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SCALAR CURVATURES ON $SU(3)/T(k,l)$

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Abstract. In this paper, we estimated the Ricci curvature and the scalar curvature on $SU(3)/T(k,l)$ under the condition $(k,l) \in \mathbb{R}^2$ $(|k|+|l| \neq 0)$, where the four isotropy irreducible representations in $SU(3)/T(k,l)$ are, not necessarily, mutually equivalent or inequivalent.

1. Introduction

Geometric properties on $SU(3)/T(k,l)$ have been investigated under the condition $k,\ l \in \mathbb{Z}$ $(|k|+|l| \neq 0)$, so far. In this paper, we estimate scalar curvatures of Riemannian homogeneous spaces $SU(3)/T(k,l)$ for $(k,l) \in \mathbb{R}^2$ $(|k|+|l| \neq 0)$.

The results of the studies on the family of the homogeneous spaces $SU(3)/T(k,l)$ $k,l \in \mathbb{Z}$ $(|k|+|l| \neq 0)$ are famous:

(i) M. Kreck and S. Stolz (cf. [4]) showed that among the family 7-dimensional homogeneous spaces $\{SU(3)/T(k,l) \mid k,l \in \mathbb{Z}$ which are mutually prime$\}$, there exist two Riemannian manifolds which are homeomorphic but not diffeomorphic.

(ii) For all (k,l), $SU(3)/T(k,l)$ admits a positively curved $SU(3)$-invariant Riemannian metric (cf. [1]).

(iii) $SU(3)/T(k,l)$ for each (k,l) admits a $SU(3)$-invariant Einstein metric (cf. [11]).

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(iv) The identity map of $SU(3)/T(k,l)$ for each (k,l) which is normal homogeneous is stable as a harmonic map (cf. [6]).

(v) The spectrum of the Laplacian of the metric in (ii) is determined (cf. [8, 9]).

Not under the condition $(k,l) \in \mathbb{Z}^2$ $(|k|+|l| \neq 0)$, the studies on the family of the homogeneous spaces $SU(3)/T(k,l)$ are rarely ever seen. Recently, one of the present authors (cf. [7]) obtained a necessary and sufficient condition for four isotropy irreducible representations in $SU(3)/T(k,l)$ to be mutually inequivalent, under the condition $(k,l) \in \mathbb{R}^2$ $(|k|+|l| \neq 0)$. And then, under this inequivalent condition, he estimated the Ricci and scalar curvatures on $SU(3)/T(k,l)$.

In this paper, we do not assume that four isotropy irreducible representations in $SU(3)/T(k,l)$ are, not necessarily, either equivalent or inequivalent. But, we assume that $(k,l) \in \mathbb{R}^2$ $(|k|+|l| \neq 0)$. Under these conditions, we estimate the Ricci curvature and the scalar curvature on $SU(3)/T(k,l)$ with a $SU(3)$-invariant Riemannian metric.

2. Scalar curvatures on $SU(3)/T(k,l)$

2.1. The curvature tensor field on a homogeneous Riemannian space

Let G be a compact connected semisimple Lie group and H a closed subgroup of G. We denote by $\mathfrak{g}$ and $\mathfrak{h}$ the corresponding Lie algebras of G and H, respectively. Let B be the negative of the Killing form of $\mathfrak{g}$. We consider the $\mathrm{Ad}(H)$-invariant decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ with $B(\mathfrak{h}, \mathfrak{m}) = 0$. Then the set of G-invariant symmetric covariant 2-tensor fields on G/H can be identified with the set of $\mathrm{Ad}(H)$-invariant symmetric bilinear forms on $\mathfrak{m}$. In particular, the set of G-invariant Riemannian metrics on G/H is identified with the set of $\mathrm{Ad}(H)$-invariant inner products on $\mathfrak{m}$ (cf. [2, 3, 6, 7]).

