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CLASSIFICATION OF SPACES IN TERMS OF BOTH A DIGITIZATION AND A MARCUS WYSE TOPOLOGICAL STRUCTURE

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Abstract. In order to examine the possibility of some topological structures into the fields of network science, telecommunications related to the future internet and a digitization, the paper studies the Marcus Wyse topological structure. Further, this paper develops the notions of *lattice based Marcus Wyse continuity* and *lattice based Marcus Wyse homeomorphism* which can be used for studying spaces $X \subset \mathbf{R}^2$ in the Marcus Wyse topological approach. By using these two notions, we can study and classify lattice based simple closed Marcus Wyse curves.

1. Introduction

In relation to the mathematical recognition of a set $X \subset \mathbf{Z}^n$ with some topological structures or some adjacency relations of $\mathbf{Z}^n$, digital topology played an important role in computer graphics, image synthesis, image analysis, network science and so forth. It grew out of discrete geometry expanded into applications where significant topological issues arise. It may be of interest both for computer scientists who try to apply topological knowledge for investigating digital spaces and for mathematicians who want to use computers to solve complicated topological problems.

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In digital geometry, we have used many tools from combinatorial topology, Marcus Wyse (briefly, MW-) topology, Khalimsky topology, digital graph theory and so forth [2, 3, 4, 5, 6, 7, 8, 10]. In particular, in order to digitize a subset of the Euclidean 2D space, many tools from MW-topology have been used [5, 6, 8]. Let us now recall some basic facts and terminology for further discussion. Let $\mathbf{Z}$, $\mathbf{N}$ and $\mathbf{Z}^2$ represent the sets of integers, natural numbers and points in the Euclidean 2D space with integer coordinates, respectively. Let $(\mathbf{R}, U)$ and $(\mathbf{R}^2, U^2)$ be the usual topology on the set of real numbers and the typical product topology of $(\mathbf{R}, U)$, respectively. Motivated by an Alexandroff space [1], the Marcus Wyse (briefly, MW-) 2D topological space, denoted by $(\mathbf{Z}^2, \gamma)$, was established. For a set $X \subset \mathbf{Z}^2$ we can take the *subspace* induced from $(\mathbf{Z}^2, \gamma)$ denoted by (X, γ_X). Let $f : (X, \gamma_X) := X \to (Y, \gamma_Y) := Y$ be a Marcus Wyse (briefly, MW-) continuous map. Both connectedness and MW-adjacency relations are symmetric, and an MW-topological space is a T_0-Alexandroff space [5]. Indeed, in MW-topology connectedness is equivalent to pathconnectedness [5]. Thus, for an MW-pathconnected subset $A \subset X$, the image by the map f, $f(A)$, is also an MW-pathconnected subset of Y. Furthermore, by using both an MW-continuous map and an MW-homeomorphism, we have efficiently studied MW-topological spaces.

In the study of the digitization of a Euclidean nD subspace, there are a number of researches. However, this paper proposes a special kind of digitization method of a set $X \subset \mathbf{R}^2$ (see Theorem 4.3) in terms of both a combinatorial and a Marcus Wyse topological tool. This paper establishes a *lattice based Marcus Wyse (briefly, LMW-)continuous map* and an *LMW-homeomorphism.* In terms of these two notions, we can efficiently study spaces $X \subset \mathbf{R}^2$ in the MW-topological approach. The rest of this paper proceeds as follows.

Section 2 provides some basic notions. Section 3 proposes the notion of *lattice based Marcus Wyse* (briefly, LMW-) continuity and investigates its various properties. In addition, in order to study spaces $X \subset \mathbf{R}^2$ based on the MW-topological structure, this section also establishes the notion of LMW-homeomorphism. Section 4 suggests a method of digitizing a Euclidean 2D space $X \subset \mathbf{R}^2$ in the MW-topological approach. Section 5 classifies LMW-curves in terms of an LMW-homeomorphism. Section 6 concludes the paper with a summary and further work.

2. Preliminaries

Let us now review some basic facts and notions from the MW-topology and the digital topology. Let $N_k(p)$ be the k-neighbor of a point $p \in \mathbf{Z}^n$ [7, 9]. The *MW-topology* on $\mathbf{Z}^2$, denoted by $(\mathbf{Z}^2, \gamma)$, is induced from the base $B = \{U\}$ in (2.1) [10], where for each point $p = (x, y) \in \mathbf{Z}^2$

$$U := \left\{ \begin{array}{l} U(p) := N_4(p) \cup \{p\} \text{ if } x + y \text{ even, and} \\ \{p\} : \text{ else.} \end{array} \right\} \tag{2.1}$$

In (2.1), the terminology *even* can be exchanged into *odd*.
In relation to the further statement of a point in $\mathbf{Z}^2$, in this paper we call a point $p = (x_1, x_2)$ *double even* if $x_1 + x_2$ is an even number such that each x_i is even $i \in \{1, 2\}$; *even* if $x_1 + x_2$ is an even number such that each x_i is odd, $i \in \{1, 2\}$; and *odd* if $x_1 + x_2$ is an odd number.

