Honam Mathematical J. **33** (2011), No. 4, pp. 557–573
http://dx.doi.org/10.5831/HMJ.2011.33.4.557

AFFINE YANG-MILLS CONNECTIONS ON NORMAL HOMOGENEOUS SPACES

Joon-Sik Park

Abstract. Let G be a compact and connected semisimple Lie group, H a closed subgroup, $\mathfrak{g}$ (resp. $\mathfrak{h}$) the Lie algebra of G (resp. H), B the Killing form of $\mathfrak{g}$, g the normal metric on the homogeneous space G/H which is induced by -B. Let D be an invarint connection with Weyl structure (D, g, ω) in the tangent bundle over the normal homogeneous Riemannian manifold $(G/H, g)$ which is projectively flat. Then, the affine connection D on $(G/H, g)$ is a Yang-Mills connection if and only if D is the Levi-Civita connection on $(G/H, g)$

§1. Introduction

The problem of finding metrics and connections which are critical points of some functional plays an important role in global analysis and Riemannian geometry. A Yang-Mills connection is a critical point of the Yang-Mills functional

$$(1.1)\qquad \mathcal{YM}(D) = \frac{1}{2}\int_M ||R^D||^2\ v_g$$

on the space $\mathfrak{C}_E$ of all connections in a smooth vector bundle E over a closed (compact and connected) Riemannian manifold (M, g), where R^D is the curvature of $D \in \mathfrak{C}_E$. Equivalently, D is a Yang-Mills connection if it satisfies the Yang-Mills equation (cf. [2,8,13,14.17])

$$(1.2)\qquad \delta_D R^D = 0,$$

(the Euler-Lagrange equations of the variational principle associated with (1.1)).

Received September 28, 2011. Accepted October 18, 2011.
2000 Mathematics Subject Classification. 53C07; 53A15.
Key words and phrases. Yang-Mills connection; Weyl structure; invariant connection; normal homogeneous Riemannian manifold.

If D is a connection in a vector bundle E with bundle metric h over a Riemannian manifold (M, g), then the connection D^* given by

$$(1.3) \quad h(D_X^* s\ ,\ t) = X(h(s\ ,\ t)) - h(s\ ,\ D_X t), \quad (X \in \mathfrak{X}(M)) \text{ and } s, t \in \Gamma(E))$$

is referred to as *conjugate* (cf. [1, 10]) to D.

Recently, using the concept of conjugate connection, the present author obtained the following

Theorem 1.1 [11]. *A connection D in a vector bundle E over a closed Riemannian manifold (M, g) is a Yang-Mills connection if and only if the conjugate connection D^* is a Yang-Mills connection.*

If a tortion free affine connection D in the tangent bundle over a Riemannian manifold (M, g) satisfies $Dg = \omega \otimes g$ for a 1-form ω on M, then (D, g, ω) is called a *Weyl structure* (cf. [4,15]). By virtue of Theorm 1.1, the present author got the following

Theorem 1.2 [12]. *Let D be a Yang-Mills connection with Wely structure (D, g, ω) in the tangent bundle TM over a closed Riemannian manifold (M, g). Then $dw = 0$.*

Let G be a compact Lie group, H a closed subgroup of G, $\mathfrak{g}$ (resp. $\mathfrak{h}$) the Lie algebra of G (resp. H), and $\mathfrak{m}$ a subspace of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$. A homogeneous Riemannian metric g on G/H is said to be *normal* homogeneous if there exists $Ad(G)$ invariant inner product $< , >$ on $\mathfrak{g}$ such that $< \mathfrak{m}, \mathfrak{h} >= 0$ and $< , >_{\mathfrak{m}}= g_{\{H\}}$.

Through this paper, let G be a compact connected semisimple Lie group, H a closed subgroup of the group G, $\mathfrak{g}$ (resp. $\mathfrak{h}$) the Lie algebra of G (resp. H), and B the Killing form of $\mathfrak{g}$, $< , >:= -B$, $\mathfrak{m}$ the subspace of $\mathfrak{g}$ such that $< \mathfrak{m}, \mathfrak{h} >= 0$ and $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$, and g the normal homogeneous Riemannian metric on the space G/H such that $< , >_{\mathfrak{m}}= g_{\{H\}}$,

By the help of the second Bianchi identity $\delta_D R^D = 0$, the following is well known:

A necessary and sufficient condition for a connection D on a closed Riemannian manifold (M, g) to be a Yang-Mills connection is that the curvature tensor field for D is harmonic.

In general, the curvature tensor R^{∇} for the Levi-Clvita conection ∇ in TM over a closed Riemannian manifold (M, g) is not harmonic, and hence ∇ is not a Yang-Mills connection.

In this paper, we get a necessary and sufficient condition for the Levi-Civita connection ∇ on the normal homogeneous Riemannian manifold $(G/H, g)$ to be a Yang-Mills connection (cf. Proposition 3.1, Theorem

3.4). And, using these results, Theorem 1.2 and the fact that the 1-form ω in the Weyl structure (D, g, ω) related to an invariant affine connection D on the normal homogeneous Riemannian manifold $(G/H, g)$ is invariant on G/H (cf. Lemma 4.2), we get the following

Theorem 4.3. *Let D be an invariant connection with Weyl structure (D, g, ω) in the tangent bundle over the normal homogeneous space $(G/H, g)$, dim $G/H = m \geq 3$. Assume the connection D is projectively flat. Then, a necessary and sufficient condition for the connection D to be a Yang-Mills connection is $\omega = 0$.*

Corollary 4.10. *Under the same situation and assumption as in* Theorem 4.3, *the connection D is a Yang-Mills connection if and only if D coincides with the Levi-Civita connection ∇ on the normal homogeneous space $(G/H, g)$.*

§2. Yang-Mills connections in a vector bundle and Weyl structures in a tangent bundle

2.1. In this subsection, we treat the Yang-Mills equation in vector bundles over a closed Riemannian manifold (M, g), using the concept of conjugate connection.

