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INTERVAL-VALUED FUZZY SUBGROUPS AND HOMOMORPHISMS

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Abstract. We obtain the interval-valued fuzzy subgroups generated by interval-valued fuzzy sets and some properties preserved by a ring homomorphism. Furthermore, we introduce the concepts of interval-valued fuzzy coset and study some of it's properties.

1. Introduction

In 1975, Zadeh[8] introduced the concept of interval-valued fuzzy sets as a generalization of fuzzy sets introduced by himself[7]. After then, Biswas[1] applied the notion of interval-valued fuzzy sets to group theory. Moreover, Gorzalczany[3] applied it to a method of inference in approximate reasoning, and Montal and Samanta[6] applied it to topology. Recently, Hur et al.[4] introduced the concept of an interval-valued fuzzy relations and obtained some of it's properties . Also, Choi et al.[2] applied it to topology in the sense of Šostak, Kang and Hur [5] applied it to algebra.

In this paper, we introduce the notion of interval-valued fuzzy cosets and investigate some of it's properties. Furthermore we obtain the interval-valued fuzzy subgroups generated by interval-valued fuzzy sets and study some properties preserved by a ring homomorphism.

2. Preliminaries

We will list some concepts and two results needed in the later sections.

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Let $D(I)$ be the set of all closed subintervals of the unit interval $I = [0, 1]$. The elements of $D(I)$ are generally denoted by capital letters $M, N, \cdots$, and note that $M = [M^L, M^U]$, where M^L and M^U are the lower and the upper end points respectively. Especially, we denoted , $\mathbf{0} = [0, 0]$, $\mathbf{1} = [1, 1]$, and $\mathbf{a}$=$[a, a]$ for every $a \in (0, 1)$, We also note that

(i) $(\forall M, N \in D(I))\ (M = N \Leftrightarrow M^L = N^L, M^U = N^U)$,

(ii) $(\forall M, N \in D(I))\ (M \leq N \Leftrightarrow M^L \leq N^L, M^U \leq N^U)$.

For every $M \in D(I)$, the *complement* of M, denoted by M^c, is defined by $M^c = 1 - M = [1 - M^U, 1 - M^L]$(See[6]).

Definition 2.1[3,8]. A mapping $A : X \to D(I)$ is called an *interval-valued fuzzy set* (in short, IVFS) in X, denoted by $A = [A^L, A^U]$, if $A^L, A^U \in I^X$ such that $A^L \leq A^U$, *i.e.*, $A^L(x) \leq A^U(x)$ for each $x \in X$, where $A^L(x)$[resp. $A^U(x)$] is called the *lower*[resp. *upper*] *end point of x to A*. For any $[a, b] \in D(I)$, the interval-valued fuzzy set A in X defined by $A(x) = [A^L(x), A^U(x)] = [a, b]$ for each $x \in X$ is denoted by $\widetilde{[a, b]}$ and if $a = b$, then the IVFS $\widetilde{[a, b]}$ is denoted by simply $\widetilde{a}$. In particular, $\widetilde{0}$ and $\widetilde{1}$ denote the *interval-valued fuzzy empty set* and the *interval-valued fuzzy whole set* in X, respectively.

We will denote the set of all IVFSs in X as $D(I)^X$. It is clear that set $A = [A^L, A^U] \in D(I)^X$ for each $A \in I^X$.

Definition 2.2[**6**]. An IVFS A is called an *interval-valued fuzzy point*(in short, IVFP) in X with the support $x \in X$ and the value $[a, b] \in D(I)$ with $b > 0$, denoted by $A = x_{[a,b]}$, if for each $y \in X$,

$$A(y) = \begin{cases} [a, b] & \text{if } y = x, \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

In particular, if $b = a$, then $x_{[a,b]}$ is denoted by $x_{\mathbf{a}}$.

We will denote the set of all IVFPs in X as $\mathrm{IVF}_P(X)$.

Definition 2.3[6]. Let $A, B \in D(I)^X$ and let $\{A_\alpha\}_{\alpha\in\Gamma} \subset D(I)^X$. Then:

(i) $A \subset B$ iff $A^L \leq B^L$ and $A^U \leq B^U$.

(ii) $A = B$ iff $A \subset B$ and $B \subset A$.

(iii) $A^c = [1 - A^U, 1 - A^L]$.

(iv) $A \cup B = [A^L \vee B^L, A^U \vee B^U]$.

(iv)′ $\bigcup_{\alpha\in\Gamma} A_\alpha = [\bigvee_{\alpha\in\Gamma} A^L_\alpha, \bigvee_{\alpha\in\Gamma} A^U_\alpha]$.

(v) $A \cap B = [A^L \wedge B^L, A^U \wedge B^U]$.

(v)′ $\bigcap_{\alpha\in\Gamma} A_\alpha = [\bigwedge_{\alpha\in\Gamma} A^L_\alpha, \bigwedge_{\alpha\in\Gamma} A^U_\alpha]$.

Result 2.A[6, Theorem 1]. Let $A, B, C \in D(I)^X$ and let $\{A_\alpha\}_{\alpha\in\Gamma} \subset D(I)^X$. Then:

(a) $\widetilde{0} \subset A \subset \widetilde{1}$.

(b) $A \cup B = B \cup A$, $A \cap B = B \cap A$.

(c) $A \cup (B \cup C) = (A \cup B) \cup C$, $A \cap (B \cap C) = (A \cap B) \cap C$.

(d) $A, B \subset A \cup B$, $A \cap B \subset A, B$.

(e) $A \cap (\bigcup_{\alpha\in\Gamma} A_\alpha) = \bigcup_{\alpha\in\Gamma}(A \cap A_\alpha)$.

(f) $A \cup (\bigcap_{\alpha\in\Gamma} A_\alpha) = \bigcap_{\alpha\in\Gamma}(A \cup A_\alpha)$.

(g) $(\widetilde{0})^c = \widetilde{1}$, $(\widetilde{1})^c = \widetilde{0}$.

(h) $(A^c)^c = A$.

(i) $(\bigcup_{\alpha\in\Gamma} A_\alpha)^c = \bigcap_{\alpha\in\Gamma} A^c_\alpha$, $(\bigcap_{\alpha\in\Gamma} A_\alpha)^c = \bigcup_{\alpha\in\Gamma} A^c_\alpha$.

Definition 2.4[7]. Let $A \in D(I)^X$ and let $x_M \in \mathrm{IVF}_P(X)$. Then:

(i) The set $\{x \in X : A^U(x) > 0\}$ is called the *support* of A and is denoted by $S(A)$.

(ii) x_M said to *belong to* A, denoted by $x_M \in A$, if $M^L \leq A^L(x)$ and $M^U \leq A^U(x)$ for each $x \in X$.

It is obvious that $A = \bigcup_{x_M\in A} x_M$ and $x_M \in A$ if and only if $x_{M^L} \in A^L$ and $x_{M^U} \in A^U$.

