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A NOTE ON THE WEIGHTED q-BERNOULLI NUMBERS AND THE WEIGHTED q-BERNSTEIN POLYNOMIALS

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Abstract. Recently, the modified q-Bernoulli numbers and polynomials with weight α are introduced in [3]. In this paper we give some interesting p-adic integral representation on $\mathbb{Z}_p$ of the weighted q-Bernstein polynomials related to the modified q-Bernoulli numbers and polynomials with weight α. From those integral representation on $\mathbb{Z}_p$ of the weighted q-Bernstein polynomials, we can derive some identities on the modified q-Bernoulli numbers and polynomials with weight α.

1. Introduction

Let p be a fixed prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of p-adic rational integers, the field of p-adic rational numbers, and the completion of algebraic closure of $\mathbb{Q}_p$, respectively. Throughout this paper, we assume that $\alpha \in \mathbb{Q}$ and $q \in \mathbb{C}_p$ with $|1 - q|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$. The q-number $[x]_q$ is defined by $[x]_q = \frac{1-q^x}{1-q}$. Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable on $\mathbb{Z}_p$ and $C(\mathbb{Z}_p)$ be the space of continuous functions on $\mathbb{Z}_p$. For $f \in C(\mathbb{Z}_p)$, the weighted q-Bernstein operator of order n for $f \in \mathbb{Z}_p$ is defined by Kim-Bayad-Kim as follows:

$$\mathbb{B}_{n,q}^{(\alpha)}(f) = \sum_{k=0}^{n} f(\frac{k}{n})\binom{n}{k}[x]_{q^\alpha}^k[1-x]_{q^{-\alpha}}^{n-k} = \sum_{k=0}^{n} f(\frac{k}{n})B_{k,n}^{(\alpha)}(x,q), \text{ (see [1,2]).} \quad (1)$$

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Here,

$$B_{k,n}^{(\alpha)}(x,q) = \binom{n}{k}[x]_{q^\alpha}^k[1-x]_{q^{-\alpha}}^{n-k}$$

is called the weighted q-Bernstein polynomials of degree n (see [1,2,7]). Recently, several authors have studied weighted q-Bernoulli and Euler numbers (see [1-10]). For $f \in UD(\mathbb{Z}_p)$, the p-adic q-integral on $\mathbb{Z}_p$ (q-Volkenborn integral on $\mathbb{Z}_p$) is defined by Kim as follows (see [4]):

$$I_q(f) = \int_{\mathbb{Z}_p} f(x)d\mu_q(x) = \lim_{N\to\infty}\frac{1}{[p^N]_q}\sum_{x=0}^{p^N-1} f(x)q^x. \tag{2}$$

Recently, Dolgy-Kim-Lee-Rim introduced the modified q-Bernoulli numbers with weight α as follows:

$$\tilde{\beta}_{0,q}^{(\alpha)} = \frac{q-1}{\log q}, \quad \text{and} \quad (q^\alpha\tilde{\beta}_q^{(\alpha)}+1)^n - \tilde{\beta}_{n,q}^{(\alpha)} = \begin{cases} \frac{\alpha}{[\alpha]_q} & \text{if } \ n=1, \\ 0 & \text{if } \ n>1 \end{cases} \tag{3}$$

with the usual convention about replacing $(\tilde{\beta}_q^{(\alpha)})^n$ by $\tilde{\beta}_{n,q}^{(\alpha)}$ (see [3,10]). From (3), they also defined the modified q-Bernoulli polynomials with weight α as follows:

$$\begin{aligned} \tilde{\beta}_{n,q}^{(\alpha)}(x) &= \sum_{l=0}^{n}\binom{n}{l}[x]_{q^\alpha}^{n-l}q^{\alpha l x}\tilde{\beta}_{l,q}^{(\alpha)} \\ &= ([x]_{q^\alpha} + q^{\alpha x}\tilde{\beta}_q^{(\alpha)})^n \text{ (see [3]).} \end{aligned} \tag{4}$$

From (3) and (4), we note that

$$\tilde{\beta}_{n,q}^{(\alpha)}(x) = \frac{\alpha}{(1-q)^n[\alpha]_q^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{\alpha l x}\frac{l}{[\alpha l]_q}. \tag{5}$$

