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A CONVERSE OF EULER'S THEOREM FOR POLYHEDRA

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Abstract. We give a converse of the well-known Euler's theorem for convex polyhedra.

Let $\mathbf{D}$ be the set of all convex polyhedra and $\Phi : \mathbf{D} \to N^3$ the map defined by $\Phi(S) = (v, e, f)$, where N denotes the set of all natural numbers, v, e, and f the number of vertices, edges, and faces of a convex polyhedron S, respectively. Then the well-known Euler's theorem for polyhedra states that the image $\mathrm{Im}\Phi$ of Φ is contained in the plane

$$\Pi = \{(v, e, f) \in N^3 | v - e + f = 2\}.$$

A number of proofs of this theorem are presented in [2]. A heuristic proof may be also found in [4]. For a brief history of the theorem, see [1].

Obviously, $\mathrm{Im}\Phi$ is a proper subset of the plane Π. Hence it is natural to ask the following:

"For what values of $(v, e, f) \in \Pi$ does there exist a convex polyhedron S with $\Phi(S) = (v, e, f)$?"

In this short note, we give a complete answer to this question. More precisely, we shall give the following:

Theorem. *The image of the map $\Phi : \mathbf{D} \to N^3$ is given by*

$$\mathrm{Im}\Phi = \{(v, e, f) \in N^3 | v - e + f = 2, 2e \geq 3f, f \geq 4\}.$$

First we give

Lemma 1. *Let v, e, and f denote the number of vertices, edges, and faces of a convex polyhedron S, respectively. Then we have*

$$2e \geq 3f, \tag{1}$$

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where the equality holds in (1) if and only if each face is a triangle.

Proof. We give a proof for completeness. Let f_n stand for the number of those faces that have precisely n sides, then we have

$$f = \sum_{n=3}^{\infty} f_n. \tag{2}$$

Furthermore, since every edge of the polyhedron is a side of exactly two faces, we also obtain

$$2e = \sum_{n=3}^{\infty} n f_n. \tag{3}$$

Then, (2) and (3) imply the following inequality:

$$2e = \sum_{n=3}^{\infty} n f_n \geq 3 \sum_{n=3}^{\infty} f_n = 3f, \tag{4}$$

which completes the proof of (1). The equality condition of (1) follows immediately from (4). □

Now we define three maps $g_i : N^3 \to N^3, i = 1, 2, 3$ as follows:

$$\begin{aligned} g_1(v, e, f) &= (v + 1, e + 1, f), \\ g_2(v, e, f) &= (v + 1, e + 2, f + 1), \\ g_3(v, e, f) &= (v + 1, e + 3, f + 2). \end{aligned} \tag{5}$$

Then we have

Lemma 2. *Each map* $g_i, i = 1, 2, 3$ *maps* $\mathrm{Im}\Phi$ *into* $\mathrm{Im}\Phi$.

Proof. For any $(v, e, f) \in \mathrm{Im}\Phi$, we consider a convex polyhedron S in $\mathbf{D}$ with $\Phi(S) = (v, e, f)$. Fix a convex n-gonal face $\sigma = P_1 P_2 \cdots P_n$ of the polyhedron S where $P_1, P_2, \cdots, P_n$ denote the consecutive vertices of the face σ. Then, without loss of generality, we may assume that the straight line $\overleftrightarrow{P_1P_2}$ does not pass through P_3. Choose an interior point P of the edge P_1P_2 and an interior point Q of the triangle $P_1P_2P_3$.

We now construct a polyhedron S_1 consisting of the vertices as P and all of vertices of S, the edges as P_1P, PP_2 and all edges of S other than P_1P_2 and the faces as all of faces of S. Then S_1 is a convex polyhedron with $\Phi(S_1) = (v + 1, e + 1, f)$. Next, we consider the polyhedron S_2 consisting of the vertices as P and other vertices of S, the edges as P_1P, PP_2, PP_3 and all edges of S other than P_1P_2 and the faces as $P_1PP_3 \cdots P_n, PP_2P_3$ and all of faces of S except σ. Then S_2 is a convex polyhedron with $\Phi(S_2) = (v+1, e+2, f+1)$. Finally, we construct the polyhedron S_3 which consists of the vertices as Q and the vertices of S, the edges as P_1Q, P_2Q, P_3Q and all of edges of S and the faces as $P_1QP_3 \cdots P_n, P_1P_2Q, P_2P_3Q$ and all of faces of S other than σ. We see that S_3 is a convex polyhedron with $\Phi(S_3) = (v+1, e+3, f+2)$.

From (5) we see that each map $g_i, i = 1, 2, 3$ satisfies $g_i(v, e, f) = \Phi(S_i)$, which completes the proof. $\Box$

We now prove our theorem. Suppose that $(v, e, f) \in N^3$ satisfies the following:

$$v - e + f = 2, \tag{6}$$

and

$$2e \geq 3f, \quad f \geq 4. \tag{7}$$

First note that for a tetrahedron Σ we have $\Phi(\Sigma) = (4, 6, 4)$.

Case 1. If $v \geq f$, then (6) shows that (v, e, f) satisfies the following:

$$(v, e, f) - (4, 6, 4) = (v - f)(1, 1, 0) + (f - 4)(1, 2, 1). \tag{8}$$

Let m and n denote the integers $v - f$ and $f - 4$, respectively. Then (7) implies that m and n are nonnegative integers. (8) shows that

$$(v, e, f) = g_2^n \circ g_1^m(4, 6, 4). \tag{9}$$

Case 2. If $v < f$, then (6) shows that (v, e, f) satisfies the following:

$$(v, e, f) - (4, 6, 4) = (2e - 3f)(1, 2, 1) + (f - v)(1, 3, 2). \tag{10}$$

Let m and n denote the integers $2e - 3f$ and $f - v$, respectively. Then (7) implies that m, n are nonnegative integers. (10) implies that

$$(v, e, f) = g_3^n \circ g_2^m(4, 6, 4). \tag{11}$$

Thus together with (9) and (11), Lemma 2 shows that $\Phi(S) = (v, e, f)$ for a convex polyhedron S, which can be constructed from a tetrahedron Σ. This together with Lemma 1 completes the proof of our theorem.

Remark. In [3], they prove as follows that Euler characteristic $\chi = v - e + f$ for polyhedra is the essentially unique topological invariant: Consider a map $g : \mathbf{D} \to R$ given by $g(S) = g(v, e, f)$, where $\Phi(S) = (v, e, f)$. Suppose that g is topologically invariant. Then g is a function of $\chi = v - e + f$.

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