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ON THE STABILITY OF THE PEXIDER EQUATION IN SCHWARTZ DISTRIBUTIONS VIA HEAT KERNEL

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Abstract. We consider the Hyers-Ulam-Rassias stability problem

$$\|u \circ A - v \circ P_1 - w \circ P_2\| \le \varepsilon(|x|^p + |y|^p)$$

for the Schwartz distributions u, v, w, which is a distributional version of the Pexider generalization of the Hyers-Ulam-Rassias stability problem

$$|f(x+y) - g(x) - h(y)| \le \varepsilon(|x|^p + |y|^p), \quad x, y \in \mathbb{R}^n,$$

for the functions $f, g, h : \mathbb{R}^n \to \mathbb{C}$.

1. Introduction

Generalizing the well known stability theorem of D. H. Hyers[12] which was motivated by S. M. Ulam[19], Th. M. Rassias[16] and Z. Gajda[9] showed the following stability theorem for the Cauchy equation:

Theorem 1.1. [16, 9] *Let f be a mapping from a normed linear space V to a Banach space X satisfying the inequality*

$$||f(x+y) - f(x) - f(y)|| \le \epsilon(\|x\|^p + \|y\|^p), \quad p \ne 1, \tag{1.1}$$

for all $x,\ y \in V$ ($x \ne 0$ and $y \ne 0$ if $p < 0$). Then there exists a unique function $g : V \to X$ satisfying

$$g(x+y) - g(x) - g(y) = 0$$

such that

$$||f(x) - g(x)|| \le \frac{2\varepsilon}{|2^p - 2|}\|x\|^p$$

for all $x \in V$ ($x \ne 0$ if $p < 0$).

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The result was further generalized by Y. H. Lee and K. W. Jun[13] for the Hyers-Ulam-Rassias stability theorem for the Pexider equation:

Theorem 1.2. [13] *Let f, g, h be mappings from a normed linear space V to a Banach space X satisfying the inequality*

$$||f(x+y) - g(x) - h(y)|| \le \epsilon(\|x\|^p + \|y\|^p), \quad p \ne 1, \tag{1.2}$$

for all $x,\ y \in V \setminus \{0\}$. Then there exists a unique function $g : V \to X$ satisfying

$$g(x+y) - g(x) - g(y) = 0$$

such that

$$||f(x) - g(x)|| \le \frac{4\epsilon(3+3^p)}{2^p|3-3^p|}\|x\|^p$$

for all $x \in X \setminus \{0\}$.

In this paper, we consider the above stability theorems in the spaces of generalized functions such as the spaces $\mathcal{S}'$ and $\mathcal{D}'$ of tempered distributions and distributions of L. Schwartz for even integers $p \ge 2$. Note that the above inequalities (1.2) makes no sense if f is a tempered distributions or distribution. Making use of the pullbacks of generalized function we extend the inequality (1.2) to distributions u, v, w as follows:

$$\|u \circ A - v \circ P_1 - w \circ P_2\| \le \varepsilon(|x|^p + |y|^p) \tag{1.3}$$

where $A(x, y) = x + y$, $P_1(x, y) = x$, $P_2(x, y) = y$, $x, y \in \mathbb{R}^n$, and $u \circ A$, $v \circ P_1$ and $w \circ P_2$ are the pullbacks of u, v, w by A, P_1 and P_2, respectively. Also $|\cdot|$ denotes the Euclidean norm and the inequality $\|\cdot\| \le \psi(x, y)$ in (1.3) means that $|\langle \cdot, \varphi\rangle| \le \|\psi\varphi\|_{L^1}$ for all test functions $\varphi(x, y)$ defined on $\mathbb{R}^{2n}$.

As the main result, we prove the following: Let $u, v, w \in \mathcal{D}'$ satisfy the inequality (1.3) for some even integer $p \ge 2$. Then, for $p > 2$, there exist a unique $a \in \mathbb{C}^n$ and complex constants c_1, c_2, c_3 such that

$$\begin{aligned}
\|u - a\cdot x - c_1\| &\le \frac{4\epsilon}{2^p - 2}|x|^p,\\
\|v - a\cdot x - c_2\| &\le \frac{\epsilon(2^p + 2)}{2^p - 2}|x|^p,\\
\|w - a\cdot x - c_2\| &\le \frac{\epsilon(2^p + 2)}{2^p - 2}|x|^p,
\end{aligned}$$

and for $p = 2$, there exist a unique $a \in \mathbb{C}^n$ and complex constants c_1, c_2, c_3 such that

$$\|u - a \cdot x - c_1\| \le 10\epsilon |x|^2,$$
$$\|v - a \cdot x - c_2\| \le 11\epsilon |x|^2,$$
$$\|w - a \cdot x - c_3\| \le 11\epsilon |x|^2.$$