In all subspaces of $(\mathbf{Z}^2, \gamma)$ of Figures 1, 2, 3, and 4, the symbols ■ and • mean a *double even point* and an *odd point*, respectively. Further, each of the *white squares* in Figures 1, 2, 3 and 4 means an *even point*. In view of (2.1), we can clearly obtain the following:

Remark 2.1. *In $(\mathbf{Z}^2, \gamma)$ the singleton with either a double even point or an even point is a closed set. In addition, the singleton with an odd point is clearly an open set.*

In this paper a set $X \subset \mathbf{Z}^2$ will be considered to be a subspace (X, γ_X) induced from $(\mathbf{Z}^2, \gamma)$. As a digital image has been often studied with digital connectivity [7, 9], it is reasonable to study a 2D MW-topological space (X, γ_X) with MW-connectedness or γ-adjacency.

Definition 1. [10] *For a set $X \subset \mathbf{Z}^2$ consider the subspace (X, γ_X) induced from $(\mathbf{Z}^2, \gamma)$. Then we call it an MW-topological space.*

Let us recall the following terminology which can be used for studying an MW-topological space. In this paper for (X, γ_X) and $x \in X$ we denote by $O(x, X)$ the smallest open set of the point x in γ_X.

Definition 2. *Let $(X, \gamma_X) := X$ be an MW-topological space. Then we define the following:*

(1) Consider two MW-topological spaces $(A, \gamma_A) := A$ and $(B, \gamma_B) := B$ such that A and B are nonempty subsets of X. Then we say that two subspaces A and B of X are not MW-connected to each other if no points $a \in A$ and $b \in B$ exist such that $a \in O(b, X)$ or $b \in O(a, X)$.

We say that two distinct points $a := \{a\}$ and $b := \{b\}$ in X are not MW-connected in (X, γ_X) if neither $a \in O(b, X)$ nor $b \in O(a, X)$.

(2) We say that a space X is MW-connected if it is not a union of two disjoint non-empty MW-spaces not MW-connected to each other.

(3) Distinct points $x, y \in X$ are called MW-path connected if there is a sequence (or a path) $(x_0, x_1, \cdots, x_m)$ on X with $\{x_0 = x, x_1, \cdots, x_m = y\}$ such that x_i and x_{i+1} are MW-connected, $i \in [0, m-1]_{\mathbf{Z}}, m \geq 1$. This sequence is called an MW-path. Furthermore, the number m is called the length of this MW-path. Furthermore, an MW-path is called a closed MW-curve if $x_0 = x_m$.

(4) For a point $x \in X$, we say that the maximal MW-connected subset of X containing the point $x \in X$ is the MW-connected component of $x \in X$.

(5) A simple MW-path in X is the sequence $(x_i)_{i \in [0,m]_{\mathbf{Z}}}$ such that x_i and x_j are MW-connected if and only if either $j = i + 1$ or $i = j + 1$. Furthermore, we say that a simple closed MW-curve with m elements $(x_i)_{i \in [0,m]_{\mathbf{Z}}}$ is a simple MW-path with $x_0 = x_m$ and that x_i and x_j are MW-connected if and only if either $j = i + 1 (mod\, m)$ or $i = j + 1 (mod\, m)$.

Let us now establish the MW-topological category and recall an MW-homeomorphism.

Definition 3. [10] *For two MW-topological spaces $(X, \gamma_X) := X$ and $(Y, \gamma_Y) := Y$, a function $f : X \to Y$ is said to be MW-continuous at a point $x \in X$ if f is continuous at the point x from the viewpoint of MW-topology.*
Furthermore, we say that a map $f : X \to Y$ is MW-continuous if it is MW-continuous at every point $x \in X$.

By using MW-continuity, we can obtain the *MW-topological category*, denoted by *MC*, consisting of two classes, as follows.

(1) A class of objects (X, γ_X),

(2) For every ordered pair of objects (X, γ_X) and (Y, γ_Y), a class of all MW-continuous maps $f : (X, \gamma_X) \to (Y, \gamma_Y)$ as morphisms.

Definition 4. [10] *For two spaces (X, γ_X) and (Y, γ_Y), a map $h : X \to Y$ is called an MW-homeomorphism if h is a MW-continuous bijection and that $h^{-1} : Y \to X$ is MW-continuous.*

In Definition 4, we denote by $X \approx_{MW} Y$ an MW-homeomorphism.

3. Lattice Based MW-continuous map and Its Properties

In this section, based on the MW-topological category, we establish the *lattice based MW-topological category* in terms of a lattice based MW-continuous map, and the notion of lattice based MW-homeomorph ism. For a non-empty set $X \subset R^2$ consider its topological *subspace* induced from $(\mathbf{R}^2, U^2)$ denoted by (X, U^2_X). In order to study a (Euclidean 2D) space $X \subset \mathbf{R}^2$ in the lattice based MW-topological approach, we define the following rule so called "*the local rule based on the MW-topology*".