By (2), we easily see that

$$\int_{\mathbb{Z}_p} f(x+n)q^{-x}d\mu_q(x) = \int_{\mathbb{Z}_p} f(x)q^{-x}d\mu_q(x) + \frac{q-1}{\log q}\sum_{l=0}^{n-1} f'(l). \tag{6}$$

By (3),(4),(5), and (6), we get

$$\begin{aligned} \tilde{\beta}_{n,q}^{(\alpha)}(x) &= \int_{\mathbb{Z}_p}[x+y]_{q^\alpha}^n q^{-y}d\mu_q(y) \\ &= \frac{\alpha}{(1-q)^n[\alpha]_q^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{\alpha l x}\frac{l}{[\alpha l]_q}. \end{aligned} \tag{7}$$

The equation (7) is important to derive the main purpose of this paper. In this paper we give a p-adic q-integral representation on $\mathbb{Z}_p$ of the weighted q-Bernstein polynomials of order n associated with the modified q-Bernoulli numbers and polynomials with weight α. From those interval representation on $\mathbb{Z}_p$, we derive some interesting identities on the modified q-Bernoulli numbers and polynomials with weight α.

2. Identities on the weighted q- Bernstein polynomials and the weighted q-Bernoulli numbers.

For $n, k \in \mathbb{Z}_+$, by (1), we get

$$B_{k,n}^{(\alpha)}(x,q) = B_{n-k,n}^{(\alpha)}(1-x,\frac{1}{q}).$$

From (7), we note that

$$\begin{aligned}\tilde{\beta}_{n,q^{-1}}^{(\alpha)}(1-x) &= \int_{\mathbb{Z}_p} [1-x+x_1]_{q^{-\alpha}}^n q^x d\mu_{q^{-1}}(x_1) \\ &= q^{\alpha n-1}(-1)^n \tilde{\beta}_{n,q}^{(\alpha)}(x).\end{aligned} \tag{8}$$

By (2), (6), and (7), we see that

$$\begin{aligned}\frac{1}{q}\int_{\mathbb{Z}_p}[1-x]_{q^{-\alpha}}^n q^{-x} d\mu_q(x) &= (-1)^n q^{\alpha n-1}\int_{\mathbb{Z}_p}[x-1]_{q^\alpha}^n d\mu_q(x) \\ &= (-1)^n q^{\alpha n-1}\tilde{\beta}_{n,q}^{(\alpha)}(-1).\end{aligned} \tag{9}$$

From (7) and (8), we can derive the following equation:

$$\frac{1}{q}\int_{\mathbb{Z}_p}[1-x]_{q^{-\alpha}}^n q^{-x} d\mu_q(x) = (-1)^n q^{\alpha n-1}\tilde{\beta}_{n,q}^{(\alpha)}(-1) = \tilde{\beta}_{n,q^{-1}}^{(\alpha)}(2). \tag{10}$$

By (3) and (4), we get

$$\begin{aligned}\tilde{\beta}_{n,q}^{(\alpha)}(2) &= (q^{2\alpha}\tilde{\beta}_q^{(\alpha)} + [2]_{q^\alpha})^n = (q^\alpha(q^\alpha\tilde{\beta}_q^{(\alpha)}+1)+1)^n \\ &= \sum_{l=0}^n \binom{n}{l} q^{\alpha l}(q^\alpha\tilde{\beta}_q^{(\alpha)}+1)^l \\ &= \tilde{\beta}_{0,q}^{(\alpha)} + nq^\alpha(\frac{\alpha}{[\alpha]_q}+\tilde{\beta}_{1,q}^{(\alpha)}) + \sum_{l=2}^n \binom{n}{l} q^{\alpha l}\tilde{\beta}_{l,q}^{(\alpha)} \\ &= nq^\alpha\frac{\alpha}{[\alpha]_q} + \sum_{l=0}^n \binom{n}{l} q^{\alpha l}\tilde{\beta}_{l,q}^{(\alpha)}.\end{aligned} \tag{11}$$

For $n \in \mathbb{N}$ with $n \geq 2$, by (3) and (11), we get

$$\tilde{\beta}_{n,q}^{(\alpha)}(2) - nq^{\alpha}\frac{\alpha}{[\alpha]_q} = \tilde{\beta}_{n,q}^{(\alpha)}. \tag{12}$$