Definition 2.5[6]. Let $f : X \to Y$ be a mapping, let $A = [A^L, A^U] \in D(I)^X$ and let $B = [B^L, B^U] \in D(I)^Y$. Then

(a) the *image of A under f*, denoted by $f(A)$, is an IVFS in Y defined as follows: For each $y \in Y$,

$$f(A^L)(y) = \begin{cases} \bigvee_{y=f(x)} A^L(x) & \text{if } f^{-1}(y) \neq \varnothing, \\ 0 & \text{otherwise.} \end{cases}$$

and

$$f(A^U)(y) = \begin{cases} \bigvee_{y=f(x)} A^U(x) & \text{if } f^{-1}(y) \neq \varnothing, \\ 0 & \text{otherwise.} \end{cases}$$

(b) the *preimage of B under f*, denoted by $f^{-1}(B)$, is an IVFS in Y defined as follows: For each $y \in Y$,

$$f^{-1}(B^L)(y) = (B^L \circ f)(x) = B^L(f(x))$$

and

$$f^{-1}(B^U)(y) = (B^U \circ f)(x) = B^U(f(x)).$$

It can be easily seen that $f(A) = [f(A^L), f(A^U)]$ and $f^{-1}(B) = [f^{-1}(B^L), f^{-1}(B^U)]$.

Result 2.B[6, Theorem 2]. Let $f : X \to Y$ be a mapping and $g : Y \to Z$ be a mapping. Then

(a) $f^{-1}(B^c) = (f^{-1}(B))^c$, $\forall B \in D(I)^Y$.
(b) $[f(A)]^c \subset f(A^c)$, $\forall A \in D(I)^Y$.
(c) $B_1 \subset B_2 \Rightarrow f^{-1}(B_1) \subset f^{-1}(B_2)$, where $B_1, B_2 \in D(I)^Y$.
(d) $A_1 \subset A_2 \Rightarrow f(A_1) \subset f(A_2)$, where $A_1, A_2 \in D(I)^X$.
(e) $f(f^{-1}(B)) \subset B$, $\forall B \in D(I)^Y$.
(f) $A \subset f(f^{-1}(A))$, $\forall A \in D(I)^Y$.
(g) $(g \circ f)^{-1}(C) = f^{-1}(g^{-1}(C))$, $\forall C \in D(I)^Z$.
(h) $f^{-1}(\bigcup_{\alpha\in\Gamma} B_\alpha) = \bigcup_{\alpha\in\Gamma} f^{-1}B_\alpha$, where $\{B_\alpha\}_{\alpha\in\Gamma} \in D(I)^Y$.
(h) $f^{-1}(\bigcap_{\alpha\in\Gamma} B_\alpha) = \bigcap_{\alpha\in\Gamma} f^{-1}B_\alpha$, where $\{B_\alpha\}_{\alpha\in\Gamma} \in D(I)^Y$.

Definition 2.6[**5**]. An interval-valued fuzzy set A in G is called an *interval-valued fuzzy subgroupoid*(in short, IVGP) in G if

$$A^L(xy) \geq A^L(x) \wedge A^L(y) \text{ and } A^U(xy) \geq A^U(x) \wedge A^U(y), \forall x, y \in G.$$

It is clear that $\widetilde{0}, \widetilde{1} \in$ IVGP(G). We will denote the IVGPs in G as IVGP(G).

3. Interval-valued fuzzy subgroup generated by an interval-valued fuzzy set

Definition 3.1[5]. Let A be an IVFS in a set X and let $[\lambda, \mu] \in D(I)$. Then the set $A^{[\lambda,\mu]} = \{x \in X : A^L(x) \geq \lambda$ and $A^U(x) \geq \mu\}$ is called a $[\lambda, \mu]$-*level subset* of A.

The following is the immediate result of Definition 3.1.

Proposition 3.2. Let A be an IVFS in a set X and let $[\lambda_1, \mu_1], [\lambda_2, \mu_2] \in$ Im(A). If $\lambda_1 < \lambda_2$ and $\mu_1 < \mu_2$, then $A^{[\lambda_1,\mu_1]} \supset A^{[\lambda_2,\mu_2]}$.

Definition 3.3[5]. Let G be a group and let $A \in D(I)^G$. Then A is called an *interval-valued fuzzy subgroup* (in short, IVG) of G if it satisfies the following conditions :

(i) $A \in$ IVGP(G), i.e., $A^L(xy) \geq A^L(x) \wedge A^L(y)$ and $A^U(xy) \geq A^U(x) \wedge A^U(y), \forall x, y \in G$.
(ii) $A^L(x^{-1}) \geq A^L(x)$ and $A^U(x^{-1}) \geq A^U(x), \forall x, y \in G$.

We will denote the set of all IVGs of G as IVG (G).

Result 3.A[1, Proposition 3.1]. Let A be an IVG of a group G. Then $A(x^{-1}) = A(x)$ and $A^L(x) \leq A^L(e), A^U(x) \leq A^U(e)$ for each $x \in G$, where e is the identity element of G.

Result 3.B[5, Proposition 4.16 and 4.17]. Let A be an IVFS in a group G. Then $A \in$ IVG(G) if and only if $A^{[\lambda,\mu]}$ is a subgroup of G for each $[\lambda, \mu] \in$ Im(A).

Definition 3.4. Let A be an IVG of a group G and $[\lambda, \mu] \in$ Im(A). Then the subgroup $A^{[\lambda,\mu]}$ is called a $[\lambda, \mu]$-*level subgroup* of A.

Lemma 3.5. Let A be any IVFS of a set X. Then $A^L(x) = \bigvee\{\lambda : x \in A^{[\lambda,\mu]}\}$ and
$A^U(x) = \bigvee\{\mu : x \in A^{[\lambda,\mu]}\}$, where $x \in X$ and $[\lambda, \mu] \in D(I)$.

Proof. Let $\alpha = \bigvee\{\lambda : x \in A^{[\lambda,\mu]}\}$, let $\beta = \bigvee\{\mu : x \in A^{[\lambda,\mu]}\}$ and let $\epsilon > 0$ be arbitrary. Then $\alpha - \epsilon < \bigvee\{\lambda : x \in A^{[\lambda,\mu]}\}$ and

$\beta - \epsilon < \bigvee\{\mu : x \in A^{[\lambda,\mu]}\}$. Thus there exist $[\lambda, \mu] \in D(I)$ such that $x \in A^{[\lambda,\mu]}, \alpha - \epsilon < \lambda$ and $\beta - \epsilon < \mu$. Since $x \in A^{[\lambda,\mu]}, A^L(x) \geq \lambda$ and $A^U(x) \geq \mu$. Thus $A^L(x) > \alpha - \epsilon$ and $A^U(x) > \beta - \epsilon$. Since $\epsilon > 0$ is arbitrary, $A^L(x) \geq \alpha$ and $A^U(x) \geq \beta$. We now show that $A^L(x) \leq \alpha$ and $A^U(x) \leq \beta$. Suppose $A^L(x) = t_1$ and $A^U(x) = t_2$. Then $[t_1, t_2] \in \text{Im(A)}$. Thus $x \in A^{[t_1,t_2]}$. So $t_1 \in \{\lambda : x \in A^{[\lambda,\mu]}\}$ and $t_2 \in \{\lambda : x \in A^{[\lambda,\mu]}\}$. So $t_1 = \bigvee\{\lambda : x \in A^{[\lambda,\mu]}\}$ and $t_2 = \bigvee\{\mu : x \in A^{[\lambda,\mu]}\}$, i.e. $A^L(x) \leq \alpha$ and $A^U(x) \leq \beta$. This completes the proof. □

We shall denote by (A) the IVG generated by the IVFS A in G. We shall use the same notation $(A^{[\lambda,\mu]})$ for the ordinary subgroup of the group G generated by the level subset $A^{[\lambda,\mu]}$.