Definition 5 (Local rule based on the MW-topology on $\mathbf{Z}^2$). *In $\mathbf{R}^2$, for each point $p \in \mathbf{Z}^2$ and $i \in \{1, 2\}$ we assume the following neighbor of p:*

$$N_M(p) := \begin{cases} \{(t_1, t_2) |\, t_i \in [p_i - \frac{1}{2}, p_i + \frac{1}{2}]\} \\ \qquad \textit{if } p = (p_1, p_2) \textit{ is a double even point, and} \\ \{(t_1, t_2) |\, t_i \in [p_i - \frac{1}{2}, p_i + \frac{1}{2}]\} \setminus \{(p_1 \pm \frac{1}{2}, p_2 \pm \frac{1}{2})\} \\ \qquad \textit{if } p = (p_1, p_2) \textit{ is an even point, and} \\ \{(t_1, t_2) |\, t_i \in (p_i - \frac{1}{2}, p_i + \frac{1}{2})\} \textit{ if } p = (p_1, p_2) \textit{ is an odd point.} \end{cases}$$

In Figure 1(a), (b) and (c), for the double even point, the even point and the odd points p we can observe their $N_M(p) \subset \mathbf{R}^2$.

Proposition 3.1. *The set $\{N_M(p) |\, p \in \mathbf{Z}^2\}$ is a partition of $\mathbf{R}^2$ in the MW-topological approach.*

In Proposition 3.1, since the current local rule of a point $p \in \mathbf{Z}^2$ is motivated from the MW-topological structure of the point $p \in \mathbf{Z}^2$, we can state that the partition of Proposition 3.1 is taken in the MW-topological approach.

In view of Remark 2.1, the local rule of Definition 5 can make substantial contributions to the study of a digitization of a Euclidean 2D space $X \subset \mathbf{R}^2$ in the MW-topological approach. In terms of (2.1), for each point $x \in (X, \gamma_X)$ we can obtain the smallest open neighborhood of x denoted by $SN(x) \subset X$ that can be used for establishing the following neighborhood in $\mathbf{R}^2$. By using $SN(x)$ in (X, γ_X), we can define the following:

Definition 6. *For a point* $x := (x_1, x_2) \in \mathbf{Z}^2$ *we define the smallest neighborhood of* x *as the set* $SN_M(x) := \cup_{q \in SN(x)} N_M(q) \subset \mathbf{R}^2$*, where* $SN(x)$ *is the smallest open neighborhood of* x *in* $(\mathbf{Z}^2, \gamma)$*.*

Remark 3.2. *In Definition 6 we can observe that for a double even point* $x \in \mathbf{Z}^2$ $SN_M(x)$ *cannot be an open set in* $(\mathbf{R}^2, U^2)$ *owing to the corner points of* $N_M(x)$*. For instance, consider the double even point* $(0,0) := x$ *of Figure 1(d). Since* $SN_M(x)$ *includes the four points* $(\pm\frac{1}{2}, \pm\frac{1}{2})$ *in* $N_M(x)$*, it cannot be an open set in* $(\mathbf{R}^2, U^2)$*.*

Definition 7. *For* (X, U^2_X) *and a point* $x \in X \cap \mathbf{Z}^2$ *we can take* $N_M(x) \cap X \subset X$ *and* $SN_M(x) \cap X \subset X$*. Then we briefly use the notations* $N_M(x) := N_M(x) \cap X \subset X$ *and* $SN_M(x) := SN_M(x) \cap X$*.*

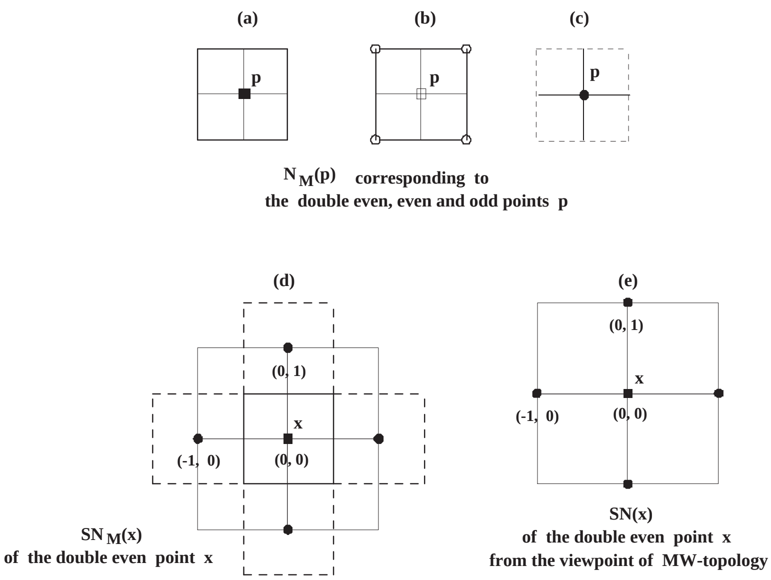

FIGURE 1. (a)(resp. (b)) Configuration of $N_M(p) \subset \mathbf{R}^2$ corresponding to the double even (resp. an even) point p. (c) $N_M(p) \subset \mathbf{R}^2$ corresponding to the odd point p. (d) Configuration of $SN_M(x)$ of the double even point x. (e) Configuration of the smallest open neighborhood of the double even point x in $(\mathbf{Z}^2, \gamma)$, $SN(x)$.

