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IDEAL THEORY OF PRE-LOGICS BASED ON $\mathcal{N}$-STRUCTURES

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Abstract. Using $\mathcal{N}$-structures, the notion of an $\mathcal{N}$-ideal in a pre-logic is introduced. Characterizations of an $\mathcal{N}$-ideal are discussed. Conditions for an $\mathcal{N}$-structure to be an $\mathcal{N}$-ideal are provided.

1. Introduction

A (crisp) set A in a universe S can be defined in the from of its characteristic function $\mu_A : X \to \{0,1\}$ yielding the value 1 for the elements belonging to the set A and the value 0 for element excluded from the set A. So far most of the generalization of the crisp set have been conducted on the unit interval $[0,1]$ and they are consistent with the asymmetry observation. In other words, the generalization of the crisp set to fuzzy sets relied on spreading positive information that fit the crisp point $\{1\}$ into the interval $[0,1]$. Because no negative meaning of information is suggested, Jun et al. [3] introduced a new function which is called a negative-valued function, and constructed $\mathcal{N}$-structures. They applied $\mathcal{N}$-structures to BCK/BCI-algebras and $\mathcal{N}$-ideals in BCK/BCI-algebras. I. Chajda and R. Halas [1] introduced the concept of a pre-logic which is an algebra weaker than a Hilbert algebra (an algebraic counterpart of intuitionistic logic) but strong enough to have deductive systems. They also studied algebraic properties of pre-logics and of lattices of their deductive systems. Young Hie Kim and Sun Shn Ahn ([4]) defined the notion of commutative pre-logic and terminal sections and investigated some of their properties. In [2], S. S. Ahn and J. K.

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Yoo defined the notion of complicated pre-logic and a special set in a pre-logic

In this paper, we introduce the notion of an $\mathcal{N}$-ideal in a pre-logic and investigate several characterizations of an $\mathcal{N}$-ideal. Also we provide conditions for a an $\mathcal{N}$-structure to be an $\mathcal{N}$-ideal.

2. Preliminaries

We recall some definitions and results (see [1]).

Definition 2.1. By a *pre-logic*, we mean a triple $(X; \cdot, 1)$ where X is a non-empty set, $\cdot$ is a binary operation on X and $1 \in X$ such that the following identities hold:

(P1) $(\forall x \in X)\ (x \cdot x = 1)$,
(P2) $(\forall x \in X)\ (1 \cdot x = x)$,
(P3) $(\forall x \in X)\ (x \cdot (y \cdot z) = (x \cdot y) \cdot (x \cdot z))$,
(P4) $(\forall x, y, z \in X)\ (x \cdot (y \cdot z) = y \cdot (x \cdot z))$.

Lemma 2.2. *Let $(X; \cdot, 1)$ be a pre-logic. Then the following hold:*

(a) $(\forall x \in X)\ (x \cdot 1 = 1)$;
(b) $(\forall x, y \in X)\ (x \cdot (y \cdot x) = 1)$;
(c) *an order relation $\leq$ on A defined by*

$$(\forall x, y \in X)\,(x \leq y \text{ if and only if } x \cdot y = 1)$$

is a quasiorder on X (i.e., a reflexive and transitive order relation on X);

(d) $1 \leq x$ *for all* $x \in X$ *implies* $x = 1$.

Remark 2.3. The quasiorder $\leq$ of Lemma 2.2(c) is called the *induced quasiorder* of a pre-logic X.

Lemma 2.4. *Let $\leq$ be the induced quasiorder of a pre-logic $X = (X; \cdot, 1)$ and let $x, y, z \in X$. If $x \leq y$, then $z \cdot x \leq z \cdot y$ and $y \cdot z \leq x \cdot z$.*

Definition 2.5. Let $X = (X; \cdot, 1)$ be a pre-logic. A non-empty subset D of X is called a *deductive system* of X if the following conditions hold:

(d1) $1 \in D$,
(d2) if $x \in D$ and $x \cdot y \in D$, then $y \in D$.