From (10) and (12), we have

$$\frac{1}{q}\int_{\mathbb{Z}_p}[1-x]_{q^{-\alpha}}^{n}q^{-x}d\mu_q(x) = \tilde{\beta}_{n,q^{-1}}^{(\alpha)}(2) = \frac{n}{q}\frac{\alpha}{[\alpha]_q} + \tilde{\beta}_{n,q^{-1}}^{(\alpha)} \quad \text{if } n \geq 2. \tag{13}$$

Let us take the p-adic q-integral on $\mathbb{Z}_p$ for one weighted q-Bernstein polynomials as follows:

$$\begin{aligned}\int_{\mathbb{Z}_p} B_{k,n}^{(\alpha)}(x,q)q^{-x}d\mu_q(x) &= \binom{n}{k}\int_{\mathbb{Z}_p}[x]_{q^{\alpha}}^{k}[1-x]_{q^{-\alpha}}^{n-k}q^{-x}d\mu_q(x)\\ &= \binom{n}{k}\sum_{l=0}^{n-k}\binom{n-k}{l}(-1)^{l}\int_{\mathbb{Z}_p}[x]_{q^{\alpha}}^{k+l}q^{-x}d\mu_q(x)\\ &= \binom{n}{k}\sum_{l=0}^{n-k}\binom{n-k}{l}(-1)^{l}\tilde{\beta}_{k+l,q}^{(\alpha)}.\end{aligned} \tag{14}$$

By the symmetry of the weighted q-Bernstein polynomials, we get

$$\begin{aligned}\int_{\mathbb{Z}_p} B_{k,n}^{(\alpha)}(x,q)q^{-x}d\mu_q(x) &= \int_{\mathbb{Z}_p} B_{n-k,n}^{(\alpha)}(1-x,\frac{1}{q})q^{-x}d\mu_q(x)\\ &= \binom{n}{k}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k+l}\int_{\mathbb{Z}_p}[1-x]_{q^{-\alpha}}^{n-l}q^{-x}d\mu_q(x)\\ &= q\binom{n}{k}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k+l}(\frac{1}{q}\int_{\mathbb{Z}_p}[1-x]_{q^{-\alpha}}^{n-l}q^{-x}d\mu_q(x))\\ &= q\binom{n}{k}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k+l}(\frac{n-l}{q}\frac{\alpha}{[\alpha]_q} + \tilde{\beta}_{n-l,q^{-1}}^{(\alpha)})\\ &\text{if } n-k > 1.\end{aligned} \tag{15}$$

Therefore, by (15), we obtain the following theorem.

Theorem 1. *For $n, k \in \mathbb{Z}_+$ with $n-k>1$, we have*

$$\int_{\mathbb{Z}_p} B_{k,n}^{(\alpha)}(x,q)q^{-x}d\mu_q(x) = \begin{cases} n\frac{\alpha}{[\alpha]_q} + q\tilde{\beta}_{n,q^{-1}}^{(\alpha)} & \text{if } k=0, \\ n(\frac{\alpha}{[\alpha]_q} - q\tilde{\beta}_{n,q^{-1}}^{(\alpha)} + q\tilde{\beta}_{n-1,q^{-1}}^{(\alpha)}) & \text{if } k=1, \\ q\binom{n}{k}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k+l}\tilde{\beta}_{n-l,q^{-1}}^{(\alpha)} & \text{if } k\geq 2. \end{cases}$$

From (14) and Theorem 1, we can derive the following corollary.

Corollary 2. *For $n, k \in \mathbb{Z}_+$ with $n-k>1$, we have*

$$\sum_{l=0}^{n-k}\binom{n-k}{l}(-1)^l\tilde{\beta}_{k+l,q}^{(\alpha)} = q\sum_{l=0}^{k}\binom{k}{l}(-1)^{k+l}\tilde{\beta}_{n-l,q^{-1}}^{(\alpha)} \quad \text{if } k\geq 2.$$