Each space in Figures 2, 3 and 4 is presented by using $N_M(p)$ of Definition 7. In Figure 1(d), for the double even point x we can observe

$SN_M(x)$ by using the smallest open neighborhood of x in an MW-topological space.
By using both the local rule of Definition 5 and the MW-continuity, we can establish the following notion.

Definition 8. *Let $F : (X, U_X^2) \to (Y, U_Y^2)$ be a map. Then we say that F is a lattice-based MW-continuous map (briefly, an LMW-continuous map) if*

(1) $F(X \cap \mathbf{Z}^2) \subset Y \cap \mathbf{Z}^2$,

(2) the restriction of F to $X \cap \mathbf{Z}^2 := X'$ with the codomain $Y \cap \mathbf{Z}^2 := Y'$, denoted by $f : X' \to Y'$ with $f(x) = F|_{X'}(x)$, is an MW-continuous map, and

(3) for each point $p \in X'$, $F(N_M(p)) \subset N_M(f(p))$.

Example 3.3. *(1) Consider two spaces (X, U_X^2) and (Y, U_Y^2) (see Figure 2 (a)), where $X = \cup_{x_i \in X'} N_M(x_i)$ and $X' = \{x_i | \, i \in [0, 7]_{\mathbf{Z}}\}$, and $Y = X - \{N_M(x_i) | \, i \in \{6, 7\}\}$ and $Y' = \{x_i | \, i \in [0, 5]_{\mathbf{Z}}\}$. Assume the map $F : (X, U_X^2) \to (Y, U_Y^2)$ given by $F(N_M(x_i)) \subset N_M(x_i), i \in [0, 5]_{\mathbf{Z}}$, $F(N_M(x_7)) \subset N_M(x_0)$ and $F(N_M(x_6)) \subset N_M(x_5)$. Further, the restriction of F to $X \cap \mathbf{Z}^2 := X'$ with the codomain $Y \cap \mathbf{Z}^2 := Y'$, denoted by $f : X' \to Y'$, is given by $f(x_i) = x_i, i \in [0, 5]_{\mathbf{Z}}$, $f(x_7) = x_0$ and $f(x_6) = x_5$. Then we can observe that F is an LMW-continuous map.*

(2) Consider two spaces (X, U_X^2) and (Z, U_Z^2) (see Figure 2 (b)), where $X = \cup_{x_i \in X'} N_M(x_i)$ and $X' = \{x_i | \, i \in [0, 7]_{\mathbf{Z}}\}$, and $Z = X - \{N_M(x_i) | \, i \in \{1, 5\}\}$ and $Z' = \{x_i | \, i \in [0, 7]_{\mathbf{Z}} - \{1, 5\}\}$. Assume the map $G : (X, U_X^2) \to (Z, U_Z^2)$ given by $G(N_M(x_i)) \subset N_M(x_i), i \in [0, 7]_{\mathbf{Z}} - \{1, 5\}$, $G(N_M(x_1)) \subset N_M(x_0)$ and $G(N_M(x_5)) \subset N_M(x_4)$. Further, the restriction of G to $X \cap \mathbf{Z}^2 := X'$ with the codomain $Z \cap \mathbf{Z}^2 := Z'$, denoted by $g : X' \to Z'$, is given by $g(x_i) = x_i, i \in [0, 7]_{\mathbf{Z}} - \{1, 5\}$, $g(x_1) = x_0$ and $g(x_5) = x_4$. Then we can observe that G cannot be an LMW-continuous map at the point $x_5 \in X'$. More precisely, while the map G satisfies the properties (1) and (3), it cannot satisfy (2) at the point x_5 because $g(SN(x_5))$ cannot be a subset of $SN(x_4) \subset Z'$, where $SN(x_5) = \{x_4, x_5, x_6\} \subset X'$ and $SN(x_4) = \{x_4\} \subset Z'$.

When showing the LMW-continuity of the map F in Figure 2(a), we can observe a big gap between two points x_0 and x_5 in Y. However, we do not need to concern it because the set $\{x_6, x_7\} \subset X'$ (resp. $\{N_M(x_6), N_M(x_7)\} \subset X$) which is the preimage of $\{x_0, x_5\} \subset Y'$ (resp. $\{N_M(x_0), N_M(x_5)\} \subset Y$) by the map f (resp. F) cannot be connected in $(X', \gamma_{X'})$ (resp. (X, U_X^2)).