Definition 2.6. Let $X = (X; \cdot, 1)$ be a pre-logic. A non-empty subset I of X is called an *ideal* of X if the following conditions are satisfied:

(I1) $x \in X$ and $y \in I$ imply $x \cdot y \in I$;
(I2) $x \in X$ and $y_1, y_2 \in I$ imply $(y_2 \cdot (y_1 \cdot x)) \cdot x \in I$.

Denote by $\mathcal{I}(X)$ the set of all ideals of X.

Theorem 2.7. *Let $X = (X; \cdot, 1)$ be a pre-logic. Then every ideal of X is a deductive system on X and conversely.*

Lemma 2.8. *Let $X = (X; \cdot, 1)$ be a pre-logic and $\leq$ its induced quasiorder. The the following hold:*

(a) $(\forall x, y \in X)\ (x \cdot ((x \cdot y) \cdot y) = 1)$,
(b) $(\forall x, y, z \in X)\ ((y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 1)$,
(c) *if D is a deductive system of X, $a \in D$, and $a \leq b$, then $b \in D$.*

3. $\mathcal{N}$-ideals

In what follows, let X denote a pre-logic and let f denote an $\mathcal{N}$-function on X unless otherwise specified.

Theorem 3.1. *A non-empty subset I of a pre-logic X is an ideal of X if and only if it satisfies the following two conditions:*

(I1′) $(1 \in I)$;
(I2′) $(\forall x, z \in X)(\forall y \in I)\ (x \cdot (y \cdot z) \in I \Rightarrow x \cdot z \in I)$.

Proof. Let I be an ideal of X. Using (P1) and (I1), we have $1 = a \cdot a \in I$ for all $a \in I$. We prove the following assertion:

$$(*) \qquad (\forall x \in I)(\forall y \in X)(x \cdot y \in I \Rightarrow y \in I).$$

Let $x \in I$ and $y \in X$ be such that $x \cdot y \in I$. Then $y = 1 \cdot y = ((x \cdot y) \cdot (x \cdot y)) \cdot y \in I$ by (I2). Now, let $x, z \in X$ and $y \in I$ be such that $x \cdot (y \cdot z) \in I$. Then $y \cdot (x \cdot z) \in I$ by (P4). Since $y \in I$, it follows from $(*)$ that $x \cdot z \in I$. Hence (I2′) is valid.

Conversely, assume that (I1′) and (I2′) are valid. Let $x \in X$ and $a \in I$. Then $x \cdot (a \cdot a) = x \cdot 1 = 1 \in I$, and so $x \cdot a \in I$ by (I2′). Since $(a \cdot x) \cdot (a \cdot x) = 1 \in I$, we have $(a \cdot x) \cdot x \in I$ by (I2′). It follows that $(a \cdot (b \cdot x)) \cdot (b \cdot x) \in I$ for all $a, b \in I$ and $x \in X$. Using (I2′), we get $(a \cdot (b \cdot x)) \cdot x \in I$. Therefore I is an ideal of X. □

Denote by $\mathcal{F}(X, [-1, 0])$ the collection of functions from a set X to $[-1, 0]$. We say that an element of $\mathcal{F}(X, [-1, 0])$ is a *negative-valued function* from X to $[-1, 0]$(briefly, an *$\mathcal{N}$-function* on X). By an *$\mathcal{N}$-structure* we mean an ordered pair (X, f) of X and an $\mathcal{N}$-function f on X.

For any $\mathcal{N}$-structure (X, f) and $t \in [-1, 0]$, the non-empty set

$$C(f; t) := \{x \in X | f(x) \leq t\}$$

Table 1. $\cdot$ -operation

$\cdot$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	d
b	1	a	1	c	c
c	1	1	b	1	b
d	1	1	1	1	1

is called a *closed* (f, t)*-cut* of (X, f).

