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A NOTE ON THE q-ANALOGUES OF EULER NUMBERS AND POLYNOMIALS

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Abstract. In this paper, we consider the q-analogues of Euler numbers and polynomials using the fermionic p-adic invariant integral on $\mathbb{Z}_p$. From these numbers and polynomials, we derive some interesting identities and properties on the q-analogues of Euler numbers and polynomials.

1. Introduction

Let p be a fixed odd prime number. Throughout this paper $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will denote the ring of p-adic rational integers, the field of p-adic rational numbers, and the completion of algebraic closure of $\mathbb{Q}_p$, respectively. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$.

Let $|\cdot|$ be a p-adic norm with $|x|_p = p^{-r}$ where $x = p^r s/t$ and $(p,s) = (s,t) = (p,t) = 1, \quad r \in \mathbb{Q}$. Let us assume that $q \in \mathbb{C}_p$ with $|1-q|_p < 1$ and $[x]_q = \frac{1-q^x}{1-q}$ (see [1-10]). Note that $\lim_{q\to 1}[x]_q = x$.

Let $C(\mathbb{Z}_p)$ be the space of continuous functions on $\mathbb{Z}_p$. For $f \in C(\mathbb{Z}_p)$, the fermionic p-adic invariant integral on $\mathbb{Z}_p$ is defined by Kim as follows :

$$I(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)(-1)^x, \text{ (see [11]).} \quad (1)$$

For $n \in \mathbb{N}$, let $f_n(x) = f(x+n)$. Then, by (1), we get

$$I(f_n) + (-1)^{n-1} I(f) = 2\sum_{l=0}^{n-1} (-1)^{n-1-l} f(l), \quad \text{(see [2, 3]).} \quad (2)$$

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In the special case, $n = 1$, we have

$$I(f_1) + I(f) = 2f(0). \tag{3}$$

From (2) and (3), we note that

$$\int_{\mathbb{Z}_p} e^{xt} d\mu_{-1}(x) = \frac{2}{e^t + 1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}, \tag{4}$$

where E_n are the n-the Euler numbers (see [1-13]).

In the viewpoint of the q-extension of (4), we consider the q-analogues of Euler numbers and polynomials. From these q-Euler numbers and polynomials, we derive some interesting identities and properties on the q-analogues of Euler numbers and polynomials.

2. q-analogues of Euler numbers and polynomials

In the viewpoint of the q-extension of (4), let us consider the following q-Euler numbers:

$$\tilde{\mathcal{E}}_{n,q} = \int_{\mathbb{Z}_p} q^x x^n d\mu_{-1}(x), \quad \text{where } n \in \mathbb{Z}_+. \tag{5}$$

Let $F_q(t) = \sum_{n=0}^{\infty} \tilde{\mathcal{E}}_{n,q} \frac{t^n}{n!}$. Then we see that

$$F_q(t) = \int_{\mathbb{Z}_p} q^x e^{xt} d\mu_{-1}(x) = \frac{2}{qe^t + 1}. \tag{6}$$

From (5) and (6), we have

$$\int_{\mathbb{Z}_p} q^x e^{xt} d\mu_{-1}(x) = \frac{2}{qe^t + 1} = \frac{1 + q^{-1}}{e^t + q^{-1}} \left(\frac{2}{1 + q}\right), \tag{7}$$

and the Frobenius-Euler numbers are defined by

$$\frac{1 - u}{e^t - u} = e^{H(u)t} = \sum_{n=0}^{\infty} H_n(u) \frac{t^n}{n!}, \tag{8}$$

with the usual convention about replacing $(H(u))^n$ by $H_n(u)$ (see [6, 11]).

By (6), (7) and (8), we obtain the following theorem.

Theorem 1. *For $n \in \mathbb{Z}_+$, we have*

$$\tilde{\mathcal{E}}_{n,q} = \frac{2}{1+q} H_n(-q^{-1}), \tag{9}$$

where $H_n(-q^{-1})$ are the n-th Frobenius-Euler numbers.

Now, we define the q-Euler polynomials as follows:

$$\tilde{\mathcal{E}}_{n,q}(x) = \int_{\mathbb{Z}_p} q^y (x+y)^n d\mu_{-1}(x), \quad \text{for } n \in \mathbb{Z}_+. \tag{10}$$

From (10), we have

$$\tilde{\mathcal{E}}_{n,q}(x) = \sum_{l=0}^{n} \binom{n}{l} x^{n-l} \tilde{\mathcal{E}}_{l,q} = \left(x + \tilde{\mathcal{E}}_q\right)^n, \tag{11}$$

with the usual convention about replacing $(\tilde{\mathcal{E}}_q)^n$ by $\tilde{\mathcal{E}}_{n,q}$.