Let $< \, , \, >$ be an inner product which is invariant with respect to $\mathrm{Ad}(H)$ on $\mathfrak{m}$, where Ad denotes the adjoint representation of H in $\mathfrak{g}$. This inner product $< \, , \, >$ determines a G-invariant Riemannian metric $g_{<,>}$ on G/H.

For the sake of the calculus, we take a neighborhood V of the identity element e in G and a subset N (resp. N_H) of G (resp. H) in such a way that

(i) $N = V \cap \exp(\mathfrak{m})$, $N_H = V \cap \exp(\mathfrak{h})$,

(ii) the map $N \times N_H \ni (c,h) \longmapsto ch \in N \cdot N_H$ is a diffeomorphism,

(iii) the projection π of G onto G/H is a diffeomorphism of N onto a neighborhood $\pi(N)$ of the origin $\{H\}$ in G/H.

Here, $\{\exp(tX) | t \in \mathbb{R}\}$ for $X \in \mathfrak{g}$ is a 1-parameter subgroup of G.

Now for an element $X \in \mathfrak{m}$, we define a vector field X^* on the neighborhood $\pi(N)$ of $\{H\}$ in G/H by

$$X^*_{\pi(c)} := (\tau_c)_* X_{\{H\}} \in T_{\pi(c)} G/H, \quad (c \in N),$$

where τ_c denotes the transformation of G/H which is induced by c. Let $\{X_i\}_i$ be an orthonormal basis of the inner product space $(\mathfrak{m}, < \, , \, >)$. Then $\{X_i^*\}_i$ is an orthonormal frame on $\pi(N) (\subset G/H)$.

On the other hand, the connection function α (cf. [5, p.43]) on $\mathfrak{m} \times \mathfrak{m}$ corresponding to the invariant Riemannian connection of $(G/H, \; g_{<,>})$ is given as follows (cf. [5, p.52]):

$$\alpha(X,Y) = \frac{1}{2}[X,Y]_{\mathfrak{m}} + U(X,Y), \quad (X, Y \in \mathfrak{m}), \tag{2.1}$$

where $U(X,Y)$ is determined by

$$2 < U(X,Y), Z > = < [Z,X]_{\mathfrak{m}}, Y > + < X, [Z,Y]_{\mathfrak{m}} > \tag{2.2}$$

for X, Y and $Z \in \mathfrak{m}$, and $X_{\mathfrak{m}}$ denotes the $\mathfrak{m}$-component of an element $X \in \mathfrak{g} = \mathfrak{h} + \mathfrak{m}$. Let ∇ be the Levi-Civita connection on the Riemannian manifold $(G/H, \; g_{<,>})$. Then on $\pi(N)$ $(\nabla_{X^*} Y^*)_{\{H\}} = \alpha(X,Y)$ $(X, Y \in \mathfrak{m})$. Moreover, the expression for the value at $p_o := \{H\} (\in G/H)$ of the curvature tensor field is as follows (cf. [5, p.47]):

$$(2.3)\qquad \begin{aligned} R(X,Y)Z &= \alpha(X,\alpha(Y,Z)) - \alpha(Y,\alpha(X,Z)) \\ &\quad - \alpha([X,Y]_{\mathfrak{m}}, Z) - [[X,Y]_{\mathfrak{h}}, Z], \quad (X,Y,Z \in \mathfrak{m}), \end{aligned}$$

where $X_{\mathfrak{h}}$ denotes the $\mathfrak{h}$-component of an element $X \in \mathfrak{g} = \mathfrak{h} + \mathfrak{m}$.

2.2. Ricci curvatures on $SU(3)/T(k,l)$

In this subsection, we use the following notations.

$G = SU(3)$, $\mathfrak{g}$: the Lie algebra of $SU(3)$, $i = \sqrt{-1}$,

$T = T(k,l) = \{diag[e^{2\pi ik\theta}, e^{2\pi il\theta}, e^{-2\pi i(k+l)\theta} | \theta \in \mathbb{R}\}$ for $(k,l) \in \mathbb{R}^2$ and $|k| + |l| \neq 0$,

$\mathfrak{t}(k,l)$: the Lie algebra of $T(k,l)$,

$(X,Y) = B(X,Y) = -6\ Trace(XY)$: the negative of the Killing form of $\mathfrak{g}$.