Denote by $\mathcal{I}(X)$ the set of all ideals of X.

Definition 3.2. By an $\mathcal{N}$*-ideal* of X we mean an $\mathcal{N}$-structure (X, f) which satisfies the following assertion:

$$(\forall t \in [-1, 0])(C(f; t) \in \mathcal{I}(X) \cup \{\emptyset\}). \tag{3.1}$$

Example 3.3. Let $X := \{1, a, b, c, d\}$ be a set with the $\cdot$ -operation given by Table 1. Then $(X; \cdot, 1)$ is a pre-logic.
(1) Consider an $\mathcal{N}$-structure (X, f) in which f is defined by

$$f(x) := \begin{cases} -0.5 & \text{if } x \in \{1, a, b\} \\ -0.2 & \text{if } x \in \{c, d\}. \end{cases}$$

Then

$$C(f; t) = \begin{cases} X & \text{if } t \in [-0.2, 0] \\ \{1, a, b\} & \text{if } t \in [-0.5, -0.2) \\ \emptyset & \text{if } t \in [-1, -0.5). \end{cases}$$

Note that $\{1, a, b\}$ and X are ideals of X and so (X, f) is an $\mathcal{N}$-ideal of X.
(2) Consider an $\mathcal{N}$-structure (X, g) in which g is defined by

$$g(x) := \begin{cases} -0.8 & \text{if } x \in \{1, b, c\} \\ -0.4 & \text{if } x \in \{a, c\}. \end{cases}$$

Then

$$C(g; t) = \begin{cases} X & \text{if } t \in [-0.4, 0] \\ \{1, b, c\} & \text{if } t \in [-0.8, -0.4) \\ \emptyset & \text{if } t \in [-1, -0.8). \end{cases}$$

Note that $\{1, b, c\}$ is not an ideal of X since $(b \cdot (a \cdot a)) \cdot a = (b \cdot 1) \cdot a = 1 \cdot a = a \notin \{1, b, c\}$. Hence (X, g) is not an $\mathcal{N}$-ideal of X.

Theorem 3.4. *For an $\mathcal{N}$-structure (X, f), the following are equivalent:*

(1) (X, f) *is an $\mathcal{N}$-ideal of X,*

(2) (X, f) *satisfies the following two conditions:*

(2.1) $(\forall x, y \in X)(f(x \cdot y) \leq f(y))$,

(2.2) $(\forall x, y, z \in X)(f((x \cdot (y \cdot z)) \cdot z) \leq \max\{f(x), f(y)\})$.

Proof. Assume that (X, f) satisfies two conditions (2.1) and (2.2). Let $t \in [-1, 0]$ be such that $C(f; t) \neq \emptyset$. Let $x \in X$ and $a \in C(f; t)$. Then $f(a) \leq t$, and so $f(x \cdot a) \leq f(a) \leq t$ by (2.1). Thus $x \cdot a \in C(f; t)$. Let $x \in X$ and $a, b \in C(f; t)$. Then $f(a) \leq t$ and $f(b) \leq t$. It follows from (2.2) that

$$f((a \cdot (b \cdot x)) \cdot x) \leq \max\{f(a), f(b)\} \leq t$$

so that $(a \cdot (b \cdot x)) \cdot x \in C(f; t)$. Hence $C(f; t)$ is an ideal of X, and therefore (X, f) is an $\mathcal{N}$-ideal of X.

Conversely, suppose that (X, f) is an $\mathcal{N}$-ideal of X. If $f(a \cdot b) > t_b := f(b)$ for some $a, b \in X$ and $t_b \in [-1, 0]$, then $b \in C(f; t_b)$, but $a \cdot b \notin C(f; t_b)$. This is a contradiction, and so (2.1) is valid. Assume that (2.2) is not valid. Then there exist $a, b, c \in X$ such that $f((a \cdot (b \cdot c)) \cdot c) > \max\{f(a), f(b)\}$. Taking $t := \max\{f(a), f(b)\}$ implies that $a, b \in C(f; t)$ and $(a \cdot (b \cdot c)) \cdot c \notin C(f; t)$. This is impossible, and thus (2.2) is true. □

Proposition 3.5. *Every $\mathcal{N}$-ideal (X, f) satisfies the following inequalities:*

(1) $(\forall x \in X)(f(1) \leq f(x))$,

(2) $(\forall x, y \in X)(f((x \cdot y) \cdot y) \leq f(x))$.