Let E be a vector bundle, with bundle metric h, over an n-dimensional closed Riemannian manifold (M, g). Let $D \in \mathfrak{C}_E$ and ∇ the Levi-Civita connection on (M, g). The pair (D, ∇) induces a connection in product bundles $\bigwedge^p TM^* \otimes E$, denoted by D, as well. Set $A^p(E) := \Gamma(\bigwedge^p TM^* \otimes E)$. We consider the differential operator

$$d_D : A^p(E) \longrightarrow A^{p+1}(E),$$

$$(d_D\varphi)(X_1, X_2, \cdots, X_{p+1}) = \sum_{i=1}^{p+1}(-1)^{i+1}(D_{X_i}\varphi)(X_1, \cdots, \widehat{X}_i, \cdots, X_{p+1}),$$

$$\varphi \in A^p(E),\ X_i \in \mathfrak{X}(M)\ (i = 1, 2, \cdots, p+1),$$

which are defined by

$$d_D(\omega \otimes \xi) : = d\omega \otimes \xi + (-1)^p \omega \wedge D\xi,$$
$$D_X(\omega \otimes \xi) : = (\nabla_X \omega) \otimes \xi + \omega \otimes D_X\xi,$$

for $\omega \in \Gamma(\bigwedge^p TM^*),\ \xi \in \Gamma(E)$ and $X \in \mathfrak{X}(M)$.

Let δ_D be the formal adjoint of d_D with respect to the L^2-inner product

$$(\varphi, \psi) = \int_M < \varphi, \psi > v_g$$

for $\varphi, \psi \in A^p(E)$. Here $<\ ,\ >$ is the bundle metric in $\bigwedge^p TM^* \otimes E$ induced by the pair (g,h) and v_g is the canonical volume form on (M,g). The following identity is elementary, yet crucial (cf. [2,3])

$$\delta_D \varphi = (-1)^{p+1}(*^{-1} \cdot d_{D^*} \cdot *)(\varphi) = (-1)^{np+1}(* \cdot d_{D^*} \cdot *)(\varphi) \tag{2.1}$$

for any $\varphi \in A^{p+1}(E)$. Here, $* : A^q(E) \longrightarrow A^{n-q}(E)$, $(0 \le q \le n)$, is the Hodge operator with respect to g. Let $\{e_i\}_{i=1}^n$ be a local orthonormal frame on (M,g) and $\{\theta^j\}_{j=1}^n$ the dual coframe. Let $\{d_\alpha\}_{\alpha=1}^r$ be a local orthonormal frame on (E,h) and $\{\sigma_\alpha\}_{\alpha=1}^r$ the dual coframe, where r is the rank of E. Note that (2.1) may also be written as (cf. [2])

$$(\delta_D \varphi)(X_1, \cdots, X_p) = -\textstyle\sum_{i=1}^n (D^*_{e_i}\varphi)(e_i, X_1, \cdots, X_p). \tag{2.2}$$

The connections D, $D^* \in \mathfrak{C}_E$ naturally induce connections, denoted by the same symbols, in $\mathrm{End}(E)$ $(:= E \otimes E^*)$. Then, a straightforward argument shows that D, $D^* \in \mathfrak{C}_{\mathrm{End}(E)}$ are conjugate connections. The following curvature property is immediate (cf. [2])

$$h(R^D(X,Y)s,t) = -h(s, R^{D^*}(X,Y)t), \text{ for } s,t \in \Gamma(E) \text{ and } X,Y \in \mathfrak{X}(M). \tag{2.3}$$

Specially in $E = TM$ over a closed Riemannian manifold (M,g), we easily find from (1.3) and (2.2) that $D \in \mathfrak{C}_E$ is a Yang-Mills connection if and only if

$$\begin{aligned}(\delta_D R^D)(X)Y &= -\sum_{i=1}^n (D^*_{e_i} R^D)(e_i, X)Y \\ &= -\sum_{i=1}^n \{(D^*_{e_i} R^D)(e_i, X)Y - R^D(\nabla_{e_i} e_i, X)Y \\ &\qquad - R^D(e_i, \nabla_{e_i} X)Y - R^D(e_i, X)D^*_{e_i} Y\} \\ &= 0,\end{aligned} \tag{2.4}$$

where $X \in \mathfrak{X}(M)$ and $Y \in \Gamma(TM)$. Moreover, $R^D \in \Gamma(\bigwedge^2 TM^* \otimes \mathrm{End}(TM))$.

2.2. In this subsection, we introduce some well known facts on a Weyl structure (D, g, ω) on a closed (compact and connected) Riemannian manifold (M,g).

Let (D, g, ω) be a Weyl structure in the tangent bundle TM over a closed Riemannian manifold $(M.g)$, i.e.,

$$Dg = \omega \otimes g, \qquad \text{and} \qquad T^D = 0 \quad \text{(torsion free)}, \tag{2.5}$$

for some 1-form on M. Then, we have for $X, Y \in \mathfrak{X}(M)$ and $Z \in \Gamma(TM)$

(2.6)
$$\begin{cases} D^*_X Z = D_X Z + \omega(X)Z, \\ R^D(X,Y) - R^\nabla(X,Y) = [\nabla_X, \alpha_Y] + [\alpha_X, \nabla_Y] + [\alpha_X, \alpha_Y] - \alpha_{[X,Y]} \end{cases}$$

where $\alpha := D - \nabla \in \Gamma(TM^* \otimes \mathrm{End}(TM))$. From (2.4) and (2.6), we have for $Y \in \mathfrak{X}(M)$ and $Z \in \Gamma(TM)$

(2.7)
$$\begin{aligned} (\delta_D R^D)(Y)Z &= -\sum_{i=1}^n (D^*_{X_i} R^D)(X_i, Y)Z \\ &= -\sum_{i=1}^n \{D_{X_i}(R^D(X_i,Y)Z) - R^D(\nabla_{X_i}X_i, Y)Z \\ &\qquad - R^D(X_i, \nabla_{X_i}Y)Z - R^D(X_i,Y)D_{X_i}Z\}, \end{aligned}$$

where $\{X_i\}_{i=1}^n$ is an (locally defined) orthonormal frame on (M,g). For an (locally defined) orthonormal frame $\{X_i\}_{i=1}^n$, let $\{\theta_j\}_{j=1}^n$ be the local orthonormal coframe on (M,g). For the frames $\{X_i\}_{i=1}^n$ and $\{\theta_j\}_{j=1}^n$, we introduce ${\Gamma_{ij}}^l := \theta^l(\nabla_{X_i}X_j)$. Then, we have