Let E_{ij} be a real 3×3 matrix with 1 on entry (i,j) and 0 elsewhere. And we put

$$(2.4)\qquad \begin{cases} X_1 = (E_{12} - E_{21})/\sqrt{12}, \quad X_2 = i(E_{12} + E_{21})/\sqrt{12}, \\ X_3 = (E_{13} - E_{31})/\sqrt{12}, \quad X_4 = i(E_{13} + E_{31})/\sqrt{12}, \\ X_5 = (E_{23} - E_{32})/\sqrt{12}, \quad X_6 = i(E_{23} + E_{32})/\sqrt{12}, \\ X_7 = i\ diag[(k+2l), -(2k+l), (k-l)]/\sqrt{36\gamma}, \\ X_8 = i\ diag[k, l, -(k+l)]/\sqrt{12\gamma}, \end{cases}$$

where $\gamma = k^2 + kl + l^2$. Then

$$(2.5)\qquad \{X_1, \cdots, X_7\} \quad (\text{resp. } \{X_8\})$$

is an orthonormal basis of $\mathfrak{m}$ (resp. $\mathfrak{t}(k,l)$) with respect to $(\cdot\ ,\ \cdot)$ such that

$$\mathfrak{g} = \mathfrak{m} + \mathfrak{t}(k,l) \text{ and } (\mathfrak{m}, \mathfrak{t}(k,l)) = 0.$$

If we put $\{X_1, X_2\}_{\mathbb{R}} = \mathfrak{m}_1$, $\{X_3, X_4\}_{\mathbb{R}} = \mathfrak{m}_2$, $\{X_5, X_6\}_{\mathbb{R}} = \mathfrak{m}_3$, and $\{X_7\}_{\mathbb{R}} = \mathfrak{m}_4$, then $\mathfrak{m}_i$ are irreducible $\mathrm{Ad}(T)$-representations.

Now, we take another $\mathrm{Ad}(T)$-invariant inner product $< , >$ on $\mathfrak{m}$ such that

$$\{X_i/\sqrt{\lambda} =: Y_i,\ (i=1,2,\cdots,6),\quad X_7 =: Y_7\} \tag{2.6}$$

is an orthonormal basis of $\mathfrak{m}$ with respect to $< , >$. This inner product $<$ $, >$ determines a G-invariant Riemannian metric g_λ on the homogeneous space G/T. Then from (2.6), we obtain