Proof. (1) Using (P1) and (2.1) in Theorem 3.4, we have $f(1) = f(x \cdot x) \leq f(x)$ for all $x \in X$.

(2) Taking $x := x, y := 1$ and $z := y$ in Theorem 3.4(2.2) and using (P2) and (1), we get

$$f((x \cdot y) \cdot y) = f((x \cdot (1 \cdot y)) \cdot y) \leq \max\{f(x), f(1)\} = f(x)$$

for all $x, y \in X$. □

Corollary 3.6. *Every $\mathcal{N}$-ideal (X, f) is order reversing.*

Proof. Let $x, y \in X$ be such that $x \leq y$. Then $x \cdot y = 1$, and so

$$f(y) = f(1 \cdot y) = f((x \cdot y) \cdot y) \leq f(x)$$

by (P2) and Proposition 3.5(2). Hence (X, f) is order reversing. □

Proposition 3.7. *An $\mathcal{N}$-structure (X, f) satisfying the first condition of Proposition 3.5 and*

$$(\forall x, y, z \in X)(f(x \cdot z) \le \max\{f(x \cdot (y \cdot z)), f(y)\}) \tag{3.2}$$

is order reversing.

Proof. Let $x, y \in X$ be such that $x \le y$. Then $x \cdot y = 1$, and so

$$f(y) = f(1 \cdot y) \le \max\{f(1 \cdot (x \cdot y)), f(x)\} = \max\{f(1 \cdot 1), f(x)\} = f(x)$$

by (P1), (P2), (3.2) and Proposition 3.5(1). Therefore (X, f) is order reversing. □

Theorem 3.8. *For any $\mathcal{N}$-structure (X, f) in a pre-logic X, the following are equivalent:*

1. *(X, f) is an $\mathcal{N}$-ideal of X.*
2. *(X, f) satisfies two conditions Proposition* 3.5(1) *and Proposition* 3.7(3.2).

Proof. Assume that (X, f) is an $\mathcal{N}$-ideal of X. It suffices to show that (X, f) satisfies (3.2). Using Lemma 2.8(b), we have

$$(y \cdot z) \cdot z \le (x \cdot (y \cdot z)) \cdot (x \cdot z), \tag{3.3}$$

i.e., $((y \cdot z) \cdot z) \cdot ((x \cdot (y \cdot z)) \cdot (x \cdot z)) = 1$ for all $x, y, z \in X$. It follows from (P2), (2.2) in Theorem 3.4 and Proposition 3.5(2) that

$$\begin{aligned} f(x \cdot z) =& f(1 \cdot (x \cdot z)) \\ =& f((((y \cdot z) \cdot z) \cdot ((x \cdot (y \cdot z)) \cdot (x \cdot z))) \cdot (x \cdot z)) \\ \le& \max\{f((y \cdot z) \cdot z), f(x \cdot (y \cdot z))\} \\ \le& \max\{f(x \cdot (y \cdot z)), f(y)\}. \end{aligned}$$

Hence (X, f) satisfies the condition (3.2).