(2.8)
$$D_{X_i}X_j = \sum_{l=1}^n {\Gamma_{ij}}^l X_l, \quad \text{and} \quad D_{X_i}\theta^j = -\sum_{i=1}^n {\Gamma_{il}}^j \theta^l.$$

By virtue of the fact $Dg = \omega \otimes g$, we have

(2.9)
$${\Gamma_{ij}}^j = -\frac{1}{2}\omega(X_i) \quad \text{for each } j, \qquad {\Gamma_{ij}}^k = -{\Gamma_{ik}}^j \quad (j \neq k).$$

Moreover we have for $X, Y \in \mathfrak{X}(M)$

(2.10)
$$\alpha_X Y := D_X Y - \nabla_X Y = \alpha_Y X,$$

since the connections are torision free. Using (2.10) and fundamental properties of a connection, we get (cf. [15])

(2.11)
$$\alpha_X Y = \frac{1}{2}\{g(X,Y)\omega^\# - \omega(X)Y - \omega(Y)X\},$$

where $\omega^\# := \sum_{l=1}^n \omega(X_i)X_i$.

§3. Yang-Mills Levi-Civita connection on $(G/H, g)$

3.1. Let G be an n-dimensional compact connected semisimple Lie group and H a closed subgroup of G. Let $\mathfrak{g}$ be the Lie algebra (the set of all left invariant vector filelds on G) of the group G, $\mathfrak{h}$ the Lie algebra of H, and B the Killing form of $\mathfrak{g}$.

We denote $-B =: < \ , \ >$. Then, the inner product $< \ , \ >$ is an $Ad(G)$-invariant inner product on $\mathfrak{g}$, and there exists the subspace $\mathfrak{m}$ of $\mathfrak{g}$ such that $\mathfrak{m} \oplus \mathfrak{h} = \mathfrak{g}$ and $< \mathfrak{m}, \mathfrak{h} >= 0$, where $Ad(G)$ denotes the adjoint representation of G in $\mathfrak{g}$. We denote by p_o the point represented by the coset H in the homogeneous space G/H. Then, the subspace $\mathfrak{m}$ of $\mathfrak{g}$ is identified with the tangent space $T_{p_o}G/H$ at p_o. Let g be the invarient Riemannian metric on the homogeneous manifold G/H which is induced from $< \ , \ > |_{\mathfrak{m}\times\mathfrak{m}}$. Then, the homogeneous Riemannian manifold $(G/H, g)$ is a normal homogeneous manifold (cf. [1, p. 3]).

Let $\{X_i\}_{i=1}^n$ be an orthonormal base on $(\mathfrak{g}, < \ , \ >)$ such that the first m elements span $\mathfrak{m}$ and the last $n-m$ elements span $\mathfrak{h}$. For the calculus, we define X^*, $\quad X \in \mathfrak{m} \ = T_{P_o}(G/H)$, on a some proper neighborhood $U = \pi(\exp V)$, $\ (0 \in V \subset \mathfrak{m} \subset \mathfrak{g})$, of p_o in G/H by

$$X^*{}_{xH} := (\tau_x)_* X \in T_{xH}(G/H), \quad x \in \exp V \subset G \text{ (cf. [9, p. 42])}.$$

Here, τ_x denotes the transformation of G/H which is induced by $x \in G$. Then, $\{X_i^*\}_{i=1}^n$ is an orthonormal frame on the neighbourhood U of p_o in G/H. Let $\{\theta^{j*}\}_{j=1}^m$ be a system of 1-forms on U which is dual to $\{X_i^*\}_{i=1}^n$. The Levi-Civita connection ∇ for the metric g is given by (cf. [9, p. 52])

$$\nabla_X Y^* = \tfrac{1}{2}[X, Y]_{\mathfrak{m}}, \quad (X, Y \in \mathfrak{m}), \text{ at } p_o = \{H\} \in G/H, \tag{3.1}$$

where $X_{\mathfrak{m}}$ denotes the $\mathfrak{m}$-component of the element $X \in \mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$. Moreover by virtue of (3.1) the curvature tensor field R^{∇} at p_o is given by (cf. [9, p. 47])

$$\begin{aligned} R^{\nabla}(X,Y)Z &:= ([\nabla_{X^*}, \nabla_{Y^*}]Z^*)_{p_o} - \nabla_{[X,Y]}Z^* \\ &= \frac{1}{4}\ \{\ [X,[Y,Z]_{\mathfrak{m}}]_{\mathfrak{m}} - [Y,[X,Z]_{\mathfrak{m}}]_{\mathfrak{m}}\} \\ &\quad - \frac{1}{2}[[X,Y]_{\mathfrak{m}}, Z]_{\mathfrak{m}} - [[X,Y]_{\mathfrak{h}}, Z], \quad (X,Y,Z \in \mathfrak{m}). \end{aligned} \tag{3.2}$$

From now on in this paper, the indices $i, j, k, l, s, t, \cdots$ run over the range $\{1, 2, 3, \cdots, m\}$, and the indices $a, b, c, \cdots$ run over the range $\{m+1, m+2, \cdots, n\}$, $(m = \dim \mathfrak{m},\ n = \dim \mathfrak{g})$, without further specification. We denote

$$[X_i, X_j] = \textstyle\sum_k C_{ij}{}^k X_k + \sum_a C_{ij}{}^a X_a \tag{3.3}$$

By virtue of (3.1) and (3.3), we have

$$(\nabla_{X_i^*} X_j^*)_{p_o} = \tfrac{1}{2}\textstyle\sum_k C_{ij}{}^k X_k. \tag{3.4}$$

We get from (3.2) and (3.3)

$$
\begin{aligned}
R^{\nabla}(X_i, X_j)X_k = & \sum_{l,t} \frac{1}{4}(C_{jk}{}^t C_{it}{}^l - C_{ik}{}^t C_{jt}{}^l - 2C_{ij}{}^t C_{tk}{}^l)X_l \\
& - \sum_l \sum_a C_{ij}{}^a C_{ak}{}^l X_l.
\end{aligned} \tag{3.5}
$$