$$\begin{aligned}
&[Y_1,Y_2] = (k+l)(2\lambda\sqrt{\gamma})^{-1}\ Y_7 + (k-l)(2\lambda\sqrt{3\gamma})^{-1}\ X_8,\\
&[Y_1,Y_3] = -(2\sqrt{3\lambda})^{-1}\ Y_5, \qquad [Y_1,Y_4] = -(2\sqrt{3\lambda})^{-1}\ Y_6,\\
&[Y_1,Y_5] = (2\sqrt{3\lambda})^{-1}\ Y_3, \qquad [Y_1,Y_6] = (2\sqrt{3\lambda})^{-1}\ Y_4,\\
&[Y_1,Y_7] = -(k+l)(2\sqrt{\gamma})^{-1}\ Y_2,\\
&[Y_1,X_8] = -(k-l)(2\sqrt{3\gamma})^{-1}\ Y_2,\\
&[Y_2,Y_3] = (2\sqrt{3\lambda})^{-1}\ Y_6, \qquad [Y_2,Y_4] = -(2\sqrt{3\lambda})^{-1}\ Y_5,\\
&[Y_2,Y_5] = (2\sqrt{3\lambda})^{-1}\ Y_4, \qquad [Y_2,Y_6] = -(2\sqrt{3\lambda})^{-1}\ Y_3,\\
&[Y_2,Y_7] = (k+l)(2\sqrt{\gamma})^{-1}\ Y_1, \quad [Y_2,X_8] = (k-l)(2\sqrt{3\gamma})^{-1}\ Y_1,\\
&[Y_3,Y_4] = l(2\lambda\sqrt{\gamma})^{-1}\ Y_7 + (2k+l)(2\lambda\sqrt{3\gamma})^{-1}\ X_8,\\
&[Y_3,Y_5] = -(2\sqrt{3\lambda})^{-1}\ Y_1, \qquad [Y_3,Y_6] = (2\sqrt{3\lambda})^{-1}\ Y_2,\\
&[Y_3,Y_7] = -l(2\sqrt{\gamma})^{-1}\ Y_4,\\
&[Y_3,X_8] = -(2k+l)(2\sqrt{3\gamma})^{-1}\ Y_4,\\
&[Y_4,Y_5] = -(2\sqrt{3\lambda})^{-1}\ Y_2, \qquad [Y_4,Y_6] = -(2\sqrt{3\lambda})^{-1}\ Y_1,\\
&[Y_4,Y_7] = l(2\sqrt{\gamma})^{-1}\ Y_3, \qquad [Y_4,X_8] = (2k+l)(2\sqrt{3\gamma})^{-1}\ Y_3,\\
&[Y_5,Y_6] = -k(2\lambda\sqrt{\gamma})^{-1}\ Y_7 + (k+2l)(2\lambda\sqrt{3\gamma})^{-1}\ X_8,\\
&[Y_5,Y_7] = k(2\sqrt{\gamma})^{-1}\ Y_6,\\
&[Y_5,X_8] = -(k+2l)(2\sqrt{3\gamma})^{-1}\ Y_6,\\
&[Y_6,Y_7] = -k(2\sqrt{\gamma})^{-1}\ Y_5, \qquad [Y_6,X_8] = (k+2l)(2\sqrt{3\gamma})^{-1}\ Y_5,\\
&[Y_7,X_8] = 0.
\end{aligned} \tag{2.7}$$

From (2.2) and (2.7), we get

$$
\begin{aligned}
&U(Y_1, Y_7) = (k+l)(\lambda-1)(4\lambda\sqrt{\gamma})^{-1}\ Y_2,\\
&U(Y_2, Y_7) = (k+l)(1-\lambda)(4\lambda\sqrt{\gamma})^{-1}\ Y_1,\\
&U(Y_3, Y_7) = l(\lambda-1)(4\lambda\sqrt{\gamma})^{-1}\ Y_4,\\
&U(Y_4, Y_7) = l(1-\lambda)(4\lambda\sqrt{\gamma})^{-1}\ Y_3,\\
&U(Y_5, Y_7) = k(1-\lambda)(4\lambda\sqrt{\gamma})^{-1}\ Y_6,\\
&U(Y_6, Y_7) = k(\lambda-1)(4\lambda\sqrt{\gamma})^{-1}\ Y_5,\\
&U(Y_i, Y_j) = 0,\ \text{otherwise.}
\end{aligned}
\tag{2.8}
$$