Conversely, suppose that (X, f) satisfies Proposition 3.5(1) and (3.2). Using (P1), Lemma 2.2(a), (3.2) and Proposition 3.5(1), we have

$$\begin{aligned} f(x \cdot y) \le& \max\{f(x \cdot (y \cdot y)), f(y)\} \\ =& \max\{f(x \cdot 1), f(y)\} \\ =& \max\{f(1), f(y)\} = f(y) \end{aligned}$$

and

$$\begin{aligned} f((x \cdot y) \cdot y) &\le \max\{f((x \cdot y) \cdot (x \cdot y)), f(x)\} \\ &= \max\{f(1), f(x)\} = f(x) \end{aligned} \tag{3.4}$$

for all $x, y \in X$. Since (X, f) is order reversing by Proposition 3.7, it follows from (3.3) that $f((y \cdot z) \cdot z) \geq f((x \cdot (y \cdot z)) \cdot (x \cdot z))$ so from (3.2) and (3.4) that

$$\begin{aligned} f((x \cdot (y \cdot z)) \cdot z) &\leq \max\{f(((x \cdot (y \cdot z)) \cdot (x \cdot z)), f(x)\} \\ &\leq \max\{f((y \cdot z) \cdot z), f(x)\} \\ &\leq \max\{f(x), f(y)\} \end{aligned}$$

for all $x, y, z \in X$. By Theorem 3.4, (X, f) is an $\mathcal{N}$-ideal of X. □

Lemma 3.9. *Every $\mathcal{N}$-ideal (X, f) satisfies the following inequality:*

$$(\forall x, y \in X)(f(y) \leq \max\{f(x \cdot y), f(x)\}). \tag{3.5}$$

Proof. Using (P1), (P2) and (2.2) in Theorem 3.4, we have

$$f(y) = f(1 \cdot y) = f((x \cdot y) \cdot (x \cdot y)) \cdot y) \leq \max\{f(x), f(x \cdot y)\}$$

for all $x, y \in X$. □

Corollary 3.10. *An $\mathcal{N}$-structure (X, f) is an $\mathcal{N}$-ideal of X if and only if (X, f) satisfies two conditions:*

(1) $(\forall x \in X)(f(1) \leq f(x))$ *and Lemma* 3.9(3.5).

Proof. Assume that an $\mathcal{N}$-structure (X, f) is an $\mathcal{N}$-ideal of X. By Lemma 3.9, (X, f) satisfies the condition (3.5).

Conversely, suppose that an $\mathcal{N}$-structure (X, f) satisfies the conditions (1) and (3.5). Then we have $f(x \cdot z) \leq \max\{f(y \cdot (x \cdot z)), f(y)\}$ for all $x, y, z \in X$. By Theorem 3.8, (X, f) is an $\mathcal{N}$-ideal of X. □

Lemma 3.11. *For any $\mathcal{N}$-structure (X, f) in a pre-logic X, the following are equivalent:*

(1) $(\forall x, y \in X)(f(y) \leq max\{f(x \cdot y), f(x)\})$,
(2) $(\forall x, y, z \in X)(f(x \cdot z) \leq max\{f(x \cdot (y \cdot z)), f(x \cdot y)\})$.

Proof. Assume that (X, f) satisfies (1). For any $x, y, z \in X$, using (P3), we have

$$\begin{aligned} f(x \cdot z) \leq& \max\{f((x \cdot y) \cdot (x \cdot z)), f(x \cdot y)\} \\ =& \max\{f(x \cdot (y \cdot z)), f(x \cdot y)\}. \end{aligned}$$

Thus (2) is valid.

Conversely, suppose that (X, f) satisfies (2). Putting $x := 1$ in (2) and using (P2), we have

$$\begin{aligned} f(1 \cdot z) \leq& \max\{f((1 \cdot (y \cdot z)), f(1 \cdot y)\} \\ =& \max\{f(y \cdot z), f(y)\}. \end{aligned}$$

Hence $f(z) \leq \max\{f(y \cdot z), f(y)\}$. Thus (1) is true. □

Proposition 3.12. *An $\mathcal{N}$-structure (X, f) is an $\mathcal{N}$-ideal of X if and only if (X, f) satisfies two conditions:*

(1) $(\forall x \in X)(f(1) \leq f(x))$
(2) $(\forall x, y, z \in X)(f(x \cdot z) \leq max\{f(x \cdot (y \cdot z)), f(x \cdot y)\})$.