By the help of (3.3), $< [X_i, X_j], X_k >= - < X_j, [X_i, X_k] >$, and $< [X_i, X_j], X_a >= -B([X_i, X_j], X_a) = B(X_j, [X_i, X_a]) = - < X_j, [X_i, X_a] >$, we have

$$
C_{ij}{}^a = -C_{ia}{}^j = -C_{ji}{}^a, \quad C_{ij}{}^k = -C_{ik}{}^j = -C_{ji}{}^k. \tag{3.6}
$$

From (3.6), $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$, $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$, and the facts

$$
\begin{aligned}
< X_i, X_j > &= -B(X_i, X_j) = -\mathrm{Tr}(adX_i adX_j) = \delta_{i,j}, \\
< X_i, X_a > &= -B(X_i, X_a) = -\mathrm{Tr}(adX_i adX_a) = 0,
\end{aligned}
$$

we obtain

$$
\textstyle\sum_{l,t} C_{lt}{}^i C_{lt}{}^j + 2\sum_l \sum_a C_{la}{}^i C_{la}{}^j = \delta_{ij}, \quad \sum_{l,t} C_{tl}{}^a C_{tl}{}^i = 0. \tag{3.7}
$$

The Ricci tensor Ric^{∇} for the Levi-Civita connection ∇ is defined by

$$
Ric^{\nabla}(Y, Z) = \mathrm{Tr}\{X \longmapsto R^{\nabla}(X, Y)Z\}. \tag{3.8}
$$

We obtain from (3.5), (3.6), (3.7) and (3.8)

$$
Ric^{\nabla}(X_j, X_k) = \tfrac{1}{4}(\delta^{jk} + 2\textstyle\sum_i \sum_a C_{ia}{}^j C_{ia}{}^k). \tag{3.9}
$$

By virtue of (3.9), the scalar curvature $S(g)$ on the normal homogeneous space $(G/H, g)$ is given as follows;

$$
S(g) = \tfrac{1}{4}(m + 2\textstyle\sum_{it} \sum_a C_{ia}{}^t C_{ia}{}^t). \tag{3.10}
$$

3.2. In this subsection, we retain the notation as in 3.1. We obtain

Proposition 3.1. *A necessary and sufficient condition for the Levi -Civita connection ∇ on the normal homogeneous Riemannian manifold $(G/H, g)$ to be a Yang-Mills connection is*

$$
\begin{aligned}
& 2\sum_{i,t} \sum_a (C_{ij}{}^t C_{tk}{}^a C_{is}{}^a + C_{ij}{}^a C_{ak}{}^t C_{is}{}^t - C_{ij}{}^a C_{ai}{}^t C_{ks}{}^t) \\
& = \sum_{i,t} \sum_a (C_{is}{}^a C_{ai}{}^t C_{kj}{}^t + C_{ik}{}^a C_{ai}{}^t C_{sj}{}^t),
\end{aligned} \tag{3.11}
$$

that is,

$$2\sum_i(<[[X_i,X_j]_{\mathfrak{m}},X_k]_{\mathfrak{h}}+[[X_i,X_j]_{\mathfrak{h}},X_k]_{\mathfrak{m}},[X_i,X_s]>$$
$$-<[[X_i,X_j]_{\mathfrak{h}},X_i],[X_k,X_s]>)=\sum_i(<[[X_i,X_s]_{\mathfrak{h}},X_i],[X_k,X_j]>$$
$$+<[[X_i,X_k]_{\mathfrak{h}},X_i],[X_s,X_j]>).$$

In order to prove this proposition, let's calculus $(\delta_\nabla R^\nabla)(X_j)X_k$. From (2.4) and (3.1), we have

$$\begin{aligned}(\delta_\nabla R^\nabla)(X_j)X_k&\\ &=-\sum_i(\nabla_{X_i}R^\nabla)(X_i,X_j)X_k\\ &=-\sum_i\{\nabla_{X_i}(R^\nabla(X_i^*,X_j^*)X_k^*)-R^\nabla(X_i,\nabla_{X_i}X_j^*)X_k\\ &\quad-R^\nabla(X_i,X_j)\nabla_{X_i}X_k^*\}.\end{aligned}\tag{3.12}$$

In order to analyze (3.12), we obtain from (3.5), (3.6) and (3.7)

Lemma 3.2. *The terms appeared in* (3.12) *are changed as follows* ;

(a) $$\sum_i\nabla_{X_i}(R^\nabla(X_i^*,X_j^*)X_k^*)$$
$$=\frac{1}{8}\sum_s\{-C_{jk}{}^s-3\sum_{i,l,t}C_{ik}{}^tC_{jt}{}^lC_{il}{}^s$$
$$+2\sum_{i,l}\sum_a(C_{jk}{}^lC_{ia}{}^lC_{ia}{}^s-2C_{ij}{}^aC_{ak}{}^lC_{il}{}^s)\}X_s,$$

(b) $$\sum_i R^\nabla(X_i,\nabla_{X_i}X_j^*)X_k$$
$$=\frac{1}{4}\sum_s(C_{jk}{}^s+\sum_{i,t,l}C_{ik}{}^tC_{jt}{}^lC_{il}{}^s-2\sum_{i,t}\sum_a C_{ia}{}^jC_{ia}{}^tC_{tk}{}^s)X_s,$$

(c) $$\sum_i R^\nabla(X_i,X_j)\nabla_{X_i}X_k^*$$
$$=\frac{1}{8}\sum_s\{C_{jk}{}^s+3\sum_{i,l,t}C_{ik}{}^tC_{jt}{}^lC_{il}{}^s$$
$$-2\sum_{i,l}\sum_a(C_{ia}{}^kC_{ia}{}^lC_{jl}{}^s+2C_{ik}{}^lC_{ij}{}^aC_{al}{}^s)\}X_s.$$