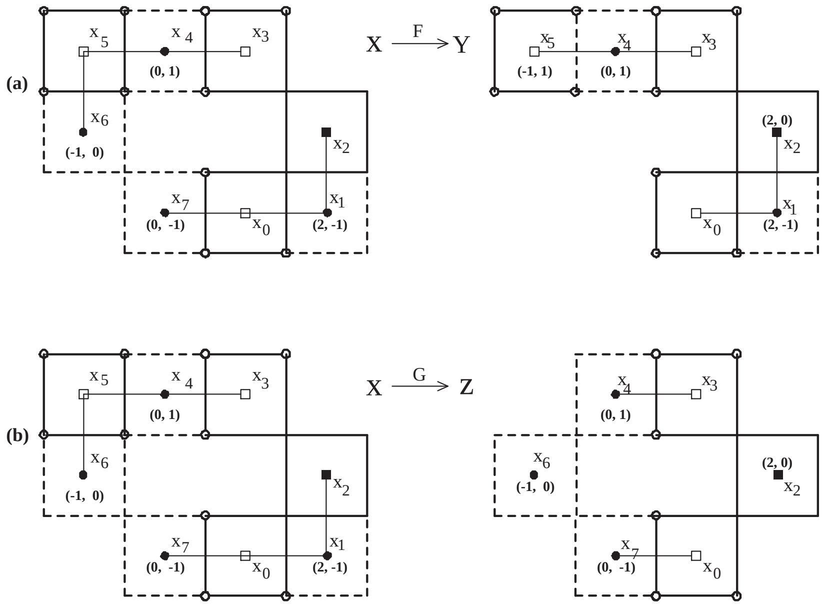

FIGURE 2. (a) LMW-continuity. (b) Non-LMW-continuity.

Remark 3.4 (Merits and limitation of LMW-continuity)**.** *(1) When digitizing a (Euclidean* $2D$*) space* $X \subset \mathbf{R}^2$ *(see Theorem 4.3) in the* MW*-topological approach, as the points* $x \in X - \cup_{p \in \mathbf{Z}^2 \cap X} N_M(p)$ *are due to be ignored, the* LMW*-continuity is meaningful to study* $X \subset \mathbf{R}^2$ *in terms of the* MW*-topological structure. Further, as the* LMW*-continuity is defined by using combinatorial and Marcus Wyse topological tools, its utility can be expanded in the fields of the combinatorial and the* MW*-topology.*

(2) There are some limitations of the LMW*-continuity, as follows. An* LMW*-continuous map* $F : (X, U_X^2) \to (Y, U_Y^2)$ *is focused on the mapping of the points* $x \in \cup_{p \in \mathbf{Z}^2 \cap X} N_M(p) \subset X$*. Thus the points* $x \in X - \cup_{p \in \mathbf{Z}^2 \cap X} N_M(p)$ *are not related to the properties (1)-(3) of Definition 8.*

We can establish the *lattice based* MW*-topological category*, briefly LMC, consisting of two things:

($*$ 1) A class $Ob(C)$ consisting of $(X, U_X^n) := X$;

($*$ 2) A class $Mor(X, Y)$ consisting of LMW-continuous maps as morphisms.

Motivated by an MW-homeomorphism, we can establish the notion of LMW-homeomorphism in LMC, as follows.

Definition 9. *Let $F : (X, U_X^2) := X \to (Y, U_Y^2) := Y$ be an LMW-continuous map. Then we say that F is an LMW-homeomorphism if*

(1) the restriction of F to $X \cap \mathbf{Z}^2 := X'$ with the codomain $Y \cap \mathbf{Z}^2 := Y'$, denoted by $f : (X', \gamma_{X'}) \to (Y', \gamma_{Y'})$ with $f(x) = F|_{X'}(x)$, is an MW-homeomorphism,

(2) the inverse of f has an extension $G : Y \to X$ such that $G \circ F = 1_X$ and $F \circ G = 1_Y$.

In Definition 9, we denote by $X \approx_{LMW} Y$ an LMW-homeomorphism. By using an LMW-homeomorphism, we can classify spaces $X \subset \mathbf{R}^2$ in the MW-topological approach.

4. Method of Digitizing a Euclidean 2D Space $X \subset \mathbf{R}^2$ by Using the Local Rule Based on the MW-Topological Structure

In this section, by using the notions discussed in Sections 2 and 3, we study a method of digitizing a space $X \subset \mathbf{R}^2$ in the MW-topological approach.

Theorem 4.1. *If $F : (X, U_X^2) \to (Y, U_Y^2)$ is an LMW-continuous map, then for every point $p \in X \cap \mathbf{Z}^2 := X'$ we obtain $F(SN_M(p)) \subset SN_M(f(p))$, where $f : (X', \gamma_{X'}) \to (Y', \gamma_{Y'})$ is the restriction of F to X' with the codomain $(Y', \gamma_{Y'})$ and $Y' := Y \cap \mathbf{Z}^2$. However the converse does not hold.*

Proof: By using the LMW-continuity of F, we can obtain the following:

$$\left\{ \begin{array}{l} F(SN_M(p)) = F(\cup_{x \in SN(p)} N_M(x)) = \cup_{f(x) \in SN(f(p))} F(N_M(x)) \\ \subset \cup_{f(x) \in SN(f(p))} N_M(f(x)) = SN_M(f(p)). \end{array} \right\} \tag{4.1}$$

By (4.1), we can observe $F(SN_M(p)) \subset SN_M(f(p))$.