Proof. It follows from Lemma 3.11 and Corollary 3.10. □

Corollary 3.13. *Every $\mathcal{N}$-ideal (X, f) satisfies the following inequality*

$$(\forall x, y \in X)(f(x \cdot y) \leq f(x \cdot (x \cdot y))).$$

Proof. Putting $x := x, z := y$ and $y := x$ in Proposition 3.12(2), we have

$$\begin{aligned} f(x \cdot y) \leq & \max\{f(x \cdot (x \cdot y)), f(x \cdot x)\} \\ = & \max\{f(x \cdot (x \cdot y)), f(1)\} \\ = & f(x \cdot (x \cdot y)), \end{aligned}$$

for all $x, y \in X$. □

For any $a, b \in X$, the set

$$A(a, b) := \{x \in X | a \cdot (b \cdot x) = 1\}$$

is called the *upper set* of a and b. Clearly, $1, a, b \in A(a, b)$ for all $a, b \in X$.

Theorem 3.14. ([2]) *Let $(X; \cdot, 1)$ be a pre-logic. Then the upper set $A(x, y)$ is a deductive system of X, where $x, y \in X$.*

Corollary 3.15. *Let $(X; \cdot, 1)$ be a pre-logic. Then the upper set $A(x, y)$ is an ideal of X, where $x, y \in X$.*

Proof. It follows from Theorem 2.7 and Theorem 3.14. □

Proposition 3.16. *If (X, f) is an $\mathcal{N}$-ideal of X, then*

(3.6) $$(\forall a, b \in X)(\forall t \in [-1, 0])(a, b \in C(f; t) \Rightarrow A(a, b) \subseteq C(f; t)).$$

Proof. Let $a, b \in C(f; t)$ for any $t \in [-1, 0]$. Then $f(a) \leq t$ and $f(b) \leq t$. If $x \in A(a, b)$, then $a \cdot (b \cdot x) = 1$. Using (P2) and Theorem 3.4(2), we have

$$f(x) = f(1 \cdot x) = f((a \cdot (b \cdot x)) \cdot x) \leq \max\{f(a), f(b)\} \leq t,$$

and so $x \in C(f; t)$. Therefore $A(a, b) \subseteq C(f; t)$. □

We now consider the converse of Proposition 3.16. Let $t \in [-1, 0]$ and (X, f) an $\mathcal{N}$-structure satisfying (3.6). Note that $1 \in A(a, b) \subseteq C(f; t)$

for all $a, b \in X$. Let $x, y, z \in X$ be such that $x \cdot (y \cdot z) \in C(f;t)$ and $y \in C(f;t)$. Using (P4) and (P1), we know that

$$(x \cdot (y \cdot z)) \cdot (y \cdot (x \cdot z)) = (x \cdot (y \cdot z)) \cdot (x \cdot (y \cdot z)) = 1.$$

Thus $x \cdot z \in A(x \cdot (y \cdot z), y) \subseteq C(f;t)$, and so $C(f;t)$ is an ideal of X by Theorem 3.1. Therefore (X, f) is an $\mathcal{N}$-ideal of X. Hence we have the following theorem.

Theorem 3.17. *If an $\mathcal{N}$-structure (X, f) satisfies* (3.6), *then (X, f) is an $\mathcal{N}$-ideal of X.*

Corollary 3.18. *For any $\mathcal{N}$-ideal (X, f), we have*

$$(\forall t \in [-1, 0])(C(f;t) \neq \emptyset \Rightarrow C(f;t) = \cup_{a,b \in C(f;t)} A(a, b)). \quad (3.7)$$

Proof. Assume that $C(f;t) \neq \emptyset$ for all $t \in [-1, 0]$. Since $1 \in C(f;t)$, we get

$$C(f;t) \subseteq \cup_{a \in C(f;t)} A(a, 1) \subseteq \cup_{a,b \in C(f;t)} A(a, b).$$