Theorem 3.6. Let G be group and let $A \in D(I)^G$. Let $A^* \in D(I)^G$ be defined as follows: For each $x \in G$,

$(A^*)^L(x) = \bigvee\{\lambda : x \in (A^{[\lambda,\mu]})\}$ and $(A^*)^U(x) = \bigvee\{\mu : x \in (A^{[\lambda,\mu]})\}$,

where $[\lambda, \mu] \in D(I)$. Then $A^* \in \text{IVG(G)}$ such that $A^* = \bigcap\{B \in \text{IVG(G)}: A \subset B\}$. In this case, A^* is called the *interval-valued fuzzy subgroup generated by A in G* and will be denoted by (A).

Proof. Let $[t_1, t_2] \in \text{Im }(A^*)$ and $\alpha = t_1 - \frac{1}{n}$ and $\alpha = t_2 - \frac{1}{n}$, where n is any sufficiently large positive integer. Let $x \in G$. Suppose $x \in {A^*}^{[t_1,t_2]}$. Then $(A^*)^L(x) \geq t_1$ and $(A^*)^U(x) \geq t_2$. Thus there exist $[\lambda, \mu] \in D(I)$ such that $\lambda > \alpha$, $\mu > \beta$ and $x \in A^{[\lambda,\mu]}$. Since $[\alpha, \beta] < [\lambda, \mu]$ and $[\alpha, \beta] \in D(I)$, by Proposition 3.2, $A^{[\lambda,\mu]} \subset A^{[\alpha,\beta]}$. So $x \in A^{[\alpha,\beta]}$, i.e., $x \in (A^{[\alpha,\beta]})$. Now suppose $x \in (A^{[\lambda,\mu]})$. Then $\alpha \in \{\lambda : x \in (A^{[\lambda,\mu]})\}$ and $\beta \in \{\mu : x \in (A^{[\lambda,\mu]})\}$. Thus $\alpha \leq \bigvee\{\lambda : x \in (A^{[\lambda,\mu]})\}$ and $\beta \leq \bigvee\{\mu : x \in (A^{[\lambda,\mu]})\}$. So $t_1 - \frac{1}{n} \leq (A^*)^L(x)$ and $t_2 - \frac{1}{n} \leq (A^*)^U(x)$, i.e., $t_1 \leq (A^*)^L(x)$ and $t_2 \leq (A^*)^U(x)$. Hence $x \in {A^*}^{[t_1,t_2]}$, i.e.,$({A^*}^{[\alpha,\beta]}) \subset {A^*}^{[t_1,t_2]}$. Therefore ${A^*}^{[t_1,t_2]} = ({A^*}^{[\alpha,\beta]})$. Since $({A^*}^{[\alpha,\beta]})$ is a subgroup of G, ${A^*}^{[t_1,t_2]}$ is a subgroup of G. By Result 3.B, $A^* \in \text{IVG(G)}$.

Now, we show that $A \subset A^*$. Let $x \in G$. Then, by Lemma 3.5, $(A^*)^L(x) = \bigvee\{\lambda : x \in A^{[\lambda,\mu]}\}$ and $(A^*)^U(x) = \bigvee\{\mu : x \in A^{[\lambda,\mu]}\}$. Thus $(A^*)^L(x) \leq \bigvee\{\lambda : x \in (A^{[\lambda,\mu]})\}$ and $(A^*)^U(x) \leq \bigvee\{\mu : x \in (A^{[\lambda,\mu]})\}$. So $A \subset A^*$. Finally, let B be any IVG of G such that $A \subset B$. We show that $A^* \subset B$. Let $x \in G$ and $A^*(x) = [t_1, t_2]$. Then ${A^*}^{[t_1,t_2]} = (A^{[\alpha,\beta]})$, where $\alpha = t_1 - \frac{1}{n}, \beta = t_2 - \frac{1}{n}$, and n is any sufficiently large positive integer. Thus $x \in (A^{[\alpha,\beta]})$. So $x = a_1 a_2 \cdots a_m$, where a_i or a_i^{-1} belongs to $A^{[\alpha,\beta]}(i = 1, \cdots, m)$.

On the other hand,

$$\begin{aligned} B^L(x) &= B^L(a_1a_2\cdots a_m) \\ &\geq B^L(a_1)\wedge B^L(a_2)\wedge\cdots\wedge B^L(a_m) \\ &\geq A^L(a_1)\wedge A^L(a_2)\wedge\cdots\wedge A^L(a_m) \\ &\geq \alpha = t_1 - \frac{1}{n}. \end{aligned}$$

By the similar arguments, we have that $B^U(x) \geq \beta = t_2 - \frac{1}{n}$. Since n is sufficiently large positive integer, $B^L(x) \geq t_1$ and $B^U(x) \geq t_2$. So $A^* \subset B$. Hence $A^* = \bigcap\{B \in$IVG(G): $A \subset B\}$. This completes the proof. □

It is possible that card Im (A^*) be less than card Im (A). Moreover, Im (A^*) need not be contained in Im (A) as shown in the following examples.

Example 3.7. let $G = \{e, a, b, c\}$ be the Klein four -group, where $a^2 = b^2 = e$ and $ab = ba$. Define an IVFS A of G by: $A(e) = [0.5, 0.5], A(a) = [0.2, 0.8], A(b) = [0.3, 0.7], A(ab) = [0.4, 0.6]$. Then $A^{[0.2,0.8]} = \{a\}, A^{[0.3,0.7]} = \{a, b\}, A^{[0.4,0.6]} = \{a, b, ab\}$ and $A^{[0.5,0.5]} = G$. Thus $(A^{[0.2,0.8]}) = \{e, a\}$ and $(A^{[0.3,0.7]}) = G$. Moreover, by definition, we have $A^*(e) = A^*(a) = [0.2, 0.8]$ and $A^*(b) = A^*(ab) = [0.3, 0.7]$. □

Now an attempt is made to obtain a necessary and sufficient condition for a p-group to be cyclic.

Lemma 3.8. Let G be a finite group. Suppose there exists $A \in$ IVG(G) satisfying the following conditions: For any $x, y \in G$,

(i) $A(x) = A(y) \Rightarrow (x) = (y)$.

(ii) $A^L(x) > A^L(y)$ and $A^U(x) > A^U(y) \Rightarrow (x) \subset (y)$.

Then G is cyclic.

Proof. Suppose A is constant on G. Then $A(x) = A(y)$ for any $x, y \in G$. By the condition (i), $(x) = (y)$. So $G = (x)$. Now suppose A is not constant on G. Let Im $(A) = \{[t_0, s_0], [t_1, s_1], \cdots, [t_n, s_n]\}$, where $t_0 > t_1 > \cdots > t_n$ and $s_0 > s_1 > \cdots > s_n$. Then, by Proposition 3.2 and Result 3.B, we obtain the chain of level subgroups of A:

$A^{[t_0,s_0]} \subset A^{[t_1,s_1]} \subset \cdots \subset A^{[t_n,s_n]} = G$.