Let us now prove that for each point $p \in X \cap \mathbf{Z}^2$ $F(SN_M(p)) \subset SN_M(f(p))$ does not imply the LMW-continuity of F. For some point $x \in N_M(p)$, since the hypothesis that $F(SN_M(p)) \subset SN_M(f(p))$ need not imply that $F(x) \in N_M(f(p))$ with $x \neq p$, the proof is completed. □

In order to digitize a space $X \subset \mathbf{R}^2$ in the MW-topological approach, we need to establish the following relation in $N_M(p)$ of Definition 7.

Definition 10. *In the set $X \subset \mathbf{R}^2$, we say that for the two points $x, y \in X$ x is related to y if $x, y \in N_M(p)$ for some point $p \in X \cap \mathbf{Z}^2$. In this case we use the notation $(x, y) \in L$ in the set (X, L).*

Lemma 4.2. *The relation L in the set (X, L) of Definition 10 is an equivalent relation.*

By Lemma 4.2, we can digitize X in an MW-topological approach, as follows.

Theorem 4.3. *For a given space $X \subset \mathbf{R}^2$ take $X \cap \mathbf{Z}^2 := X'$. Let us proceed the following three steps:*

(1) Delete the points $x \in X - \cup_{p \in X'} N_M(p)$ from the given space (X, U_X^2).

(2) By Lemma 4.2, consider $N_M(p)$ to be the equivalence class of the point p, i.e., $N_M(p) := [p] = p, p \in X'$.

(3) Assume $D(X) := \cup\{p | p \in X'\}$ to be the digitizing space of X.

Then there is a functor $D : LMC \to MC$ given by both $D((X, U_X^2)) = (D(X), \gamma_{D(X)})$ and $D(F) = f$, where $F : (X, U_X^2) := X \to (Y, U_Y^2) := Y \in LMC$ and f is defined to be the restriction of F to $(D(X), \gamma_{D(X)})$ with the codomain $(D(Y), \gamma_{D(Y)})$.
Concretely, we can say that f is a digitizing map of F in the MW-topological approach and this functor D is a digitizing functor based on the MW-topology.

Proof: Before proving Theorem 4.3, it is helpful to write an algorithm for digitizing a space $X \in LMC$ into a space $D(X) \in MC$. Namely, for $(X, U_X^2) \in LMC$ we can proceed its digitization in the MW-topological approach with the following steps (see Figure 3):

(Step 1) Take the points $p \in \mathbf{Z}^2 \cap X := X'$.

(Step 2) For each point $p \in X'$ take $N_M(p)$.

(Step 3) Delete the points $x \in X - \cup_{p \in X'} N_M(p)$.

(Step 4) By Lemma 4.2, assume $N_M(p)$ to be the quotient space p, i.e., $N_M(p) := p$.

(Step 5) Assume the set $D(X) := \cup\{p | p \in X'\}$ with the MW-topology such as $(D(X), \gamma_{D(X)}) \in MC$.

Let us now consider a space $(X, U_X^2) \in LMC$. After digitizing X in terms of the functor of this theorem, we denote it by $D(X) \subset X$.
To prove the assertion, we only suffice to show that

$$\left\{ \begin{array}{l} D(1_{(X, U_X^2)}) = 1_{(D(X), \gamma_{D(X)})} \text{ and} \\ D(G \circ F) = D(G) \circ D(F), \end{array} \right\}$$

where $(X, U_X^2) \in LMC$ and $G, F \in LMC$.
First, we can clearly observe that $D(1_{(X,U_X^2)}) = 1_{(D(X),\gamma_{D(X)})}$.
Second, in LMC take $F : (X, U_X^2) := X \to (Y, U_Y^2) := Y$ and $G : (Y, U_Y^2) := Y \to (Z, U_Z^2) := Z$. Then, in MC we can obtain the maps

$$\left\{ \begin{array}{l} D(F) := f : (D(X), \gamma_{D(X)}) \to (D(Y), \gamma_{D(Y)}) \text{ and} \\ D(G) := g : (D(Y), \gamma_{D(Y)}) \to (D(Z), \gamma_{D(Z)}). \end{array} \right\}$$

Consider the composition $G \circ F : X \to Z$. Then we can observe that $D(G \circ F) = g \circ f = D(G) \circ D(f)$. □