Now, let $x \in \cup_{a,b \in C(f;t)} A(a, b)$. Then there exist $u, v \in C(f;t)$ such that $x \in A(u, v) \subseteq C(f;t)$. Hence $\cup_{a,b \in C(f;t)} A(a, b) \subseteq C(f;t)$. This completes the proof. □

4. Positive implicative $\mathcal{N}$-ideals

Definition 4.1. A non-empty subset I of X is a *positive implicative ideal* of a pre-logic X if it satisfies (I1′) and

(I3) $(\forall y, z \in X)(\forall x \in I)\ (x \cdot ((y \cdot z) \cdot y) \in I \Rightarrow y \in I)$.

Example 4.2. Consider a pre-logic $X = \{1, a, b, c, d\}$ as in Example 3.3(1). It is easy to check that $I = \{1, a, b\}$ is a positive implicative ideal of X.

Theorem 4.3. *Every positive implicative ideal of a pre-logic X is an ideal of X.*

Proof. Let I be a positive implicative ideal of X and let $x \cdot y \in I$ and $x \in I$. Then $x \cdot ((y \cdot y) \cdot y) = x \cdot (1 \cdot y) = x \cdot y \in I$. Since I is a positive implicative ideal of X, $y \in I$. Hence I is a deductive system of X. By Theorem 2.7, I is an ideal of X. □

Denote by $\mathcal{I}_p(X)$ the set of all positive implicative ideals of X.

Definition 4.4. By a *positive implicative $\mathcal{N}$-ideal* of X we mean an $\mathcal{N}$-structure (X, f) which satisfies the following assertion:

$$(\forall t \in [-1, 0])(C(f;t) \in \mathcal{I}_p(X) \cup \{\emptyset\}). \quad (4.1)$$

Example 4.5. Let $X = \{1, a, b, c, d\}$ be a pre-logic as in Example 3.3(1).
(1) Consider an $\mathcal{N}$-structure (X, f) as in Example 3.3(1). Then (X, f) is a positive implicative $\mathcal{N}$-ideal of X, since $\{1, a, b\}$ is a positive implicative ideal of X.
(2) Consider an $\mathcal{N}$-structure (X, g) in which g is defined by

$$g(x) := \begin{cases} -0.7 & \text{if } x \in \{1, b\} \\ -0.5 & \text{if } x \in \{a, c, d\}. \end{cases}$$

Then

$$C(g; t) = \begin{cases} X & \text{if } t \in [-0.5, 0] \\ \{1, b\} & \text{if } t \in [-0.7, -0.5) \\ \emptyset & \text{if } t \in [-1, -0.7). \end{cases}$$

Note that $J := \{1, b\}$ is an ideal of X but not a positive implicative ideal of X, since $b \cdot ((a \cdot d) \cdot a) = b \cdot (d \cdot a) = b \cdot 1 = 1 \in J$ and $b \in J$ but $a \notin J$. Hence (X, f) is an $\mathcal{N}$-ideal of X, but not a positive implicative $\mathcal{N}$-ideal of X.

Proposition 4.6. *Every positive implicative $\mathcal{N}$-ideal (X, f) is an $\mathcal{N}$-ideal.*

Proof. Straightforward by Theorem 4.3 and Definition 4.4. □

The converse of Proposition 4.6 is not true in general (see Example 4.5(2)).

Theorem 4.7. *For an $\mathcal{N}$-structure (X, f), the following are equivalent:*

(1) (X, f) *is a positive implicative $\mathcal{N}$-ideal of X.*
(2) (X, f) *satisfies the following two conditions:*

(2.1) $(\forall x \in X)(f(1) \leq f(x))$
(2.2) $(\forall x, y, z \in X)(f(y) \leq \max\{f(x \cdot ((y \cdot z) \cdot y)), f(x)\}$.

