t_1,t_2)\to^{\mathcal{D}} N(0,1). \tag{4.1}$$

From (3.2) and (3.3), there exists a positive constant D_q such that, for any $q>2$

(4.2)
$$E|\max_{1\le k_1\le n_1,1\le k_2\le n_2}\sum_{t_1=1}^{k_1}\sum_{t_2=1}^{k_2}\xi_\gamma(t_1,t_2)|^q\le D_q|\mathbf{n}|^{\frac{q}{2}}E|\xi(t_1,t_2)|^q,\ \mathbf{n}=(n_1,n_2).$$

If we apply Lemma 2.1 to the backshift binomial $A(B_1, B_2)$, then the following equality holds almost surely:

$$\begin{aligned} X(t_1, t_2) &= A(1,1)\xi(t_1, t_2) + (B_1 - 1)A_1(B_1, 1)\xi(t_1, t_2) \\ &+ (B_2 - 1)A_2(1, B_2)\xi(t_1, t_2) + (B_1 - 1)(B_2 - 1)A_{12}(B_1, B_2)\xi(t_1, t_2) \end{aligned}$$

which implies that,

$$\begin{aligned} (4.3) \qquad & (n_1 n_2)^{-\frac{1}{2}} \sum_{t_1=1}^{n_1} \sum_{t_2=1}^{n_2} X(t_1, t_2) \\ &= (n_1 n_2)^{-\frac{1}{2}} \{ \sum_{t_1=1}^{n_1} \sum_{t_2=1}^{n_2} A(1,1)\xi(t_1, t_2) - \sum_{t_2=1}^{n_2} \xi_1(n_1, t_2) \\ &+ \sum_{t_2=1}^{n_2} \xi_1(0, t_2) - \sum_{t_1=1}^{n_1} \xi_2(t_1, n_2) + \sum_{t_1=1}^{n_1} \xi_2(t_1, 0) \\ &- \xi_{12}(0, n_2) + \xi_{12}(0,0) - \xi_{12}(n_1, 0) + \xi_{12}(n_1, n_2) \} \\ &= (n_1 n_2)^{-\frac{1}{2}} \{ \sum_{t_1=1}^{n_1} \sum_{t_2=1}^{n_2} A(1,1)\xi(t_1, t_2) + R_{n_1, n_2} \}. \end{aligned}$$

Note that $\xi_1(\cdot, \cdot), \xi_2(\cdot, \cdot)$ and $\xi_{12}(\cdot, \cdot)$ are LNQD.

From Markov's inequality, and (4.2),

$$P\{ \max_{1 \le k_2 \le n_2} (n_1 n_2)^{-\frac{1}{2}} | \sum_{t_2=1}^{k_2} \xi_1(n_1, t_2)| > \delta \} \le \frac{E \max_{1 \le k_2 \le n_2} | \sum_{t_2=1}^{k_2} \xi_1(n_1, t_2)|^q}{(n_1 n_2)^{\frac{q}{2}} \delta^q}$$

$$(4.4) \qquad \le C n_1^{-\frac{q}{2}} = o(1)$$

as $n_1 \to \infty$. We can also apply exactly the same argument to establish

$$(4.5) \qquad P\{ \max_{1 \le k_1 \le n_1} (n_1 n_2)^{-1} | \sum_{t_1=1}^{k_1} \xi_2(t_1, n_2)| > \delta \} = o(1) \ as \ n_2 \to \infty.$$

By Lemma 3.2 we have for $q > 2$

$$E|\xi_{12}(n_1, n_2)|^q < \infty$$

and hence by the same argument as above we also have

$$(4.6) \qquad P\{ \max_{n_1 \ge 1, n_2 \ge 1} (n_1 n_2)^{-\frac{1}{2}} |\xi_{12}(n_1, n_2)| > \delta \} = o(1) \ as \ \mathbf{n} \to \infty.$$

Thus, we have

$$\sup_{n_1 \ge 1, n_2 \ge 1} |(n_1 n_2)^{-\frac{1}{2}} R_{n_1, n_2}| = o(1),$$

which yields

$$\sigma^{-1}|\mathbf{n}|^{-\frac{1}{2}}\sum_{t_1=1}^{n_1}\sum_{t_2=1}^{n_2}X(t_1,t_2)\rightarrow^{\mathcal{D}} A(1,1)N(0,1)\ as\ \mathbf{n}\rightarrow\infty$$

by Theorem 4.1 of Billingsley(1968).