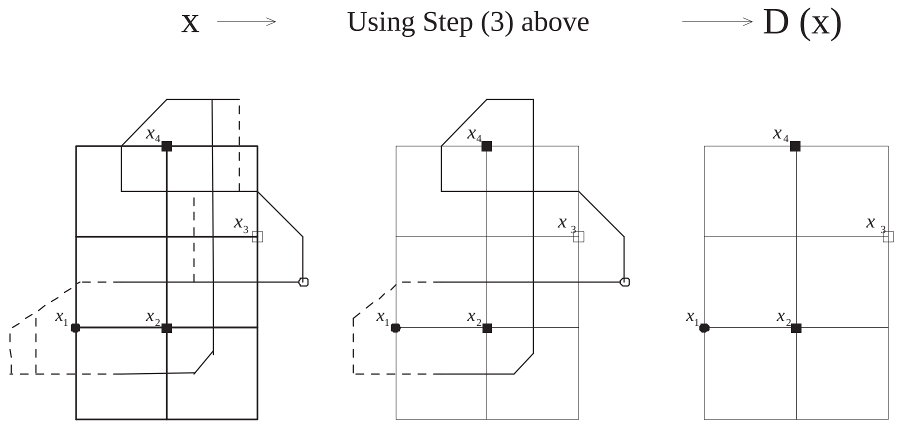

Figure 3. Configuration of the digitization followed from Theorem 4.3.

Lemma 4.4. *Let (A, U_A^2) be a connected nonempty subset of (X, U_X^2) in LMC. Then its digitization $(D(A), \gamma_{D(A)})$ of Theorem 4.3 need not be MW-connected in $(D(X), \gamma_{D(X)})$.*

Proof: Consider the subset (A, U_A^2) in Example 3.3 (see the set $A \subset X$ Figure 2(a)), where $A = \cup_{i\in\{2,3\}} N_M(x_i)$. While the set A is connected, $D(A)$ cannot be MW-connected. □

Corollary 4.5. *Let $F : (X, U_X^2) \to (Y, U_Y^2)$ be an LMW-continuous map in LMC. If $A \subset X$ is connected in LMC, then $f(A \cap \mathbf{Z}^2)$ need not be MW-connected, where $D(F) = f$ is the digitizing map in Theorem 4.3.*

Proof: Owing to the LMW-continuity of F, by Theorem 4.3, the map $D(F) := f$ is an MW-continuous map. But, as discussed in Lemma 4.4, for a connected subset A of $(X, U_X^2) \in LMC$ since $A \cap \mathbf{Z}^2$ need not be MK-connected, the image $f(A \cap \mathbf{Z}^2)$ by the MK-continuous map f need not be MK-connected. □

5. Classification of Spaces in terms of an LMW-homeomorph ism

In this section we classify spaces $X \subset \mathbf{R}^2$ in terms of an LMW-homeomorphism.

We recall that a *simple closed MW-curve with l elements* in $\mathbf{Z}^2$ (briefly, $SC_M^{2,l} := (x_i)_{i\in[0,l-1]_{\mathbf{Z}}}$) is a simple MW-path $(x_i)_{i\in[0,l]_{\mathbf{Z}}}$ with $x_0 = x_l$ and x_i and x_j are MW-connected if and only if either $j = i + 1(mod\, l)$ or $i = j + 1(mod\, l)$.

Definition 11. *For the functor $D : LMC \to MC$ of Theorem 4.3, if $D(X)$ is an $SC_M^{2,l}$, then we say that the space $X' := \cup_{p\in\mathbf{Z}^2\cap X} N_M(p)$ is an $SC_{LM}^{2,l}$ named by a simple closed LM-curve in $\mathbf{R}^2$ such that the set $D(X) := \{p|\, p \in X'$ has the cardinality l.*

Theorem 5.1. *If S_{LM}^{n,l_0} is LM-homeomorphic to S_{LM}^{2,l_1}, then $D(S_{LM}^{n,l_0})$ is MW-homeomorphic to $D(S_{LM}^{2,l_1})$. But the converse does not hold.*

Proof: For two S_{LM}^{2,l_0} and S_{LM}^{2,l_1} that are LM-homeomorphic to each other, by using the digitizing functor followed from Theorem 4.3, we can take two simple closed MW-curves with l_0 and l_1 elements in $\mathbf{Z}^2$ such as $S_M^{2,l_0}(\subset S_{LM}^{2,l_0})$ and $S_M^{2,l_1}(\subset S_{LM}^{2,l_1})$. By the hypothesis that $S_{LM}^{2,l_0} \approx_{LMW} S_{LM}^{2,l_1}$, we can obtain that $S_M^{2,l_0} \approx_{MW} S_M^{2,l_1}$, where $D(S_{LM}^{2,l_0}) = S_M^{2,l_0}$ and $D(S_{LM}^{2,l_1}) = S_M^{2,l_1}$.