Let $x \in G - A^{[t_{n-1},s_{n-1}]}$. We show that $G = (x)$. Let $g \in G - A^{[t_{n-1},s_{n-1}]}$. Since $t_0 > t_1 > \cdots > t_n$ and $s_0 > s_1 > \cdots > s_n, A(g) = A(x) =$

$A^{[t_{n-1},s_{n-1}]}$. By the condition (i), $(g) = (x)$. Thus $G - A^{[t_{n-1},s_{n-1}]} \subset (x)$. Now let $g \in A^{[t_{n-1},s_{n-1}]}$. Then $A^L(g) \geq t_{n-1} > t_n = A^L(x)$ and $A^U(g) \geq s_{n-1} > s_n = A^U(x)$. By the condition (ii), $(g) = (x)$. Thus $A^{[t_{n-1},s_{n-1}]} \subset (x)$. So $G = (x)$. Hence, in either cases, G is cyclic. □

Lemma 3.9. Let G be a cyclic group of order p^n, where p is prime. Then there exists $A \in$ IVG(G) satisfying the following conditions: For any $x, y \in G$,

(i) $A(x) = A(y) \Rightarrow (x) = (y)$.

(ii) $A^L(x) > A^L(y)$ and $A^U(x) > A^U(y) \Rightarrow (x) \subset (y)$.

Proof. Consider the following chain of subgroups of G:

$(e) = G_0 \subset G_1 \subset \cdots \subset G_{n-1} \subset G_n = G$,

where G_i is the subgroup of G generated by an element of order p^i, $i = 0, 1, \cdots, n$ and e is the identity of G. We define a mapping $A : G \to D(I)$ as follows: For each $x \in G$, $A(e) = [t_0, s_0]$ and $A(x) = [t_i, s_i]$ if $x \in G_i - G_{i-1}$ for any $i = 1, 2, \cdots, n$, where $[t_i, s_i] \in D(I)$ such that $t_0 > t_1 > \cdots > t_n$ and $s_0 > s_1 > \cdots > s_n$. Then we can easily check that $A \in$ IVG(G) satisfying the conditions (i) and (ii). □

From Lemmas 3.8 and 3.9, we obtain the following.

Theorem 3.10. Let G be a group of order p^n. Then G is cyclic if and only if there exists $A \in$ IVG(G) satisfying the following conditions: For any $x, y \in G$,

(i) $A(x) = A(y) \Rightarrow (x) = (y)$.

(ii) $A^L(x) > A^L(y)$ and $A^U(x) > A^U(y) \Rightarrow (x) \subset (y)$.

4. Interval-valued fuzzy ideals and homomorphisms

Definition 4.1[5]. Let $(R, +, \cdot)$ be a ring and let $\tilde{0} \neq A \in D(I)^R$. Then A is called an *interval- valued fuzzy subring* (in short, *IVR*) in R if it satisfies the following conditions:

(i) A is an IVG in R with respect to the operation "+"(in the sense of Definition 3.3).

(ii) A is an IVGP in R with respect to the operation " $\cdot$ "(in the sense of Definition 2.6).

It is clear that subrings of R are IVRs of R. We will denote the set of all IVRs of R as IVR(R).

Definition 4.2[5]. Let R be a ring and let $\tilde{0} \neq A \in D(I)^R$. Then A is called an *interval- valued fuzzy ideal* (in short, IVI) of R if it satisfies the following conditions:

(i) A is an IVR of R.

(ii) $A^L(xy) \geq A^L(x), A^U(xy) \geq A^U(x)$ and $A^L(xy) \geq A^L(y), A^U(xy) \geq A^U(y)$ for any $x, y \in R$.

We will denote the set of all IVIs of R as IVI(R).

Result 4.A[5, Proposition 6.5]. Let R be a ring and let $\tilde{0} \neq A \in D(I)^R$. Then $A \in \text{IVR}(R)$ if and only if for any $x, y \in R$,

(i)$A^L(x-y) \geq A^L(x) \wedge A^L(y)$ and $A^U(x-y) \geq A^U(x) \wedge A^U(y)$.

(ii)$A^L(xy) \geq A^L(x) \wedge A^L(y)$ and $A^U(xy) \geq A^U(x) \wedge A^U(y)$.

It is clear that if A is an IVI(R), then $A(-x) = A(x) \leq A(0)$ for each $x \in R$, where 0 is the identity in R with respect to "+".

Proposition 4.3. Let A be an IVFS in a ring R. Then $A \in$ IVI(R) if and only if $A^{[\lambda,\mu]}$ is an ideal of R for each $[\lambda, \mu] \in$ Im(A).

Proof. $(\Rightarrow)$: Suppose $A \in$ IVI(R). For each $[\lambda, \mu] \in \text{Im}(A)$, let $x, y \in A^{[\lambda,\mu]}$. Then $A^L(x) \geq \lambda, A^U(x) \geq \mu$ and $A^L(y) \geq \lambda, A^U(y) \geq \mu$. By Result 4.A (i), $A^L(x-y) \geq A^L(x) \wedge A^L(y)$ and $A^U(x-y) \geq A^U(x) \wedge A^U(y)$. Thus $A^L(x-y) \geq \lambda$ and $A^U(x-y) \geq \mu$. So $x - y \in A^{[\lambda,\mu]}$. Let $x \in R$ and $y \in A^{[\lambda,\mu]}$. Then $A^L(y) \geq \lambda$ and $A^U(y) \geq \mu$. Since $A \in$ IVI(R), by Result 4.A (ii), $A^L(xy) \geq A^L(x) \wedge A^L(y)$ and $A^U(xy) \geq A^U(x) \wedge A^U(y)$. Thus $A^L(xy) \geq \lambda$ and $A^U(xy) \geq \mu$. So $xy \in A^{[\lambda,\mu]}$. Similarly, we have $yx \in A^{[\lambda,\mu]}$. Hence $A^{[\lambda,\mu]}$ is an ideal of R.

$(\Leftarrow)$: Suppose the necessary holds. For any $x, y \in R$, let $A(x) = [t_1, s_1]$ and $A(y) = [t_2, s_2]$. Then clearly $x \in A^{[t_1,s_1]}$ and $y \in A^{[t_2,s_2]}$. Since $A^{[t_1,s_1]}$ is an ideal of R, $x - y \in A^{[t_1,s_1]}$. Then $A^L(x-y) \geq t_1 \geq t_1 \wedge t_2 = A^L(x) \wedge A^L(y)$ and $A^U(x-y) \geq s_1 \geq s_1 \wedge s_2 = A^U(x) \wedge A^U(y)$. Thus A satisfies the condition (i) of Result 4.A. Now for each $x \in R$, let $A(x) = [\lambda, \mu]$. Then clearly $x \in A^{[\lambda,\mu]}$. Let $y \in R$. Since $A^{[\lambda,\mu]}$ is an ideal of R, $xy \in A^{[\lambda,\mu]}$ $yx \in A^{[\lambda,\mu]}$. Then $A^L(xy) \geq \lambda = A^L(x), A^U(xy) \geq \mu = A^U(x)$ and $A^L(yx) \geq \lambda = A^L(y), A^U(yx) \geq \mu = A^U(y)$. Thus A satisfies the condition (ii) of Definition 4.2. Hence A is an IVI of R. □

Example 4.4. Let R denote the ring of real numbers under the usual operations of addition and multiplication. We define a mapping $A : R \to D(I)$ as follows: For each $x \in R$,

$$A(x) = \begin{cases} [t, s] & \text{if x is rational,} \\ \\ [t', s'] & \text{if x is irrational} \end{cases}$$

where $[t, s], [t', s'] \in D(I)$ such that $t > t'$ and $s > s'$. Then we can see that $A \in$ IVR(R) but $A \notin$ IVI(R). □

Definition 4.5[5]. Let X and Y be sets, let $f : X \to Y$ be a mapping and let $A \in D(I)^X$. Then A is said to be *interval-valued fuzzy invariant*(in short, *IVF-invariant*) if $f(x) = f(y)$ implies $A(x) = A(y)$, i.e., $A^L(x) = A^L(y)$ and $A^U(x) = A^U(y)$.