By virtue of (3.12) and Lemma 3.2, we obtain

$$
\begin{aligned}
&(\delta_\nabla R^\nabla)(X_j)X_k \\
&\quad = \frac{1}{4}\sum_s \{2C_{jk}{}^s + 4\sum_{i,l,t} C_{ik}{}^t C_{jt}{}^l C_{il}{}^s \\
&\qquad + \sum_{i,l}\sum_a (2C_{ij}{}^a C_{ak}{}^l C_{il}{}^s - 2C_{ik}{}^l C_{ij}{}^a C_{al}{}^s \\
&\qquad - 2C_{ia}{}^j C_{ia}{}^l C_{lk}{}^s - C_{ia}{}^l C_{ia}{}^s C_{jk}{}^l - C_{ia}{}^k C_{ia}{}^l C_{jl}{}^s)\} X_s.
\end{aligned} \tag{3.13}
$$

In order to analyze (3.13), we get

Lemma 3.3

$$
\begin{aligned}
2\sum_{i,l,t} C_{ik}{}^t C_{jt}{}^l C_{il}{}^s = -C_{jk}{}^s + \sum_{i,l}\sum_a (&C_{ij}{}^l C_{lk}{}^a C_{ai}{}^s + C_{jk}{}^l C_{ia}{}^l C_{ia}{}^s \\
&+ 2C_{ij}{}^a C_{ak}{}^l C_{li}{}^s - C_{ki}{}^l C_{lj}{}^a C_{as}{}^i).
\end{aligned}
$$

Proof. By virtue of (3.3), (3.6), (3,7) and $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$, we get

$$
\begin{aligned}
&\sum_{i,l,t} C_{ik}{}^t C_{jt}{}^l C_{il}{}^s \\
&\quad = \sum_i < [X_i, [X_j, [X_i, X_k]]], X_s > - \sum_{i,l}\sum_a (C_{ik}{}^a C_{ja}{}^l C_{il}{}^s + C_{ik}{}^l C_{jl}{}^a C_{ia}{}^s) \\
&\quad = -\sum_i < [X_j, [X_i, X_k]], [X_i, X_s] > - \sum_{i,l}\sum_a (C_{ik}{}^a C_{ja}{}^l C_{il}{}^s + C_{ik}{}^l C_{jl}{}^a C_{ia}{}^s) \\
&\quad = \sum_i < [X_i, [X_k, X_j]] + [X_k, [X_j, X_i]], [X_i, X_s] > \\
&\qquad - \sum_{i,l}\sum_a (C_{ik}{}^a C_{ja}{}^l C_{il}{}^s + C_{ik}{}^l C_{jl}{}^a C_{ia}{}^s) \\
&\quad = -\sum_{i,l,t} C_{ik}{}^t C_{jt}{}^l C_{il}{}^s - C_{jk}{}^s \\
&\qquad + \sum_{i,l}\sum_a (C_{ij}{}^l C_{lk}{}^a C_{ai}{}^s + C_{jk}{}^l C_{ia}{}^l C_{ia}{}^s + 2C_{ij}{}^a C_{ak}{}^l C_{li}{}^s - C_{ki}{}^l C_{lj}{}^a C_{as}{}^i).
\end{aligned}
$$

Hence, the proof of this Lemma is completed.

By the help of (3.13) and Lemma 3.3, we obtain

$$(3.14)\quad \begin{aligned}(\delta_\nabla R^\nabla)(X_j)X_k = \frac{1}{4}\sum_{i,l,s}\sum_a (&2C_{ij}{}^aC_{ak}{}^lC_{is}{}^l - 2C_{ia}{}^jC_{ia}{}^lC_{lk}{}^s \\ &+ 2C_{ij}{}^lC_{lk}{}^aC_{ai}{}^s + C_{jk}{}^lC_{ia}{}^lC_{ia}{}^s \\ &- C_{ia}{}^kC_{ia}{}^lC_{jl}{}^s)X_s.\end{aligned}$$

By virtue of (3.6) and (3.14), we obtain Proposition 3.1.

3.3. In this subsection, we retain the notations as in 3.1 and 3.2. We obtain

Theorem 3.4. *Assume the normal homogeneous Riemannian manifold $(G/H, g)$ is Einstein. Then, a necessary and sufficient condition for the Levi-Civita connection ∇ on $(G/H, g)$ to be a Yang-Mills connection is*

(3.15) $\sum_{i,l}\sum_a C_{ia}{}^lC_{ia}{}^lC_{kj}{}^s = m\sum_{i,l}\sum_a (C_{ji}{}^aC_{ak}{}^lC_{is}{}^l + C_{ji}{}^lC_{lk}{}^aC_{is}{}^a)$,

that is,

$$\begin{aligned}&\sum_i\sum_a < [X_i, X_a], [X_i, X_a] >< [X_k, X_j], X_s > \\ &\quad = m\sum_i < [[X_j, X_i]_{\mathfrak{h}}, X_k] + [[X_j, X_i]_{\mathfrak{m}}, X_k]_{\mathfrak{h}}, [X_i, X_s] >,\end{aligned}$$

*where $m =$*dim *G/H.*

Proof. By the assumption $Ric^\nabla = cg$, so $c = \frac{S(g)}{m}$. By the help of this fact and (3.10), we have

(3.16) $c = \frac{1}{4m}(m + 2\sum_{i,l}\sum_a C_{ia}{}^lC_{ia}{}^l)$.

By virtue of the fact $Ric^\nabla = cg$, (3.9) and (3.16), we get

(3.17) $\sum_i\sum_a C_{ia}{}^jC_{ia}{}^k = \frac{1}{m}\sum_{i,l}\sum_a C_{ia}{}^lC_{ia}{}^l\delta_{jk}$.

From (3.14) and (3.17), we have

$$(3.18)\quad \begin{aligned}(\delta_\nabla R^\nabla)(X_j)X_k = \frac{1}{2}\sum_{i,l,s}\sum_a (&C_{ij}{}^aC_{ak}{}^lC_{is}{}^l + C_{ij}{}^lC_{lk}{}^aC_{ai}{}^s \\ &- \frac{1}{m}C_{ia}{}^lC_{ia}{}^lC_{jk}{}^s)X_s.\end{aligned}$$

By virtue of (3.6) and (3.18), the proof of this theorem is completed.

§4. Invariant Yang-Mills connections in normal homogeneous spaces

In this section, we retain the notations as in §3.

4.1. In this subsection, we show that the 1-form ω in the Weyl structure (D, g, ω) related to an invariant affine connection D on the space $(G/H, g)$ is invariant on G/H.