Proof. Assume that (X, f) satisfies two conditions (2.1) and (2.2). Let $t \in [1-, 0]$ be such that $C(f; t) \neq \emptyset$. Then there exists $a \in C(f; t)$. By (2.1), $f(1) \leq f(a) \leq t$. Thus $1 \in C(f; t)$. Let $x \cdot ((y \cdot z) \cdot y), x \in C(f; t)$. Then $f(x \cdot ((y \cdot z) \cdot y)) \leq t$ and $f(x) \leq t$. It follows from (2.2) that

$$f(y) \leq \max\{f(x \cdot ((y \cdot z) \cdot y)), f(x)\} \leq t$$

so that $y \in C(f; t)$. Hence $C(f; t)$ is a positive implicative ideal of X and therefore (X, f) is a positive implicative $\mathcal{N}$-ideal of X.

Conversely, suppose that (X, f) is a positive implicative $\mathcal{N}$-ideal of X. If $f(1) > f(a) := t_a$ for some $a \in X$ and so $t_a \in [-1, 0]$, then

$a \in C(f; t_a)$, but $1 \notin C(f; t_a)$. This is a contradiction, and so (2.1) is true. Assume that (2.2) is not valid. Then there exist $a, b, c \in X$ such that

$$f(b) > \max\{f(a \cdot ((b \cdot c) \cdot b)), f(a)\}.$$

Taking $t := \max\{f(a \cdot ((b \cdot c) \cdot b)), f(a)\}$ implies that $a \cdot ((b \cdot c) \cdot b), a \in C(f; t)$ and $b \notin C(f; t)$. This is impossible and thus (2.2) is valid. □

Proposition 4.8. *For an $\mathcal{N}$-ideal (X, f), the following are equivalent:*

(1) *(X, f) is a positive implicative $\mathcal{N}$-ideal of X*
(2) $(\forall x, y \in X)(f(x) \leq f((x \cdot y) \cdot x))$.

Proof. Assume (X, f) is a positive implicative $\mathcal{N}$-ideal of X. Putting $x := 1, y := x$, and $z := z$ in Theorem 4.7(2.2), we have

$$\begin{aligned} f(x) \leq & \max\{f(1 \cdot ((x \cdot z) \cdot x)), f(1)\} \\ = & f((x \cdot z) \cdot x). \end{aligned}$$

Hence (2) holds.

Conversely, Suppose that an $\mathcal{N}$-ideal (X, f) satisfies (2). By Corollary 3.10, for any $x, y, z \in X$ we have

$$\begin{aligned} f(y) \leq & f((y \cdot z) \cdot y) \\ \leq & \max\{f(x \cdot ((y \cdot z) \cdot y)), f(x)\}. \end{aligned}$$

By Theorem 4.7, (X, f) is a positive implicative $\mathcal{N}$-ideal of X. □

Corollary 4.9. *Any positive implicative $\mathcal{N}$-ideal satisfies the following property:*

$$(\forall x, y \in X)(f((y \cdot x) \cdot x) \leq f((x \cdot y) \cdot y)).$$

Proof. Since $x \leq (y \cdot x) \cdot x$ for all $x, y \in X$, it follows from Lemma 2.4 that $((y \cdot x) \cdot x) \cdot y \leq x \cdot y$. Then

$$\begin{aligned} (x \cdot y) \cdot y \leq & (y \cdot x) \cdot ((x \cdot y) \cdot x) \\ = & (x \cdot y) \cdot ((y \cdot x) \cdot x) \\ \leq & (((y \cdot x) \cdot x) \cdot y) \cdot ((y \cdot x) \cdot x). \end{aligned}$$

By Corollary 3.6 and Proposition 4.8, we have $f((x \cdot y) \cdot y) \geq f((((y \cdot x) \cdot x) \cdot y) \cdot ((y \cdot x) \cdot x)) \geq f((y \cdot x) \cdot x)$, for all $x, y \in X$. This completes the proof. □

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