It is clear that if A is IVF-invariant, then $f^{-1}(f(A)) = A$.

Definition 4.6[**5**]. Let $(X, \circ)$ be a groupoid and let $A, B \in D(I)^X$. Then the *interval-valued fuzzy product* of A and B, $A \circ B$, is defined as follow : For each $x \in X$,

$$(A \circ B)^L(x) = \begin{cases} \bigvee_{(y,z) \in X \times X} (A^L(y) \wedge B^L(z)) & \text{if x=yz,} \\ \\ 0 & \text{otherwise} \end{cases}$$

and

$$(A \circ B)^U(x) = \begin{cases} \bigvee_{(y,z) \in X \times X} (A^U(y) \wedge B^U(z)) & \text{if x=yz,} \\ \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, we have the following definition.

Definition 4.7. Let A and B be any two IVIs of a ring R. Then the *interval-valued fuzzy sum of* A and B, $A + B$, is defined as follow : For each $x \in X$,

$$(A + B)^L(x) = \begin{cases} \bigvee_{(y,z) \in X \times X} (A^L(y) \wedge B^L(z)) & \text{if x=y+z,} \\ \\ 0 & \text{otherwise} \end{cases}$$

and

$$(A+B)^U(x) = \begin{cases} \bigvee_{(y,z)\in X\times X} (A^U(y)\wedge B^U(z)) & \text{if x=y+z,} \\ 0 & \text{otherwise} \end{cases}$$

This section reflects the effect of a homomorphism on the sum, product and intersection of any two IVIs of a ring.

Proposition 4.8. Let $f : R \to R'$ be a ring epimorphism. If A and B are IVIs of R, then

(a) $f(A+B) = f(A) + f(B)$,

(b) $f(A \circ B) = f(A) \circ f(B)$,

(c) $f(A \cap B) = f(A) \cap f(B)$, with equality if at least one of A or B is IVF-invariant.

Proof. (a) Let $y \in R'$ and let $\epsilon > 0$ be arbitrary. Let $[\alpha, \alpha^{'}] = f(A+B)(y)$ and let $[\beta, \beta^{'}] = (f(A)+f(B))(y)$.
Then

$$\begin{aligned} \alpha &= f(A+B)^L(y) = \bigvee_{z\in f^{-1}(y)} (A+B)^L(z), \\ \alpha^{'} &= f(A+B)^U(y) = \bigvee_{z\in f^{-1}(y)} (A+B)^U(z) \end{aligned}$$

and

$$\begin{aligned} \beta &= (f(A)+f(B))^L(y) = \bigvee_{y=z+z'} (f(A)^L(z)\wedge f(B)^L(z')), \\ \beta^{'} &= (f(A)+f(B))^U(y) = \bigvee_{y=z+z'} (f(A)^U(z)\wedge f(B)^U(z')). \end{aligned}$$

Thus $\alpha - \epsilon < \bigvee_{z\in f^{-1}(y)}(A+B)^L(z)$ and $\alpha^{'} - \epsilon < \bigvee_{z\in f^{-1}(y)}(A+B)^U(z)$. So there exist $z_0, z_0^{'} \in R$ with $f(z_0) = y$ and $f(z_0^{'}) = y$ such that $\alpha - \epsilon < (A+B)^L(z_0)$ and $\alpha - \epsilon < (A+B)^U(z_0^{'})$. By the definition of sum,

$\alpha - \epsilon < \bigvee_{z_0=a+b}(A^L(a)\wedge B^L(b))$ and $\alpha^{'} - \epsilon < \bigvee_{z_0^{'}=a^{'}+b^{'}}(A^L(a^{'})\wedge B^L(b^{'}))$.

Then there exist $a_0, b_0 \in R$ with $z_0 = a_0 + b_0$ such that $\alpha - \epsilon < (A^L(a_0)\wedge B^L(b_0))$ and there exist $a_0^{'}, b_0^{'} \in R$ with $z_0^{'} = a_0^{'} + b_0^{'}$ such that $\alpha^{'} - \epsilon < (A^U(a_0^{'})\wedge B^U(b_0^{'}))$.

On the other hand,

$$\begin{aligned}\beta &\geq f(A)^L(f(a_0)) \wedge f(B)^L(f(b_0)) \\ &= f(A^L)(f(a_0)) \wedge f(B^L)(f(b_0)) \\ &= f^{-1}(f(A^L))(a_0) \wedge f^{-1}(f(B^L))(b_0) \\ &\geq A^L(a_0) \wedge B^L(b_0).\end{aligned}$$

Similarly, we have $\beta' \geq A^U(a_0') \wedge B^U(b_0')$. So $\beta > \alpha - \epsilon$ and $\beta' > \alpha' - \epsilon$. Since ϵ is arbitrary, $\beta \geq \alpha$ and $\beta' \geq \alpha'$. Hence

$[f(A) + f(B)]^L(y) \geq f(A+B)^L(y)$ for each $y \in R'$. (4.1)

Now we will show that $\beta \leq \alpha$ and $\beta' \leq \alpha'$. Clearly,

$\beta - \epsilon < \bigvee_{y=z+z'}(f(A)^L(z) \wedge f(B)^L(z'))$

and

$\beta - \epsilon < \bigvee_{y=z+z'}(f(A)^U(z) \wedge f(B)^U(z'))$.

Then there exist $z_1, z_1' \in R'$ with $y = z_1 + z_1'$ such that

$\beta - \epsilon < f(A)^L(z_1) = \bigvee_{x \in f^{-1}(z_1)} A^L(x)$

and

$\beta - \epsilon < f(B)^L(z_1) = \bigvee_{x \in f^{-1}(z_1')} A^L(x)$.

Hence there exist $z_0, z_0' \in R'$ with $y = z_0 + z_0'$ such that

$\beta - \epsilon < f(A)^U(z_0) = \bigwedge_{x \in f^{-1}(z_0)} A^U(x)$

and

$\beta - \epsilon < f(B)^U(z_0') = \bigwedge_{x \in f^{-1}(z_0')} B^U(x)$.

Thus there exist $x_1, x_1' \in R$ with $f(x_1) = z_1,\ f(x_1') = z_1'$ such that

$\beta - \epsilon < A^L(x_1), \beta - \epsilon < B^L(x_1')$

and

there exist $x_0, x_0' \in R$ with $f(x_0) = z_0,\ f(x_0') = z_0'$ such that

$\beta - \epsilon < f^U(x_0), \beta - \epsilon < B^U(x_0')$. So

$$\begin{aligned}\beta - \epsilon &< A^L(x_1) \wedge B^L(x_1') \leq (A+B)^L(x_1 + x_1') \\ &\leq \bigvee_{x \in f^{-1}(y)} (A+B)^L(x) = f(A+B)^L(y)\end{aligned}$$

and

$$\begin{aligned}\beta' - \epsilon &< A^U(x_0) \wedge B^U(x_0') \leq (A+B)^U(x_0 + x_0') \\ &\leq \bigvee_{x \in f^{-1}(y)} (A+B)^U(x) = f(A+B)^U(y).\end{aligned}$$

Hence $\beta - \epsilon < \alpha$ and $\beta' - \epsilon < \alpha'$. Since $\epsilon > 0$ is arbitrary, $\beta \leq \alpha$ and $\beta' \leq \alpha'$. So

$(f(A) + f(B))(y) \leq f(A + B)(y)$ for each $y \in R'$. (4.2)

Therefore, by (4.1) and (4.2), $f(A) + f(B) = f(A + B)$.