Let

$$F_q(x,t) = \sum_{n=0}^{\infty} \tilde{\mathcal{E}}_{n,q}(x) \frac{t^n}{n!}.$$

Then we have

$$F_q(x,t) = \int_{\mathbb{Z}_p} q^y e^{(x+y)t} d\mu_{-1}(y) = \frac{2}{qe^t+1} e^{xt}. \tag{12}$$

From (6) and (12), we note that

$$qF_q(1,t) + F_q(t) = 2. \tag{13}$$

Thus, by (13), we obtain the following theorem.

Theorem 2. *For $n \in \mathbb{Z}_+$, we have*

$$q(\tilde{\mathcal{E}}_q + 1)^n + \tilde{\mathcal{E}}_{n,q} = \begin{cases} 2, & \text{if } n = 0, \\ 0, & \text{if } n > 0. \end{cases} \tag{14}$$

From Theorem 2, we get

$$\tilde{\mathcal{E}}_{0,q} = \frac{2}{1+q}, \quad q(\tilde{\mathcal{E}}_q + 1)^n + \tilde{\mathcal{E}}_{n,q} = 0 \quad \text{if } n > 0.$$

By (5) and (6), we get

$$q \int_{\mathbb{Z}_p} e^{-(1-x+x_1)t} q^{x_1} d\mu_{-1}(x_1) = \int_{\mathbb{Z}_p} q^{-x_1} e^{(x+x_1)t} d\mu_{-1}(x_1). \tag{15}$$

Thus, by (15), we have

$$\frac{2q}{qe^{-t}+1} e^{-(1-x)t} = \frac{2}{q^{-1}e^t+1} e^{xt}. \tag{16}$$

From (15) and (16), we can derive the following functional equation:

$$\sum_{n=0}^{\infty} q\tilde{\mathcal{E}}_{n,q}(1-x)\frac{(-1)^n t^n}{n!} = \sum_{n=0}^{\infty} \tilde{\mathcal{E}}_{n,q^{-1}}(x)\frac{t^n}{n!}. \tag{17}$$

By comparing the coefficients on the both sides in (17), we obtain the following theorem.

Theorem 3. *For $n \in \mathbb{Z}_+$, we have*

$$(-1)^n q\tilde{\mathcal{E}}_{n,q}(1-x) = \tilde{\mathcal{E}}_{n,q^{-1}}(x).$$

From Theorem 3, we have

$$(-1)^n q \int_{\mathbb{Z}_p} (1-x+x_1)^n q^{x_1} d\mu_{-1}(x_1) = \int_{\mathbb{Z}_p} (x+x_1)^n q^{-x_1} d\mu_{-1}(x_1).$$

By (14), we get

$$\begin{aligned}
q^2\tilde{\mathcal{E}}_{n,q}(2) &= q^2\Big((\tilde{\mathcal{E}}_q+1)+1\Big)^n = q^2\sum_{l=0}^{n}\binom{n}{l}\Big(\tilde{\mathcal{E}}_q+1\Big)^l \\
&= q\left(2-\tilde{\mathcal{E}}_{0,q}-\sum_{l=1}^{n}\binom{n}{l}\tilde{\mathcal{E}}_{l,q}\right) \\
&= 2q - q\sum_{l=0}^{n}\binom{n}{l}\tilde{\mathcal{E}}_{l,q} = 2q - q(\tilde{\mathcal{E}}_q+1)^n \\
&= \begin{cases} 2q+\tilde{\mathcal{E}}_{0,q}-2, & \text{if } n=0, \\ 2q+\tilde{\mathcal{E}}_{n,q}, & \text{if } n>0. \end{cases}
\end{aligned} \tag{18}$$

Therefore, by (18), we obtain the following theorem.

Theorem 4. *For $n \in \mathbb{Z}_+$, we have*

$$q^2\tilde{\mathcal{E}}_{n,q}(2) = \begin{cases} 2q-\frac{2q}{1+q}, & \textit{if } n=0, \\ 2q+\tilde{\mathcal{E}}_{n,q}, & \textit{if } n>0. \end{cases}$$

It is easy to show that

$$\int_{\mathbb{Z}_p} q^y(x+y)^n d\mu_{-1}(y) = d^n\sum_{a=0}^{d-1}(-1)^a q^a \int_{\mathbb{Z}_p} q^{yd}\left(\frac{x+y}{d}+y\right)^n d\mu_{-1}(y),$$

where $d \in \mathbb{N}$ with $d \equiv 1 \pmod 2$. Therefore, by (19), we obtain the following proposition.

Proposition 5. *For $n \in \mathbb{Z}_+$ and $d \in \mathbb{N}$ with $d \equiv 1 \pmod 2$, we have*

$$\tilde{\mathcal{E}}_{n,q}(x) = d^n \sum_{a=0}^{d-1} (-1)^a q^a \tilde{\mathcal{E}}_{n,q^d}\left(\frac{x+a}{d}\right).$$

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