2. Schwartz distributions

We briefly introduce the space $\mathcal{D}'(\mathbb{R}^n)$ of distributions and the space $\mathcal{S}'(\mathbb{R}^n)$ of tempered distributions. Here we use the multi-index notations, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\alpha! = \alpha_1! \cdots \alpha_n!$, $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$, for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$, where $\mathbb{N}_0$ is the set of non-negative integers and $\partial_j = \frac{\partial}{\partial x_j}$. We also denote by $C_c^\infty(\mathbb{R}^n)$ the set of all infinitely differentiable functions on $\mathbb{R}^n$ with compact supports.

Definition 2.1. *A distribution u is a linear form on $C_c^\infty(\mathbb{R}^n)$ such that for every compact set $K \subset \mathbb{R}^n$ there exist constants $C > 0$ and $k \in \mathbb{N}_0$ such that*

$$|\langle u, \varphi \rangle| \le C \sum_{|\alpha| \le k} \sup |\partial^\alpha \varphi|$$

for all $\varphi \in C_c^\infty(\mathbb{R}^n)$ with supports contained in K. The set of all distributions is denoted by $\mathcal{D}'(\mathbb{R}^n)$.

Definition 2.2. *We denote by $\mathcal{S}$ or $\mathcal{S}(\mathbb{R}^n)$ the Schwartz space of all infinitely differentiable functions φ in $\mathbb{R}^n$ such that*

$$\|\varphi\|_{\alpha,\beta} = \sup_x |x^\alpha \partial^\beta \varphi(x)| < \infty \tag{2.1}$$

for all α, $\beta \in \mathbb{N}_0^n$, equipped with the topology defined by the seminorms $\|\cdot\|_{\alpha,\beta}$. The elements of $\mathcal{S}$ are called rapidly decreasing functions and the elements of the dual space $\mathcal{S}'$ are called tempered distributions.

We denote by Ω_j open subsets of $\mathbb{R}^{n_j}$ for $j = 1, 2$, with $n_1 \ge n_2$.

Definition 2.3. *Let $u_j \in \mathcal{D}'(\Omega_j)$ and $\lambda : \Omega_1 \to \Omega_2$ a smooth function such that for each $x \in \Omega_1$ the derivative $\lambda'(x)$ is surjective, that is, the Jacobian matrix $\nabla\lambda$ of λ has rank n_2. Then there exists a unique continuous linear map $\lambda^* : \mathcal{D}'(\Omega_2) \to \mathcal{D}'(\Omega_1)$ such that $\Lambda^* u = u \circ \lambda$ when u is a continuous function. We call $\lambda^* u$ the pullback of u by λ and often denoted by $u \circ \lambda$.*

In particular if λ is a diffeomorphism (a bijection with λ, λ^{-1} smooth functions) the pullback $u \circ \lambda$ can be written as follows:

$$\langle u \circ \lambda, \varphi \rangle = \langle u, (\varphi \circ \lambda^{-1})(x) |(\nabla \lambda^{-1}(x)| \rangle. \tag{2.2}$$

As a matter of fact, the pullbacks $u \circ A, u \circ P_1$, $u \circ P_2$ can be written in a transparent way as

$$\langle u \circ A\,,\, \varphi(x,y)\rangle = \langle u\,, \int \varphi(x-y,y)\,dy\rangle, \tag{2.3}$$

$$\langle u \circ P_1\,,\, \varphi(x,y)\rangle = \langle u\,, \int \varphi(x,y)\,dy\rangle, \tag{2.4}$$

$$\langle u \circ P_2\,,\, \varphi(x,y)\rangle = \langle u\,, \int \varphi(x,y)\,dx\rangle \tag{2.5}$$

for all test functions $\varphi \in \mathcal{S}(\mathbb{R}^{2n})$.

For more details of distributions we refer the reader to [11, 17].