In particular, when $k=0$ and $k=1$, one has

$$\sum_{l=0}^{n-k}\binom{n-k}{l}(-1)^l\tilde{\beta}_{k+l,q}^{(\alpha)} = \begin{cases} n\frac{\alpha}{[\alpha]_q} + q\tilde{\beta}_{n,q^{-1}}^{(\alpha)} & \text{if } k=0, \\ \frac{\alpha}{[\alpha]_q} - q\tilde{\beta}_{n,q^{-1}}^{(\alpha)} + q\tilde{\beta}_{n-1,q^{-1}}^{(\alpha)} & \text{if } k=1. \end{cases}$$

Let $m, n, k \in \mathbb{Z}_+$ with $m+n-2k>1$. Then we get

$$\begin{aligned} &\int_{\mathbb{Z}_p} B_{k,n}^{(\alpha)}(x,q)B_{k,m}^{(\alpha)}(x,q)q^{-x}d\mu_q(x) \\ &= \binom{n}{k}\binom{m}{k}\int_{\mathbb{Z}_p}[x]_{q^\alpha}^{2k}[1-x]_{q^{-\alpha}}^{n+m-2k}q^{-x}d\mu_q(x) \\ &= \binom{n}{k}\binom{m}{k}\sum_{l=0}^{2k}\binom{2k}{l}(-1)^{l+2k}\int_{\mathbb{Z}_p}[1-x]_{q^{-\alpha}}^{n+m-l}q^{-x}d\mu_q(x) \\ &= q\binom{n}{k}\binom{m}{k}\sum_{l=0}^{2k}\binom{2k}{l}(-1)^{l+2k}(\frac{1}{q}\int_{\mathbb{Z}_p}[1-x]_{q^{-\alpha}}^{n+m-l}q^{-x}d\mu_q(x)). \end{aligned} \quad (16)$$

By (13) and (16), we get

$$\int_{\mathbb{Z}_p} B_{k,n}^{(\alpha)}(x,q)B_{k,m}^{(\alpha)}(x,q)q^{-x}d\mu_q(x)$$

$$= q\binom{n}{k}\binom{m}{k}\sum_{l=0}^{2k}\binom{2k}{l}(-1)^{l+2k}\left(\frac{(n+m-l)\alpha}{q[\alpha]_q}+\tilde{\beta}_{n+m-l,q^{-1}}^{(\alpha)}\right)$$

$$= \begin{cases} (n+m)\frac{\alpha}{[\alpha]_q}+q\tilde{\beta}_{n+m,q^{-1}}^{(\alpha)} & \text{if } k=0, \\ q\binom{n}{k}\binom{m}{k}\sum_{l=0}^{2k}\binom{2k}{l}(-1)^{l+2k}\tilde{\beta}_{n+m-l,q^{-1}}^{(\alpha)} & \text{if } k>0. \end{cases} \tag{17}$$

Therefore, by (17), we obtain the following theorem.

Theorem 3. *For $m, n, k \in \mathbb{Z}_+$ with $m+n-2k>1$, we have*

$$\int_{\mathbb{Z}_p} B_{k,n}^{(\alpha)}(x,q)B_{k,m}^{(\alpha)}(x,q)q^{-x}d\mu_q(x)$$

$$= \begin{cases} (n+m)\frac{\alpha}{[\alpha]_q}+q\tilde{\beta}_{n+m,q^{-1}}^{(\alpha)} & \text{if } k=0, \\ q\binom{n}{k}\binom{m}{k}\sum_{l=0}^{2k}\binom{2k}{l}(-1)^{l+2k}\tilde{\beta}_{n+m-l,q^{-1}}^{(\alpha)} & \text{if } k>0. \end{cases}$$

For $m, n, k \in \mathbb{Z}_+$, it is easy to show that

$$\begin{aligned} &\int_{\mathbb{Z}_p} B_{k,n}^{(\alpha)}(x,q)B_{k,m}^{(\alpha)}(x,q)q^{-x}d\mu_q(x) \\ &= \binom{n}{k}\binom{m}{k}\int_{\mathbb{Z}_p}[x]_{q^\alpha}^{2k}[1-x]_{q^{-\alpha}}^{n+m-2k}q^{-x}d\mu_q(x) \\ &= \binom{n}{k}\binom{m}{k}\sum_{l=0}^{n+m-2k}\binom{n+m-2k}{l}(-1)^l \\ &\quad \int_{\mathbb{Z}_p}[x]_{q^\alpha}^{2k+l}q^{-x}d\mu_q(x) \\ &= \binom{n}{k}\binom{m}{k}\sum_{l=0}^{n+m-2k}\binom{n+m-2k}{l}(-1)^l\tilde{\beta}_{2k+l,q}^{(\alpha)}. \end{aligned} \tag{18}$$

By comparing the coefficients on the both sides of Theorem 3 and (18), we obtain the following corollary.