Let us now prove that the converse does not hold with the following example. Consider the two spaces X and Y in LMC (see Figure 4) which are $S_{LM}^{2,10}$. Then we can take $D(X)$ (resp. $D(Y)$) in MC from X (resp. Y) so that $D(X)$ is MW-homeomorphic to $D(Y)$ as $S_M^{2,10}$. However, we can observe that the two spaces X and Y cannot be LMW-homeomorphic to each other. More precisely, we can observe that X has four double even points, five odd points and the only one even point. In addition, Y has the only one double even point, four even points and five odd points. Further, each of both the four double even points and one even point $x_i \in D(X)$ has its smallest open neighborhood $SN(x_i) = \{x_{i(mod\,10)-1}, x_i, x_{i(mod\,10)+1}\} \subset D(X)$, and each of both the four double even points and one even point $y_i \in D(Y)$ has its smallest open neighborhood $SN(y_i) = \{y_{i(mod\,10)-1}, y_i, y_{i(mod\,10)+1}\} \subset D(Y)$. In addition, each of the odd points in $D(X)$ (resp. $D(Y)$) has $SN(x_i) = \{x_i\} \subset D(X)$ (resp. $SN(y_i) = \{y_i\} \subset D(Y)$). In order for the spaces X and Y to be LMW-homeomorphic to each other, suppose that there is an LMW-homeomorphism $F : X \to Y$. Then, for each odd point $x_i \in D(X)$ $F(N_M(x_i))$ should be mapped into $N_M(f(x_i))$, where the point $f(x_i)$ is an odd point in $D(Y)$. If not, the map F cannot be an LMW-homeomorphism. Thus, for convenience, we can assume that $F(x_i) = y_i, i \in \{1, 3, 5, 7, 9\}$. Then, by Theorem 4.1, for each point $x_i, i \in [0, 9]_{\mathbf{Z}}$ we obtain $F(SN_M(x_i)) \subset SN_M(f(x_i))$. Thus the map F cannot be an LMW-homeomorphism between X and Y with the following reason. Consider the even points $x_4 \in D(X)$ and $y_4 \in D(Y)$, then $N_M(y_4) \subset Y$ cannot be homeomorphic to the singleton $\{x_4\} \subset X$, contrary to the property (3) of Definition 9. □

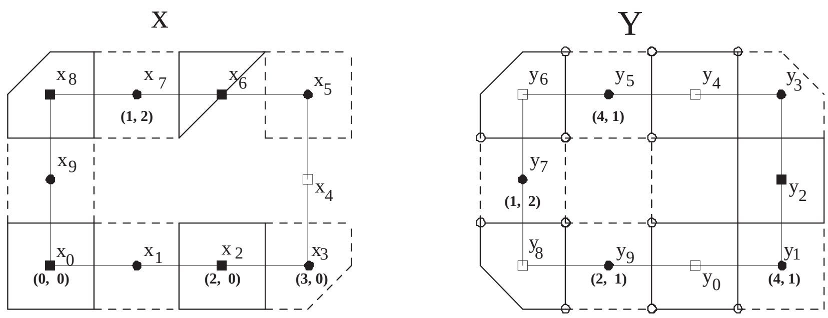

FIGURE 4. Comparison between S_{LM}^{2,l_0} and S_{LM}^{2,l_1} with $l_0 = l_1$.

Corollary 5.2. *Consider two* S_{LM}^{2,l_0} *and* S_{LM}^{2,l_1}. *Even if* $l_0 = l_1$, S_{LM}^{n,l_0} *need not be LM-homeomorphic to* S_{LM}^{2,l_1}.

Proof: As an example consider the two space X and Y in Figure 4 in which they have ten elements as $S_{LM}^{2,10}$. However, they cannot be LM-homeomorphic to each other. □

In view of Theorem 5.1 and Corollary 5.2, we need to establish the following relation.

Definition 12. *Let* $L(SC_M^{2,l})$ *be the set of all spaces* $X \subset \mathbf{R}^2$ *for which each of their digitizations followed from Theorem 4.3 is an* $SC_M^{2,l}$. *If* X *and* Y *belong to* $L(SC_M^{2,l})$, *then we say that* X *is related to* Y.

The relation of Definition 12 is clearly an *equivalence relation*. By $[X]$ we denote the *equivalence class* of $X \in L(SC_M^{2,l})$. According to Definition 12 and Theorem 5.1, we can obtain the following:

Corollary 5.3. *For two spaces $X, Y \in L(SC_M^{2,l})$ even if X and Y need not be LMW-homeomorphic to each other, we obtain $[X] = [Y]$.*

In view of Corollary 5.3, although two spaces X and Y in Figure 4 cannot be LMW-homeomorphic to each other, we can conclude that $[X] = [Y]$.

6. Summary and Further Work

We have studied the set of points displayed on $\mathbf{R}^2$ in terms of a digitization followed from the Marcus Wyse topological approach. In order to examine the possibility of a usage of some topological structures into the field of telecommunications related to the future internet and to study spaces $X \subset \mathbf{R}^2$ in the Marcus Wyse topological approach. the paper develops the notions of a local rule related to the Marcus Wyse topological structure, lattice based Marcus Wyse continuity and lattice based Marcus Wyse homeomorphism. As a further work we need to formulate various topological tools which can support the study of the future internet.

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