The following Lemma is well known (cf. [9, Theorem 8.1])

Lemma 4.1. *In the normal homoreneous space $(G/H, g)$, there exists a one-to-one correspondence between the set of all invariant affine connections on (G/H) and the set of all bilinear functions β on $\mathfrak{m} \times \mathfrak{m}$ with values in $\mathfrak{m}$ which are invariant by $Ad(H)$, that is, $Ad(h)\ \beta(X, Y) = \beta(Ad(h)X, Ad(h)Y)$ for $X, Y \in \mathfrak{m}$ and $h \in H$. The correspondence is given by*

$$\beta(X, Y) = (D_{X^*}Y^*)_{p_o}. \tag{4.1}$$

For the sake of simplicity, we call such a bilinear function β on $\mathfrak{m} \times \mathfrak{m}$ a *connection function* on $\mathfrak{m} \times \mathfrak{m}$. Each invariant affine connection D on G/H naturally induces an invariant connection, in terms of the connection function β, in various product vector bundles generated by the tangent bundle and the cotangent bundle over G/H. By virtue of (2.10), (2.11) and Lemma 4.1, we have

Lemma 4.2. *Let D be an invariant affine connection on the normal homogeneous space $(G/H, g)$ which admits a Weyl structure (D, g, ω), i.e., $Dg = \omega \otimes g$ and $T^D = 0$ where ω is a 1-form on the space G/H. Then, ω is invariant, that is, $\tau_x^*\omega = \omega$ $(x \in G)$.*

4.2. In this subsection, we prove the following main result in this paper.

Theorem 4.3. *Let D be an invariant affine connection with Weyl structure (D, g, ω) in the tangent bundle over the normal homogeneous space $(G/H, g)$, dim $G/H = m \geq 3$. Assume the connection D is projectively flat. Then, a necessary and sufficient condition for the connection D to be a Yang-Mills connection is $\omega = 0$.*

In order to prove the above main theorem, assume $d\omega = 0$. This condition $d\omega = 0$ is a necessary condition for the connection D to be a Yang-Mills connection (cf. Theorem 1.2). Then, we have

$$\textstyle\sum_{k=1}^m \omega_k {C_{ij}}^k = 0, \tag{4.2}$$

where each $\omega_k := \omega({X_k}^*)$ for the orthonormal frame $\{{X_i}^*\}_{i=1}^m$ on the neighborhod $U := \pi(\exp V)$ of p_o. In fact, from (3.3) we have $(d\omega({X_i}^*, {X_j}^*))_{p_o} = ({X_i}^*\omega({X_j}^*) - {X_j}^*\omega({X_i}^*) - d\omega([{X_i}^*, {X_j}^*]))_{p_o} = -\sum_k {C_{ij}}^k \omega_k = 0$, since $\omega({X_i}^*)$ is constant on U by the help of Lemma 4.2. Using

$Dg = \omega \otimes g$, $T^D = 0$ and fundamental properties of a connection, we get

(4.3) $(\alpha_{X_i}{}^* X_k{}^*)_{p_o} = (D_{X_i}{}^* X_k{}^* - \nabla_{X_i}{}^* X_k{}^*)_{p_o} = \frac{1}{2}(\delta_{ik}\omega^\sharp - \omega_i X_k - \omega_k X_i).$

By virtue of (2.8), (3.1), and Lemma 4.1, we have at the point p_o

$$(4.4)\quad \begin{aligned}(\delta_D R^D)(X_j)X_k = -\sum_{i=1}^{m}\{&\beta(X_i, R^D(X_i, X_j)X_k)\\ &-\frac{1}{2}\sum_{l=1}^{m} C_{ij}{}^l R^D(X_i, X_l)X_k - R^D(X_i, X_j)\beta(X_i, X_k)\}.\end{aligned}$$

From (3.1), (3.3), (3.6) and (4.3), we obtain

(4.5) $\Gamma_{ik}{}^l = \frac{1}{2}(C_{ik}{}^l + \delta_{ik}\omega_l - \omega_i\delta_k{}^l - \omega_k\delta_i{}^l)$, $\quad \sum_{i=1}^m \beta(X_i, X_j) = \frac{1}{2}(m-2)\omega^\sharp$,

where $\theta^{l*}(D_{X_i{}^*}X_k{}^*) =: \Gamma_{ik}{}^l = \theta^l(\beta(X_i, X_j))$ on the neighbourhood $U = \pi(\exp V)$ of p_o, $(0 \in V \subset \mathfrak{m})$. Moreover, we obtain the following expression for the value at p_o of the curvature tensor field for the invariant connection D (cf. [9, 16])

(4.6) $R^D(X_i, X_j)X_k = \sum_l\{\sum_t(\Gamma_{jk}{}^t\Gamma_{it}{}^l - \Gamma_{ik}{}^t\Gamma_{jt}{}^l - C_{ij}{}^t\Gamma_{tk}{}^l) - \sum_a C_{ij}{}^a C_{ak}{}^l\}X_l.$

Using (4.2), (4.5) and (4.6), we get the following

Lemma 4.4. *Let D be an invariant affine connection with Weyl structure (D, g, ω) in the tangent bundle over the normal homogeneous space $(G/H, g)$. Assume $d\omega = 0$. Then we have*

$$(4.7)\quad \begin{aligned}&R^D(X_i, X_j)X_k\\ &= \frac{1}{4}\sum_l\{\sum_t(C_{jk}{}^t C_{it}{}^l - C_{ik}{}^t C_{jt}{}^l - 2C_{ij}{}^t C_{tk}{}^l) - 4\sum_a C_{ij}{}^a C_{ak}{}^l\}X_l\\ &+ \frac{1}{4}\{(\omega_j\omega_k - \delta_{jk}||\omega||_g^2)X_i + (\delta_{ik}||\omega||_g^2 - \omega_i\omega_k)X_j + (\delta_{jk}\omega_i - \delta_{ik}\omega_j)\omega^\sharp\}.\end{aligned}$$