(b) Let $y \in R'$ and let $\epsilon > 0$ be arbitrary. Let $[\alpha, \alpha'] = f(A \circ B)(y)$ and $[\beta, \beta'] = (f(A) \circ f(B))(y)$. Then

$$\begin{aligned} \alpha &= f(A \circ B)^L(y) = \bigvee_{x \in f^{-1}(y)} (A \circ B)^L(z), \\ \alpha' &= f(A \circ B)^U(y) = \bigvee_{x \in f^{-1}(y)} (A \circ B)^U(z) \ (4.3) \end{aligned}$$

and

$$\begin{aligned} \beta &= (f(A) \circ f(B))^L(y) = \bigvee_{y = y_1 y_2} (f(A)^L(y_1) \wedge f(B)^L(y_2)), \\ \beta' &= (f(A) \circ f(B))^U(y) = \bigvee_{y = y_1 y_2} (f(A)^U(y_1) \wedge f(B)^U(y_2)). \ (4.4) \end{aligned}$$

In (4.3), $\alpha - \epsilon < \bigvee_{z \in f^{-1}(y)} (A \circ B)^L(z)$ and $\alpha' - \epsilon < \bigvee_{z \in f^{-1}(y)} (A \circ B)^U(z)$. Thus there exist $x, x' \in f^{-1}(y)$ such that $\alpha - \epsilon < (A \circ B)^L(x)$ and $\alpha' - \epsilon < (A \circ B)^U(x)$. Since $(A \circ B)^L(x) = \bigvee_{x = x_1 x_2} (A^L(x_1) \wedge B^L(x_2))$ and $(A \circ B)^U(x') = \bigvee_{x' = x_1' x_2'} (A^U(x_1') \wedge B^U(x_2'))$, there exist $x_1, x_2, x_1', x_2' \in R$ with $x = x_1 x_2$ and $x' = x_1' x_2'$ such that $\alpha - \epsilon < A^L(x_1) \wedge B^L(x_2)$ and $\alpha' - \epsilon < A^U(x_1') \wedge B^U(x_2')$. Since $A \subset f^{-1}(f(A))$, by Result 2.B(f), $A^L \leq f^{-1}(f(A))^L$ and $A^U \leq f^{-1}(f(A))^U$. On the other hand,

$f^{-1}(f(A))^L = f^{-1}(f(A)^L) = f^{-1}(f(A^L))$

and

$f^{-1}(f(A))^U = f^{-1}(f(A)^U) = f^{-1}(f(A^U))$. Thus

$$\begin{aligned} \alpha - \epsilon &< f^{-1}(f(A)^L)(x_1) \wedge f^{-1}(f(B)^L)(x_2) \\ &= f(A)^L(f(x_1)) \wedge f(B)^L(f(x_2)) \\ &\leq \bigvee_{y = y_1 y_2} (f(A)^L(y_1) \wedge f(B)^L(y_2)) \\ &= (f(A) \circ f(B))^L(y) = \beta. \end{aligned}$$

By the similar arguments, we have that $\alpha^{'} - \epsilon \leq (f(A) \circ f(B))^U(y) = \beta^{'}$. Since $\epsilon > 0$ is arbitrary, $\alpha \leq \beta$ and $\alpha^{'} \leq \beta^{'}$. In (4.4),

$$\begin{aligned}\beta - \epsilon &< \bigvee_{y=y_1y_2} (f(A)^L(y_1) \wedge f(B)^L(y_2))(y_2) \\ &= \bigvee_{y=y_1y_2} ((\bigvee_{z_1 \in f^{-1}(y_1)} A^L(z_1)) \wedge (\bigvee_{z_2 \in f^{-1}(y_2)} B^L(z_2)))\end{aligned}$$

and

$$\begin{aligned}\beta^{'} - \epsilon &< \bigvee_{y=y_1y_2} (f(A)^U(y_1) \wedge f(B)^U(y_2)) \\ &= \bigvee_{y=y_1y_2} ((\bigvee_{z_1 \in f^{-1}(y_1)} A^U(z_1)) \wedge (\bigvee_{z_2 \in f^{-1}(y_2)} B^U(z_2))).\end{aligned}$$

Thus there exist $y_1, y_2 \in R'$ with $y = y_1y_2$ such that

$$\begin{aligned}\beta - \epsilon &< (\bigvee_{z_1 \in f^{-1}(y_1)} A^L(z_1)) \wedge (\bigvee_{z_2 \in f^{-1}(y_2)} B^L(z_2)) \\ &= \bigvee_{z_1 \in f^{-1}(y_1)} \bigvee_{z_2 \in f^{-1}(y_2)} (A^L(z_1) \wedge B^L(z_2))\end{aligned}$$

and

$$\begin{aligned}\beta^{'} - \epsilon &< (\bigvee_{z_1 \in f^{-1}(y_1)} A^U(z_1)) \wedge (\bigvee_{z_2 \in f^{-1}(y_2)} B^U(z_2)) \\ &= \bigvee_{z_1 \in f^{-1}(y_1)} \bigvee_{z_2 \in f^{-1}(y_2)} (A^U(z_1) \wedge B^U(z_2)).\end{aligned}$$

So there exist $x_1 \in f^{-1}(y_1)$ and $x_2 \in f^{-1}(y_2)$ such that $\beta - \epsilon < A^L(x_1) \wedge B^L(x_2)$ and $\beta - \epsilon < A^U(x_1) \wedge B^U(x_2)$.

Let $x = x_1x_2$. Since f is a ring homomorphism, $y = y_1y_2 = f(x_1x_2) = f(x)$. Thus

$$\begin{aligned}A^L(x_1) \wedge B^L(x_2) &\leq \bigvee_{x=x_1x_2} (A^L(x_1) \wedge B^L(x_2)) \\ &= (A \circ B)^L \leq \bigvee_{x \in f^{-1}(y)} (A \circ B)^L(x) \\ &= f(A \circ B)^L(y) = \alpha\end{aligned}$$

By the similar arguments, we have that $A^U(x_1) \wedge B^U(x_2) \leq f(A \circ B)^U(y) = \alpha^{'}$. So $\beta - \epsilon < \alpha$ and $\beta^{'} - \epsilon < \alpha^{'}$. Since $\epsilon > 0$ is arbitrary, $\beta \leq \alpha$ and $\beta^{'} \leq \alpha^{'}$. Hence $[\alpha, \beta] = [\alpha^{'}, \beta^{'}]$. Therefore $f(A \circ B) = f(A) \circ f(B)$.