By the help of (2.1), (2.7) and (2.8), we have

$$
\begin{aligned}
&\alpha(Y_i, Y_i) = 0, \quad (i = 1, 2, \cdots, 7),\\
&\alpha(Y_1, Y_2) = (k+l)(4\lambda\sqrt{\gamma})^{-1}\ Y_7,\\
&\alpha(Y_1, Y_3) = -(4\sqrt{3\lambda})^{-1}\ Y_5, && \alpha(Y_1, Y_4) = -(4\sqrt{3\lambda})^{-1}\ Y_6,\\
&\alpha(Y_1, Y_5) = (4\sqrt{3\lambda})^{-1}\ Y_3, && \alpha(Y_1, Y_6) = (4\sqrt{3\lambda})^{-1}\ Y_4,\\
&\alpha(Y_1, Y_7) = -(k+l)(4\lambda\sqrt{\gamma})^{-1}\ Y_2,\\
&\alpha(Y_2, Y_3) = (4\sqrt{3\lambda})^{-1}\ Y_6, && \alpha(Y_2, Y_4) = -(4\sqrt{3\lambda})^{-1}\ Y_5,\\
&\alpha(Y_2, Y_5) = (4\sqrt{3\lambda})^{-1}\ Y_4, && \alpha(Y_2, Y_6) = -(4\sqrt{3\lambda})^{-1}\ Y_3,\\
&\alpha(Y_2, Y_7) = (k+l)(4\lambda\sqrt{\gamma})^{-1}\ Y_1,\\
&\alpha(Y_3, Y_4) = l(4\lambda\sqrt{\gamma})^{-1}\ Y_7, && \alpha(Y_3, Y_5) = -(4\sqrt{3\lambda})^{-1}\ Y_1,\\
&\alpha(Y_3, Y_6) = (4\sqrt{3\lambda})^{-1}\ Y_2, && \alpha(Y_3, Y_7) = -l(4\lambda\sqrt{\gamma})^{-1}\ Y_4,\\
&\alpha(Y_4, Y_5) = -(4\sqrt{3\lambda})^{-1}\ Y_2, && \alpha(Y_4, Y_6) = -(4\sqrt{3\lambda})^{-1}\ Y_1,\\
&\alpha(Y_4, Y_7) = l(4\lambda\sqrt{\gamma})^{-1}\ Y_3, && \alpha(Y_5, Y_6) = -k(4\lambda\sqrt{\gamma})^{-1}\ Y_7,\\
&\alpha(Y_5, Y_7) = k(4\lambda\sqrt{\gamma})^{-1}\ Y_6, && \alpha(Y_6, Y_7) = -k(4\lambda\sqrt{\gamma})^{-1}\ Y_5.
\end{aligned}
\tag{2.9}
$$

From (2.3),(2.7) and (2.9), we get

$$
\begin{aligned}
&R(Y_2, Y_1)Y_1 = \{16\gamma\lambda - 9(k+l)^2\}(48\gamma\lambda^2)^{-1}\ Y_2,\\
&R(Y_3, Y_1)Y_1 = (48\lambda)^{-1}\ Y_3, && R(Y_4, Y_1)Y_1 = (48\lambda)^{-1}\ Y_4,\\
&R(Y_5, Y_1)Y_1 = (48\lambda)^{-1}\ Y_5, && R(Y_6, Y_1)Y_1 = (48\lambda)^{-1}\ Y_6,
\end{aligned}
\tag{2.10}
$$