The Ricci tenser Ric^D for the connection D is defined by

$$(4.8)\qquad Ric^D(Y, Z) = trace\{X \mapsto R^D(X, Y)Z\}.$$

By virtue of (3.7),(4.2) and (4.7), we get

$$(4.9)\quad \begin{aligned}&Ric^D(X_j, X_k) = Ric^D(X_k, X_j)\\ &= \frac{1}{4}\{\delta_{jk} + (m-2)(\omega_j\omega_k - \delta_{jk}||\omega||_g^2) + 2(\sum_i\sum_a C_{ia}{}^j C_{ia}{}^k\}.\end{aligned}$$

Since the connection D is projectively flat by the assumption of the main theorem,

$$R^D(X,Y)Z = \tfrac{1}{(m-1)}\{Ric^D(Y,Z)X - Ric^D(X,Z)Y\}. \tag{4.10}$$

From (4.9)and (4.10), we obtain

$$\begin{aligned} R^D(X_i,X_j)X_k = &\frac{1}{4(m-1)}[\delta_{jk}\{1+(2-m)||\omega||_g^2\} \\ &+ (m-2)\omega_j\omega_k + 2\sum_t\sum_a C_{ta}{}^j C_{ta}{}^k]X_i \\ &-\frac{1}{4(m-1)}[\delta_{ik}\{1+(2-m)||\omega||_g^2\} \\ &+ (m-2)\omega_i\omega_k + 2\sum_t\sum_a C_{ta}{}^i C_{ta}{}^k]X_j. \end{aligned} \tag{4.11}$$

Since $\dim(G/H) = m \geq 3$ by the assumption, comparing coefficients of vector X_i in (4.7) and (4.11) for indices i,j,k $(i \neq j, j \neq k, i \neq k)$, we obtain from (3.6) and (3.7)

Lemma 4.5. *Under the same situation as in Lemma* 4.4*, if the connection D is projectively flat and $d\omega = 0$, then for indices $j,k,(j \neq k)$*

$$2\sum_l\sum_a C_{la}{}^j C_{la}{}^k = -\sum_{l,t} C_{lt}{}^j C_{lt}{}^k = (2-m)\omega_j\omega_k. \tag{4.12}$$

Using (3.6), (4.4), (4.5) and (4.10), we have

$$\begin{aligned} (\delta_D R^D)(X_j)X_k = &\frac{(2-m)}{2(m-1)}Ric^D(X_j,X_k)\omega^\sharp \\ &+ \frac{1}{(m-1)}\sum_{i,l}\{Ric^D(X_i,X_k)(\Gamma_{ij}{}^l - C_{ij}{}^l) \\ &+ Ric^D(X_j,X_i)\Gamma_{lk}{}^i\}X_l \\ &- \frac{1}{(m-1)}\sum_{i,t} Ric^D(X_i,X_t)\Gamma_{ik}{}^t X_j. \end{aligned} \tag{4.13}$$

From (3.6), we have

$$\sum_s\sum_a C_{sa}{}^i C_{sa}{}^t C_{it}{}^k = 0. \tag{4.14}$$

In order to analyze (4.13), we obtain from (3.6), (4.2), (4.5), (4.9) and (4.14)

Lemma 4.6. *Under the same situation and assumption as in Lemma 4.4,*

$$(a)\ 4\ Ric^D(X_j, X_k)\omega^\sharp = \{\delta_{jk} + (m-2)(\omega_j\omega_k - \delta_{jk}||\omega||_g^2) + 2\sum_i\sum_a (C_{ia}{}^j C_{ia}{}^k\}\omega^\sharp,$$

$$\begin{aligned}(b)\ 8 \sum_{i,l} & Ric^D(X_i, X_k)(\Gamma_{ij}{}^l - C_{ij}{}^l)X_l \\ &= \sum_l [\{1 + (2-m)||\omega||_g^2\}C_{jk}{}^l - 2\sum_{i,t}\sum_a (C_{ta}{}^i C_{ta}{}^k C_{ij}{}^l + \omega_j C_{ta}{}^k C_{ta}{}^l)]X_l \\ &+ [\{1 + (2-m)||\omega||_g^2\}\delta_{jk} + 2\sum_t\sum_a C_{ta}{}^j C_{ta}{}^k]\omega^\sharp \\ &- \{\omega^k + 2\sum_{s,t}\sum_a \omega_t C_{sa}{}^t C_{sa}{}^k\}X_j - \{\omega_j + (2-m)\omega^j||\omega||_g^2\}X_k,\end{aligned}$$

$$\begin{aligned}(c)\ 8 \sum_{i,l} & Ric^D(X_j, X_i)\Gamma_{lk}{}^i X_l \\ &= \sum_l [\{1 + (2-m)||\omega||_g^2\}C_{lk}{}^j - 2\sum_t\sum_a (\omega_k C_{ta}{}^j C_{ta}{}^l \\ &+ \sum_i C_{ta}{}^j C_{ta}{}^i C_{ta}{}^l)]X_l \\ &- [\{1 + (2-m)||\omega||_g^2\}\delta_{kj} + 2(m-2)\omega_j\omega_k + 2\sum_t\sum_a C_{ta}{}^j C_{ta}{}^k]\omega^\sharp \\ &+ \{(m-2)||\omega||_g^2 - 1\}\omega_k X_j + (\omega_j + 2\sum_{i,t}\sum_a \omega_i C_{ta}{}^j C_{ta}{}^i)X_k,\end{aligned}$$

$$\begin{aligned}(d)\ 8 \sum_{i,t} & Ric^D(X_i, X_t)\Gamma_{ik}{}^t X_j \\ &= \{(m-1)(m-2)\omega_k||\omega||_g^2 - m\omega_k - 2\sum_{s,t}\sum_a \omega_k C_{sa}{}^t C_{sa}{}^t\}X_j.\end{aligned}$$