Proof of Theorem 3.7: If we apply Lemma 2.1 to the backshift polynomial $A(B_1,B_2)$, we find that the following equality holds

$$\begin{aligned} X(t_1,t_2) &= A(1,1)\xi(t_1,t_2)+(B_1-1)A_1(B_1,1)\xi(t_1,t_2)\\ &+ (B_2-1)A_2(1,B_2)\xi(t_1,t_2)+(B_1-1)(B_2-1)A_{12}(B_1,B_2)\xi(t_1,t_2)\end{aligned}$$

which implies that

$$\begin{aligned} &(n_1n_2)^{-1}\sum_{t_1=1}^{n_1}\sum_{t_2=1}^{n_2}X(t_1,t_2)\\ &=(n_1n_2)^{-1}\{\sum_{t_1=1}^{n_1}\sum_{t_2=1}^{n_2}A(1,1)\xi(t_1,t_2)-\sum_{t_2=1}^{n_2}\xi_1(n_1,t_2)+\sum_{t_2=1}^{n_2}\xi_1(0,t_2)\\ &-\sum_{t_1=1}^{n_1}\xi_2(t_1,n_2)+\sum_{t_1=1}^{n_1}\xi_2(t_1,0)-\xi_{12}(0,n_2)\\ &+\xi_{12}(0,0)-\xi_{12}(n_1,0)+\xi_{12}(n_1,n_2)\}\\ &=(n_1n_2)^{-1}\{\sum_{t_1=1}^{n_1}\sum_{t_2=1}^{n_2}A(1,1)\xi(t_1,t_2)+R_n(t_1,t_2)\},\ \text{where } \mathbf{n}=(n_1,n_2).\end{aligned} \tag{4.7}$$

First we obtain

$$|\mathbf{n}|^{-1}\sum_{t_1=1}^{n_1}\sum_{t_2=1}^{n_2}A(1.1)\xi(t_1,t_2)\rightarrow 0\ a.s.\ as\ \mathbf{n}\rightarrow\infty \tag{4.8}$$

by Theorem 1.2. It follows from Lemmas 3.2 and 3.6 that

$$(n_1n_2)^{-1}\sum_{t_1=1}^{n_1}\sum_{t_2=1}^{n_2}\xi_1(n_1,t_2)=n_2^{-1}\sum_{t_2=1}^{n_2}\xi_1(n_1,t_2)\rightarrow 0\ a.s.\ as\ n_2\rightarrow\infty,$$

$$(n_1n_2)^{-1}\sum_{t_1=1}^{n_1}\sum_{t_2=1}^{n_2}\xi_1(0,t_2)=n_2^{-1}\sum_{t_2=1}^{n_2}\xi_1(0,t_2)\rightarrow 0\ a.s.\ as\ n_2\rightarrow\infty,$$

$$(n_1n_2)^{-1}\sum_{t_1=1}^{n_1}\sum_{t_2=1}^{n_2}\xi_2(t_1,n_2)=n_1^{-1}\sum_{t_1=1}^{n_1}\xi_2(t_1,n_2)\rightarrow 0\ a.s.\ as\ n_1\rightarrow\infty,$$

and

$$(n_1 n_2)^{-1} \sum_{t_1=1}^{n_1} \sum_{t_2=1}^{n_2} \xi_2(t_1, 0) = n_1^{-1} \sum_{t_1=1}^{n_1} \xi_2(t_1, 0) \to 0 \; a.s. \; as \; n_1 \to \infty.$$

Finally, we have $(n_1 n_2)^{-1}\xi_{12}(0, n_2) \to 0$ $a.s.$, $(n_1 n_2)^{-1}\xi_{12}(0, 0) \to 0$ $a.s.$,
$(n_1 n_2)^{-1}\xi_{12}(n_1, 0) \to 0$ $a.s.$ and $(n_1 n_2)^{-1}\xi_{12}(n_1, n_2) \to 0$ $a.s.$ as $\mathbf{n} \to \infty$.

Hence,

$$|\mathbf{n}|^{-1} R_n(t_1, t_2) \to 0 \; a.s. \; \mathbf{n} \to \infty, \tag{4.9}$$

which implies

$$|\mathbf{n}|^{-1} \sum_{t_1=1}^{n_1} \sum_{t_2=1}^{n_2} X(t_1, t_2) \to 0 \; a.s. \; as \; \mathbf{n} \to \infty$$

together with (4.7) and (4.8).

Remark We only consider linear random fields on $\mathbb{Z}^2$ because they are the most popular and useful model in practice, and we focus on $\mathbb{Z}^2$ instead of the more general case $\mathbb{Z}^d, d > 2$, merely for the ease of presentation. The asymptotic results stated in Section 3 can be shown to also hold for $\mathbb{Z}^d, d > 2$, with only straight forward but tedious modifications.

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