$$
\begin{aligned}
&R(Y_7,Y_1)Y_1 = (k+l)^2(16\gamma\lambda^2)^{-1}\ Y_7,\\
&R(Y_1,Y_2)Y_2 = \{16\gamma\lambda{-}9(k+l)^2\}(48\gamma\lambda^2)^{-1}\ Y_1,\\
&R(Y_3,Y_2)Y_2 = (48\lambda)^{-1}\ Y_3, \qquad R(Y_4,Y_2)Y_2 = (48\lambda)^{-1}\ Y_4,\\
&R(Y_5,Y_2)Y_2 = (48\lambda)^{-1}\ Y_5, \qquad R(Y_6,Y_2)Y_2 = (48\lambda)^{-1}\ Y_6,\\
&R(Y_7,Y_2)Y_2 = (k+l)^2(16\gamma\lambda^2)^{-1}\ Y_7,\\
&R(Y_1,Y_3)Y_3 = (48\lambda)^{-1}\ Y_1, \qquad R(Y_2,Y_3)Y_3 = (48\lambda)^{-1}\ Y_2,\\
&R(Y_4,Y_3)Y_3 = (16\gamma\lambda - 9l^2)(48\gamma\lambda^2)^{-1}\ Y_4,\\
&R(Y_5,Y_3)Y_3 = (48\lambda)^{-1}\ Y_5, \qquad R(Y_6,Y_3)Y_3 = (48\lambda)^{-1}\ Y_6,\\
&R(Y_7,Y_3)Y_3 = l^2(16\gamma\lambda^2)^{-1}\ Y_7, \qquad R(Y_1,Y_4)Y_4 = (48\lambda)^{-1}\ Y_1,\\
&R(Y_2,Y_4)Y_4 = (48\lambda)^{-1}\ Y_2,\\
&R(Y_3,Y_4)Y_4 = (16\gamma\lambda - 9l^2)(48\gamma\lambda^2)^{-1}\ Y_3,\\
&R(Y_5,Y_4)Y_4 = (48\lambda)^{-1}\ Y_5, \qquad R(Y_6,Y_4)Y_4 = (48\lambda)^{-1}\ Y_6,\\
&R(Y_7,Y_4)Y_4 = l^2(16\gamma\lambda^2)^{-1}\ Y_7, \qquad R(Y_1,Y_5)Y_5 = (48\lambda)^{-1}\ Y_1,\\
&R(Y_2,Y_5)Y_5 = (48\lambda)^{-1}\ Y_2, \qquad R(Y_3,Y_5)Y_5 = (48\lambda)^{-1}\ Y_3,\\
&R(Y_4,Y_5)Y_5 = (48\lambda)^{-1}\ Y_4,\\
&R(Y_6,Y_5)Y_5 = (16\gamma\lambda{-}9k^2)(48\gamma\lambda^2)^{-1}\ Y_6,\\
&R(Y_7,Y_5)Y_5 = k^2(16\gamma\lambda^2)^{-1}\ Y_7, \qquad R(Y_1,Y_6)Y_6 = (48\lambda)^{-1}\ Y_1,\\
&R(Y_2,Y_6)Y_6 = (48\lambda)^{-1}\ Y_2, \qquad R(Y_3,Y_6)Y_6 = (48\lambda)^{-1}\ Y_3,\\
&R(Y_4,Y_6)Y_6 = (48\lambda)^{-1}\ Y_4,\\
&R(Y_5,Y_6)Y_6 = (16\gamma\lambda{-}9k^2)(48\gamma\lambda^2)^{-1}\ Y_5,\\
&R(Y_7,Y_6)Y_6 = k^2(16\gamma\lambda^2)^{-1}\ Y_7,\\
&R(Y_1,Y_7)Y_7 = (k+l)^2(16\gamma\lambda^2)^{-1}\ Y_1,\\
&R(Y_2,Y_7)Y_7 = (k+l)^2(16\gamma\lambda^2)^{-1}\ Y_2,\\
&R(Y_3,Y_7)Y_7 = l^2(16\gamma\lambda^2)^{-1}\ Y_3, \qquad R(Y_4,Y_7)Y_7 = l^2(16\gamma\lambda^2)^{-1}\ Y_4,\\
&R(Y_5,Y_7)Y_7 = k^2(16\gamma\lambda^2)^{-1}\ Y_5, \qquad R(Y_6,Y_7)Y_7 = k^2(16\gamma\lambda^2)^{-1}\ Y_6.
\end{aligned}
\tag{2.10}
$$

In general, the Ricci tensor field Ric of type (0,2) on a Riemannian manifold (M, g) is defined by

$$Ric(Y,Z) = Trace\,\{X \mapsto R(X,Y)Z\}, \quad (X,Y,Z \in \mathfrak{X}(M)). \tag{2.11}$$

From (2.10), we obtain

Theorem 2.1. *The Ricci tensor on* $(SU(3)/T(k,l),\ g_\lambda)$*,* $(k,l) \in \mathbb{R}^2$ $(|k|+|l| \neq 0)$*, is given by*