By virtue of (4.13) and Lemma 4.6, we get

Lemma 4.7. *Under the same situation and assumption as in Lemma* 4.4

$$\begin{aligned} & 8(m-1)(\delta_D R^D)(X_j)X_k \\ &= \sum_l [(2-m)\delta_{jk}\omega_l + m(2-m)\omega_j\omega_k\omega_l + (2-m)^2\delta_{jk}\omega_l||\omega||_g^2 \\ &+ 2\sum_i\sum_a \{(2-m)\omega_l C_{ia}{}^j C_{ia}{}^k - \omega_j C_{ia}{}^k C_{ia}{}^l - \omega_k C_{ia}{}^j C_{ia}{}^l\} \\ &- 2\sum_{i,t}\sum_a (C_{ta}{}^j C_{ta}{}^i C_{ik}{}^l + C_{ta}{}^k C_{ta}{}^i C_{ij}{}^l)]X_l \end{aligned} \tag{4.15}$$

$$+\{(m-2)\omega_k-(m-2)^2\omega_k||\omega||_g^2+2\sum_{i,t}\sum_a(\omega_k C_{ia}{}^t C_{ia}{}^t$$
$$-\omega_i C_{ta}{}^i C_{ta}{}^k)\}X_j$$
$$+\{(m-2)\omega_j||\omega||_g^2+2\sum_{i,t}\sum_a\omega_i C_{ta}{}^j C_{ta}{}^i\}X_k.$$

By virtue of Lemma 4.5, the coefficient of vector X_k appeared in (4.15) can be changed as follows;
(4.16)$(m-2)\omega_j||\omega||_g^2+2\sum_{i,t}\sum_a\omega_i C_{ta}{}^j C_{ta}{}^i=(m-2)\omega_j^3+2\sum_t\sum_a\omega_j C_{ta}{}^j C_{ta}{}^j$.

Now, we assume $\delta_D R^D=0$. Then, the coefficient of vector X_l appeared in (4.15) is 0, since $\dim G/H=m\geq 3$. And then, the coefficients of vectors X_j and X_k appeared in (4.15) are 0. So, using (4.15), (4.16) and $\delta_D R^D=0$, we have

(4.17) $(m-2)\omega_j^2+2\sum_t\sum_a C_{ta}{}^j C_{ta}{}^j=0$ for each j.

Summing over j at (4.17), we get

(4.18) $(m-2)||\omega||_g^2+2\sum_{j,t}\sum_a C_{ta}{}^j C_{ta}{}^j=0$.

From (4.18) and the fact $m\geq 3$, we obtain $\omega=0$. Hence we have

Proposition 4.8. *Let (D,g,ω) be the Weyl structure related to an invariant affine connection D on the normal homogeneous space $(G/H,g)$. If the connection D is a projective flat Yang-Mills connection , then $\omega=0$.*

Conversely, assume $\omega=0$. Then, the connection D is a metric connection. Hence D coincides with the Levi-Civita connection ∇ on $(G/H,g)$, since D is torsion free and metric. On the other hand, the following facts are well known, in general:

(i) *the Levi-Civita connection on a Riemannian manifold is projectively flat if and only if the the Riemannian manifold is a space of constant curvature.*

(ii) *if a Riemannian manifold is a space of constant curvature, then the Levi-Civita connection is a Yang-Mills connection.*

By the help of the above facts, we obtain

Proposition 4.9. *Let (D,g,ω) be the Weyl structure related to an invariant affine connection D on the normal homogeneous space $(G/H,g)$. Assume D is projectively flat. Then if $\omega=0$, D is a Yang-Mills connection.*

Thus, by virtue of Theorem 1.2, Propositions 4.8 and 4.9, we obtain Theorem 4.3.

Finally, we get from Theorem 4.3

Corollary 4.10. *Let D be an invariant affine connection with Weyl structure (D, g, ω) in the tangent bundle over the mormal homogeneous space $(G/H, g)$, $dim G/H = m \geq 3$. Assume the connection D is projectively flat. Then, D is a Yang-Mills connection if and only if D is the Levi-Civita connection on the space $(G/H, g)$.*

References

[1] J.E. D'Atri and W.Ziller, Naturally reductive metrics and Einstein metrics on compact Lie groups, Memoirs of Amer. Math. Soc. 18 (1979), no. 215, 1-72.

[2] S. Dragomir, T. Ichiyama and H. Urakawa, Yang-Mills theory and conjugate connections, Differential Geom. Appl. 18 (2003), no.2, 229-238.

[3] S. Helgason, Differential Geometry, Lie Groups and Symmetric Spaces, Academic Press, New York, 1978.

[4] M. Itoh, Compact Einstein-Weyl manifolds and the associated constant, Osaka J. Math. 35 (1998), no.3, 567-578.

[5] S. Kobayashi and K. Nomizu, Foundation of Differential Geometry, Vol.I, Wiley-Interscience, New York, 1963.

[6] Y. Matsushima, Differentiable Manifolds, Marcel Dekker, Inc., 1972.

[7] J. Milnor, Curvatures of left invariant metrics on Lie groups, Adv. Math. 21 (1976), 293-329.

[8] I. Mogi and M. Itoh, Differential Geometry and Gauge Theory (in Japanese), Kyoritsu Publ., 1986.

[9] K. Nomizu, Invariant affine connections on homogeneous spaces, Amer. J. Math. 76 (1954), 33-65.

[10] K. Nomizu and T. Sasaki, Affine Differential Geometry-Geometry of Affine Immersions, Cambridge Univ. Press, Cambridge, 1994.

[11] J.-S. Park, The conjugate connection of a Yang-Mills connection, Kyushu J. of Math. 62 (2008), no.1, 217-220.

[12] J.-S. Park, Yang-Mills connections with Weyl structure, Proc. Japan Acad., 84(A) (2008), no.7, 129-132.

[13] J.-S. Park, Projectively flat Yang-Mills connections, Kyushu J. of Math. 64 (1) (2010), 49-58.

[14] J.-S. Park, Invariant Yang-Mills connections with Weyl structure, J. of Geometry and Physics 60 (2010), 1950-1957.

[15] H. Pedersen, Y. S. Poon, A. Swann, Einstein-Weyl deformations and submanifolds, Internat. J. Math. 7 (1996), no.5, 705-719.

[16] Walter A. Poor, Differential Geometric Structures, McGraw-Hill, Inc., 1981.

[17] Y. Sakane and M. Takeuchi, Geometry on Yang-Mills fields, Prof. Bourguignon' Lecture Note (1979), Osaka university, Sugaku 44-63, in Japanese.

[18] N. R. Wallach, Harmonic Analysis on Homogeneous Spaces, Marcel Dekker, New York, 1973.

Joon-Sik Park
Department of Mathematics, Pusan University of Foreign Studies,
Pusan 608-738, Korea.
E-mail: iohpark@pufs.ac.kr