(c) Clearly, $A \cap B \subset A$ and $A \cap B \subset B$. By Result 2.B(d), $f(A \cap B) \subset f(B)$. So $f(A \cap B) \subset f(A) \cap f(B)$. Suppose B is IVF-invariant. Then clearly, $f^{-1}(f(B)) = B$. Let $y \in R^{'}$ and let $\epsilon > 0$ is arbitrary. Let $[\alpha, \beta] = (f(A) \cap f(B))(y)$ and let $[\alpha^{'}, \beta^{'}] = (f(A) \cap f(B))(y)$. Then

$\alpha = (f(A) \cap f(B))^L(y) = (\bigvee_{x \in f^{-1}(y)} A^L(x)) \wedge f(B)^L(y)$

and

$\beta = (f(A) \cap f(B))^U(y) = (\bigvee_{x \in f^{-1}(y)} A^U(x)) \wedge f(B)^U(y)$.

Thus $\alpha - \epsilon < (\bigvee_{x \in f^{-1}(y)} A^L(x)) \wedge f(B)^L(y)$ and $\beta - \epsilon < (\bigvee_{x \in f^{-1}(y)} A^U(x)) \wedge f(B)^U(y)$. So there exists an $x \in f^{-1}(y)$ such that

$\alpha - \epsilon < A^L(x), \alpha - \epsilon < f(B)^L(y)$

and

$\beta - \epsilon < A^U(x), \alpha - \epsilon < f(B)^U(y)$.

Since B is IVF-invariant,$f^{-1}(f(B)) = B$. Then

$f(B)^L(y) = f(B)^L(f(x)) = f^{-1}(f(B)^L)(x) = f^{-1}(f(B^L))(x) = B^L(x)$

and

$f(B)^U(y) = f(B)^U(f(x)) = f^{-1}(f(B)^U)(x) = f^{-1}(f(B^U))(x) = B^U(x)$.

Thus $\alpha - \epsilon < A^L(x), \alpha - \epsilon < B^L(x)$ and $\beta - \epsilon < A^U(x), \beta - \epsilon < B^U(x)$. So $\alpha - \epsilon < A^L(x) \wedge B^L(x) = (A \cap B)^L(x)$ and $\beta - \epsilon < A^U(x) \wedge B^U(x) = (A \cap B)^L(x)$.

Hence

$\alpha - \epsilon < \bigvee_{x \in f^{-1}(y)} (A \cap B)^L(x) = (f(A \cap B)^L)(y) = \alpha^{'}$

and

$\alpha - \epsilon < \bigvee_{x \in f^{-1}(y)} (A \cap B)^U(x) = (f(A \cap B)^U)(y) = \beta^{'}$. Since $\epsilon > 0$ is arbitrary, $\alpha \leq \alpha^{'}$ and $\beta \leq \beta^{'}$. Thus $f(A) \cap f(B) \subset f(A \cap B)$. Therefore $f(A) \cap f(B) = f(A \cap B)$. □

5. Interval-valued fuzzy cosets

Definition 5.1. Let A be any IVI of a ring R and let $x \in R$. Then $A_x \in D(I)^R$ is called the *interval-valued fuzzy coset* determined by x and A if $A_x(r) = A(r - x)$ for each $r \in R$.

Proposition 5.2. Let R be any IVI of a ring R and let R/A the set of all interval-valued fuzzy cosets of A in R. Then R/A is a ring under the following operations:

$A_x + A_y = A_{x+y}$ and $A_x A_y = A_{xy}$ for any $x, y \in R$.

Proof. For any $a, b, c, d \in R$, suppose $A_a = A_b$ and $A_c = A_d$. Then
$A(r-a) = A(r-b)$ for each $r \in R$ (5.1)
and
$A(r-c) = A(r-d)$ for each $r \in R$. (5.2)
Let $r = a+c-d$ in (5.1), $r = c$ in (5.2) and $r = a$ in (5.1). Then
$A(a+c-d-a) = A(a+c-d-b) = A(c-d)$,
$A(c-c) = A(c-d) = A(0)$ (5.3)
and
$A(a-a) = A(a-b) = A(0)$. (5.4) On the other hand,

$$\begin{aligned}(A_a + A_c)^L(r) &= A^L_{a+c}(r) = A^L(r-a-c) \\ &= A^L((r-b-d)-(a+c-b-d)) \\ &\geq A^L(r-b-d) \wedge A^L(a+c-b-d) \\ &= A^L(r-b-d) \wedge A^L(0) \text{ (By (5.3))} \\ &= A^L(r-b-d) \\ &= A^L_{b+d}(r) = (A_b + A_d)^L(r).\end{aligned}$$

By the similar arguments, we have that $(A_a + A_c)^U(r) = (A_b + A_d)^U(r)$. Thus $A_a + A_d \subset A_a + A_c$. Similarly, we have $A_a + A_c \subset A_b + A_d$. So $A_a + A_c = A_b + A_d$. Hence addition is well-defined. Also,

$$\begin{aligned}(A_aA_c)^L(r) &= A^L_{ac}(r) = A^L(r-ac) \\ &= A^L((r-bd)-(ac-bd)) \\ &\geq A^L(r-bd) \wedge A^L(ac-bd) \\ &= A^L(r-bd) \wedge A^L((a-b)c - b(d-c)) \text{ (By (5.3) and (5.4))} \\ &\geq A^L(r-bd) \wedge A^L(a-b)A^L(d-c)) \\ &= A^L(r-bd) \wedge A^L(0)A^L(0) \text{ (By (5.4) and (5.5))} \\ &= A^L(r-bd) = A^L_{bd}(r) = A^L_b A^L_d(r).\end{aligned}$$

By the similar arguments, we have that $(A_aA_c)^U(r) = (A_bA_d)^U(r)$. Thus $A_bA_d \subset A_aA_c$. Similarly, we have $A_aA_c \subset A_bA_d$. So $A_bA_d = A_aA_c$. Hence multiplication is well-defined. Clearly, $A_0 (= A)$ acts as the additive identity, A_e as the multiplicative identity (where e is the multiplicative identity of R) and A_{-x} as additive inverse of A_x. It is now a purely routine matter to verify the other properties. This completes the proof. □

Lemma 5.3. Let A be any IVR or an IVI of a ring R. If there exist $x, y \in R$ such that $A^L(x) < A^L(y)$ and $A^U(x) < A^U(y)$, then $A(x-y) = A(x) = A(y-x)$.

Proof. Since A is an IVG of R with respect to "+", by Result 4.A, $A(x-y) = A(y-x)$. Thus it is sufficient to show that $A(x-y) = A(x)$. Since $A^L(x) < A^L(y), A^U(x) < A^U(y)$ and A is an IVR or an IVI of R, $A^L(x-y) \geq A^L(x) \wedge A^L(y) = A^L(x)$ and $A^U(x-y) \geq A^U(x) \wedge A^U(y) = A^U(x)$. On the other hand, $A^L(x) = A^L(x-y+y) \geq A^L(x-y) \wedge A^L(y)$ and $A^U(x) = A^U(x-y+y) \geq A^U(x-y) \wedge A^U(y)$.Thus $A^L(x) \geq A^L(x-y)$ and $A^U(x) \geq A^U(x-y)$. So $A^L(x-y) = A(x)$. This completes the proof. □

Lemma 5.4. If A is any IVI of a ring R, then $A(x) = A(0)$ if and only if $A_x = A_0$, where $x \in R$.

Proof. $(\Rightarrow)$: Suppose $A(x) = A(0)$. Since A is an IVG of R with respect to "+", $A(r) \leq A(0) = A(r)$, i.e., $A^L(r) \leq A^L(0) = A^L(r)$ and $A^U(r) \leq A^U(0) = A^U(r)$ for each $r \in R$.