(a) $Ric(Y_i, Y_j) = 0, \quad (i \neq j),$
(b) $r_1 := Ric(Y_1, Y_1) = Ric(Y_2, Y_2) = \{10\gamma\lambda - 3(k+l)^2\}/(24\gamma\lambda^2),$
(c) $r_2 := Ric(Y_3, Y_3) = Ric(Y_4, Y_4) = (10\gamma\lambda - 3l^2)/(24\gamma\lambda^2),$
(d) $r_3 := Ric(Y_5, Y_5) = Ric(Y_6, Y_6) = (10\gamma\lambda - 3k^2)/(24\gamma\lambda^2),$
(e) $r_4 := Ric(Y_7, Y_7) = 1/(4\lambda^2),$
where $\gamma = k^2 + kl + l^2$.

A Riemannian homogeneous space $(G/H, g)$ is called a *normal homogeneous manifold* if the metric g is induced from an $\mathrm{Ad}(G)$-invariant inner product $(\ ,\ )$ on the Lie algebra $\mathfrak{g}$ such that $T_eG = \mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ and $(\mathfrak{h}, \mathfrak{m}) = 0$. The *Ricci curvature* r of a Riemannian manifold (M, g) with respect to a nonzero vector $v \in TM$ is defined by $r(v) := Ric(v,v)/\|v\|^2$, and a manifold of constant Ricci curvature, (i.e., $Ric = cg$ for some constant c), is called an *Einstein manifold*.

If $(SU(3)/T(k,l), g_\lambda)$, $(k,l) \in \mathbb{R}^2$ $(|k|+|l| \neq 0)$, is a normal homogeneous manifold, then $\lambda = 1$.

From Theorem 2.1, we obtain the following

Corollary 2.2. *For each* $(k,l) \in \mathbb{R}^2$ $(|k|+|l| \neq 0)$*, any normal homogeneous manifold* $SU(3)/T(k,l)$ *is not an Einstein manifold.*

Furthermore, from Theorem 2.1, we have

Corollary 2.3. *The Ricci curvature* r *on the Riemannian homogeneous space* $(SU(3)/T(k,l), g_\lambda)$*,* $(k,l) \in \mathbb{R}^2$ $(|k|+|l| \neq 0)$*, is estimated as follows*:

(a) *if* $k > l > 0$ *and* $\lambda > (9k^2 + 12kl + 9l^2)/10(k^2 + kl + l^2)$*, then* $r_4 \leq r \leq r_2$.

(b) *if* $k = l$ *and* $\lambda > 1$, *then* $1/(4\lambda^2) \leq r \leq (10\lambda - 1)/(24\lambda^2)$.

The trace of the Ricci tensor field *Ric* of a Riemannian manifold (M, g), (i.e., $\sum_j Ric(e_j, e_j)$, where $\{e_j\}_j$ is an (locally defined) orthonormal frame on (M, g)), is called the *scalar curvature* of (M, g).

By the help of Theorem 2.1, we get the following

Theorem 2.4. *The scalar curvature* $S(g_\lambda)$ *on* $(SU(3)/T(k,l),\ g_\lambda)$, $(k, l \in \mathbb{R},\ |k| + |l| \neq 0)$, *is given by*

$$S(g_\lambda) = (10\lambda - 1)/(4\lambda^2). \tag{2.12}$$

From Theorem 2.4, we obtain the following

Corollary 2.5. *The scalar curvature* $S(g_\lambda)$ *on* $(SU(3)/T(k,l),\ g_\lambda)$, $(k, l \in \mathbb{R},\ |k| + |l| \neq 0)$, *is estimated as follows*:

(a) $S(g_\lambda) > 0$ *if and only if* $\lambda > 1/10$,
(b) $S(g_\lambda) = 0$ *if and only if* $\lambda = 1/10$,
(c) $S(g_\lambda) < 0$ *if and only if* $\lambda < 1/10$.

Corollary 2.6. *For each* $(k, l) \in \mathbb{R}^2$ $(|k| + |l| \neq 0)$, *the scalar curvature of the normal homogeneous Riemannian manifold* $SU(3)/T(k,l)$ *is* $9/4$.

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