Corollary 4. *For $m, n, k \in \mathbb{Z}_+$ with $m+n-2k>1$, we have*

$$\sum_{l=0}^{n+m}\binom{n+m}{l}(-1)^l\tilde{\beta}_{2k+l,q}^{(\alpha)}=(n+m)\frac{\alpha}{[\alpha]_q}+q\tilde{\beta}_{n+m,q^{-1}}^{(\alpha)}.$$

In particular, if $k>0$, then we get

$$\sum_{l=0}^{n+m-2k}\binom{n+m-2k}{l}(-1)^l\tilde{\beta}_{2k+l,q}^{(\alpha)}=q\sum_{l=0}^{2k}\binom{2k}{l}(-1)^{l+2k}\tilde{\beta}_{n+m-l,q^{-1}}^{(\alpha)}.$$

By induction hypothesis, we obtain the following theorem.

Theorem 5. *For $s\in\mathbb{N}$ and $k, n_1, n_2, \cdots, n_s \in \mathbb{Z}_+$ with $n_1+n_2+\cdots+n_s-sk>1$, we have*

$$\int_{\mathbb{Z}_p}(\prod_{i=1}^{s}B_{k,n_i}^{(\alpha)}(x,q))q^{-x}d\mu_q(x)$$
$$=\begin{cases}\frac{(n_1+n_2+\cdots+n_s)\alpha}{[\alpha]_q}+q\tilde{\beta}_{n_1+n_2+\cdots+n_s,q^{-1}}^{(\alpha)} & \text{if } k=0,\\ q(\prod_{i=1}^{s}\binom{n_i}{k})\sum_{l=0}^{sk}\binom{sk}{l}(-1)^{l+sk}\tilde{\beta}_{\sum_{i=1}^{s}n_i-l,q^{-1}}^{(\alpha)} & \text{if } k>0.\end{cases}$$

For $s\in\mathbb{N}$ and $k, n_1, n_2, \cdots, n_s \in \mathbb{Z}_+$ with $n_1+n_2+\cdots+n_s-sk>1$, By simple calculation, we easily get

$$\int_{\mathbb{Z}_p}(\prod_{i=1}^{s}B_{k,n_i}^{(\alpha)}(x,q))q^{-x}d\mu_q(x)$$
$$=(\prod_{i=1}^{s}\binom{n_i}{k})\sum_{l=0}^{n_1+\cdots+n_s-sk}\binom{n_1+\cdots+n_s-sk}{l}(-1)^l\tilde{\beta}_{l+sk,q}^{(\alpha)}. \tag{19}$$

By comparing the coefficients on the both sides of Theorem 5 and (19), we obtain the following corollary.

Corollary 6. *For $s\in\mathbb{N}$ and $k, n_1, \cdots, n_s \in \mathbb{Z}_+$ with $n_1+\cdots+n_s-sk>1$,we have*

$$q\sum_{l=0}^{sk}\binom{sk}{l}(-1)^{l+sk}\tilde{\beta}_{n_1+\cdots+n_s-l,q^{-1}}^{(\alpha)}$$
$$=\sum_{l=0}^{\sum_{i=1}^{s}n_i-sk}\binom{\sum_{i=1}^{s}n_i-sk}{l}(-1)^l\tilde{\beta}_{l+sk,q}^{(\alpha)} \text{ if } k>0.$$

In particular, $k=0$, we get

$$\sum_{l=0}^{\sum_{i=1}^{s} n_i} \binom{\sum_{i=1}^{s} n_i}{l} (-1)^l \tilde{\beta}_{l,q}^{(\alpha)} = \frac{(n_1+n_2+\cdots+n_s)\alpha}{[\alpha]_q} + q\tilde{\beta}_{n_1+n_2+\cdots+n_s,q^{-1}}^{(\alpha)}.$$

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