Case (i): Suppose $A(r) < A(x)$. Then, by Lemma 5.3, $A(r-x) = A(x)$. Thus $A_x(r) = A_0(r)$ for each $r \in R$.

Case (ii): Suppose $A(r) = A(x)$. Then $x, r \in A^{[\lambda,\mu]}$, where $[\lambda, \mu] = A(0)$. Since A is an IVG of R, $A^{[\lambda,\mu]}$ is a subgroup of R. Thus $x - r \in A^{[\lambda,\mu]}$. Thus $A^L(x-r) \leq \lambda = A^L(0)$ and $A^U(x-r) \geq \mu = A^U(0)$. Since $A^L(x-r) \leq A^L(0)$ and $A^U(x-r) \leq A^U(0)$,$A^L(x-r) = A^L(0)$ and $A^U(x-r) = A^U(0)$. Thus $A(x-r) = A(0) = A(x) = A(r)$, i.e.,$A_x(r) = A_0(r)$ for each $r \in R$. In either case, $A_x(r) = A_0(r)$ for each $r \in R$. Hence $A_x = A_0$ for each $r \in R$.

$(\Leftarrow)$: It is straightforward. □

Proposition 5.5. Let A be any IVI of a ring R and let $A(0) = [\lambda, \mu]$. Then $R/A^{[\lambda,\mu]} \cong R/A$.

Proof. Define a mapping $f : R \to R/A$ by $f(x) = A_x$ for each $r \in R$. Then it is easy to check that f is a ring epimorphism. By Lemma 5.4,

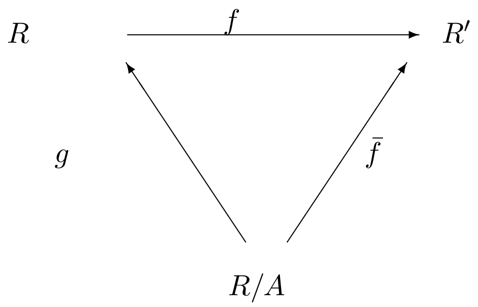
$$\begin{aligned} Ker f &= \{x \in R : f(x) = A_0\} = \{x \in R : A_x = A_0\} \\ &= \{x \in R : A(x) = A_0\} = A^{[\lambda,\mu]}. \end{aligned}$$

Hence $R/A^{[\lambda,\mu]} \cong R/A$. □

Proposition 5.6. Let $f : R \to R'$ be a ring epimorphism and let A be an IVI of R such that $A^{[\lambda,\mu]} \subset Kerf$, where $[\lambda, \mu] = A(0)$. Then there exists a unique epimorphism $\overline{f} : R/A \to R'$ such that $f = \bar{f} \circ g$, where $g(x) = A_x$ for each $r \in R$.

Proof. Define a mapping $\bar{f} : R_A \to R'$ by $\bar{f}(A_x) = f(x)$ for each $r \in R$. Suppose $A_x = A_y$. Then $A_{x-y} = A_0 = A_x = A_y$. By Lemma 5.4, $A(x-y) = A(x)$. Then $x - y \in A^{[\lambda,\mu]}$. Since $A^{[\lambda,\mu]} \subset Kerf$, $x - y \in Kerf$. Thus $f(x) = f(y)$, i.e., $\bar{f}(A_x) = \bar{f}(A_y)$. So $\bar{f}$ is well-defined. Furthermore, since f is surjective, $\bar{f}$ is also surjective. Moreover, it is easy to see that $\bar{f}$ is a homomorphism.

Consider the following diagram:



Let $x \in R$. Then $f(x) = \bar{f}(A_x) = \bar{f}(g(x)) = (\bar{f} \circ g)(x)$. Thus the above diagram commutes, i.e., $f = \bar{f} \circ g$.

Suppose there exists an epimorphism $h : R/A \to R'$ such that $f = h \circ g$. Let $x \in R$. Then $\bar{f}(A_x) = f(x) = (h \circ g)(x) = h(g(x)) = h(A_x)$. Thus $\bar{f} = h$. So $\bar{f}$ is unique. This completes the proof. □

Corollary 5.6. The induced homomorphism $\bar{f}$ is an isomorphism if and only if A is IVF-invariant.

Proof. ($\Rightarrow$) : Suppose $\bar{f}$ is an isomorphism, i.e., $\bar{f}$ is injective. For any $x, y \in R$, let $f(x) = f(y)$. Then $\bar{f}(A_x) = \bar{f}(A_y)$. Since $\bar{f}$ is injective, $A_x = A_y$. Thus $A_{x-y} = A_0$. By Lemma 5.4, $A(x-y) = A(0)$. By Proposition 4.7 in [5], $A(x) = A(y)$. So A is IVF-invariant.

($\Leftarrow$) : Suppose A is IVF-invariant and $\bar{f}(A_x) = \bar{f}(A_y)$. Then $f(x) = f(0)$. Since A is IVF-invariant, $A(x) = A(0)$. By Lemma 5.4, $A_x = A_0$. So $\bar{f}$ is injective. This completes the proof. □

Proposition 5.7. Let $f : R \to R^{'}$ be a ring epimorphism and let A be an IVF-invariant IVI of R. Then $R/A = R^{'}/f(A)$.

Proof. Since A is IVF-invariant, $Kerf \subset A^{[\lambda,\mu]}$, where $[\lambda, \mu] = A(0)$. Consider $f(A)(0^{'}) = [f(A^L)(0^{'}), f(A^U)(0^{'})]$, where $0^{'}$ denotes the additive identity in $R^{'}$. Then

$$f(A^L)(0') = \bigvee_{x \in f^{-1}(0')} A^L(x) \text{ and } f(A^U)(0^{'}) = \bigvee_{x \in f^{-1}(0')} A^U(x)$$

Since $f(0) = 0^{'}$ and $A(x) \le A(0)$, i.e., $A^L(x) \le A^L(0), A^U(x) \le A^U(0)$ for each $x \in R$, $A^L(x) = A^L(0)$ and $A^U(x) = A^U(0)$, i.e., $f(A)(0^{'}) = A(0) = [\lambda, \mu]$. Now,

$$\begin{aligned} f(x) \in [f(A)]^{[\lambda,\mu]} &\Leftrightarrow f(A)^L(f(X)) \ge \lambda \text{ and } f(A)^U(f(X)) \ge \mu \\ &\Leftrightarrow f(A^L)(f(x)) \ge \lambda \text{ and } f(A^U)(f(x)) \ge \mu \\ &\Leftrightarrow f^{-1}(f(A^L))(x) \ge \lambda \text{ and } f^{-1}(f(A^U))(x) \ge \mu \\ &\Leftrightarrow A^L(x) \ge \lambda \text{ and } A^U(x) \ge \mu \text{ (by Result 4.B)} \\ &\Leftrightarrow x \in A^{[\lambda,\mu]} \\ &\Leftrightarrow f(x) \in f(A^{[\lambda,\mu]}) \text{ (Since } Kerf \subset A^{[\lambda,\mu]}). \end{aligned}$$

So $[f(A)]^{\lambda,\mu]} = f(A^{[\lambda,\mu]})$. By Proposition 5.5, $R/A \cong R/A^{[\lambda,\mu]}$ and $R'/f(A) \cong R/[f(A)]^{[\lambda,\mu]}$. Hence $R/A \cong R'/f(A)$. This completes the proof. □

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