3. Stability in $\mathcal{S}'$

We consider the inequality (1.3) in the space $\mathcal{S}'$ of Schwartz tempered distributions. We employ the n-dimensional heat kernel $E_t(x)$ given by

$$E_t(x) = (4\pi t)^{-n/2} \exp(-|x|^2/4t),\ x \in \mathbb{R}^n,\ t > 0. \tag{3.1}$$

It is easy to see that the heat kernel $E_t(\cdot)$ belongs to the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ for each $t > 0$. Let $u \in \mathcal{S}'$. Then its *Gauss transform*

$$\tilde{u}(x,t) = (u * E_t)(x) = \langle u_y, E_t(x-y)\rangle, \quad x \in \mathbb{R}^n,\ t > 0,$$

is well defined. As a matter of fact the following result holds[10]:

Lemma 3.1. [14] *Let $u \in \mathcal{S}'(\mathbb{R}^n)$. Then its Gauss transform $\tilde{u}(x,t)$ is a C^∞–solution of the heat equation satisfying:*

(i) *There exist positive constants C, M, N and δ such that*

$$|\tilde{u}(x,t)| \le Ct^{-M}(1+|x|)^N \quad in \quad \mathbb{R}^n \times (0,\, \delta), \tag{3.2}$$

(ii) *$\tilde{u}(x,t) \to u$ as $t \to 0^+$ in the sense that for every $\varphi \in \mathcal{S}$,*

$$\langle u, \varphi\rangle = \lim_{t\to 0^+} \int \tilde{u}(x,t)\varphi(x)\,dx.$$

Conversely, every C^∞–solution $U(x,t)$ of the heat equation satisfying the estimate (3.2) can be uniquely expressed as $U(x,t) = \tilde{u}(x,t)$ for some $u \in \mathcal{S}'$.

We refer the reader to ([11], chapter VI) for pullbacks of distributions and to [10, 13] for more details of distributions and tempered distributions.

It is well known that the *weak semigroup property* of the heat kernel

$$(E_t * E_s)(x) = E_{t+s}(x) \tag{3.3}$$

holds for convolution. This semigroup property will be very useful later.

Throughout the paper, we denote by $H_{2\gamma}$ the heat polynomial of degree $2\gamma \in \mathbb{N}_0^n$ which is given by

$$H_{2\gamma}(x,t) = [\xi^{2\gamma} * E_t(\xi)](x) = (2\gamma)! \sum_{0\le\alpha\le\gamma} \frac{t^{|\alpha|}x^{2\gamma-2\alpha}}{\alpha!(2\gamma-2\alpha)!}. \tag{3.4}$$

Note that if $|\gamma| = 1$ we have

$$H_{2\gamma}(x,t) = x^{2\gamma} + 2t = x_j^2 + 2t$$

where j-th coordinate of γ equals 1, and for $|\gamma| = 1, 2, \ldots$

$$H_{2\gamma}(x,0) = x^{2\gamma}, \quad H_{2\gamma}(0,t) = \frac{(2\gamma)!t^{|\gamma|}}{\gamma!}.$$

We first prove the following stability theorem.

Lemma 3.2. *Let f, g, $h : \mathbb{R}^n \times (0,\infty) \to \mathbb{C}$ be continuous functions satisfying the inequality*

$$|f(x+y,t+s) - g(x,t) - h(y,s)| \le \epsilon(H_{2\gamma}(x,t) + H_{2\gamma}(y,s)) \tag{3.5}$$

for all $x, y \in \mathbb{R}^n$, $t, s > 0$ and $|\gamma| \ge 1$. Then, for $|\gamma| > 1$, there exist a unique $a \in \mathbb{C}^n$, a unique $b \in \mathbb{C}$ and complex constants c_1, c_2 and c_3 such that

$$\begin{aligned} |f(x,t) - a\cdot x - bt - c_1| &\le \epsilon\psi_{1,\gamma}(x,t), \\ |g(x,t) - a\cdot x - bt - c_2| &\le \epsilon\psi_{2,\gamma}(x,t), \\ |h(x,t) - a\cdot x - bt - c_3| &\le \epsilon\psi_{2,\gamma}(x,t), \end{aligned}$$

for all $x \in \mathbb{R}^n$, $t > 0$, where

$$\psi_{1,\gamma}(x,t) = (2\gamma)! \sum_{0\le\alpha\le\gamma} \frac{2^{|\alpha|+2}t^{|\alpha|}x^{2\gamma-2\alpha}}{(2^{|2\gamma|} - 2^{|\alpha|+1})\alpha!(2\gamma-2\alpha)!},$$

$$\psi_{2,\gamma}(x,t) = (2\gamma)! \sum_{0\le\alpha\le\gamma} \frac{(2^{|2\gamma|} + 2^{|\alpha|+1})t^{|\alpha|}x^{2\gamma-2\alpha}}{(2^{|2\gamma|} - 2^{|\alpha|+1})\alpha!(2\gamma-2\alpha)!},$$

and for $|\gamma| = 1$, *there exist a unique* $a \in \mathbb{C}^n$, c_1, $c_2, c_3 \in \mathbb{C}$ *and* $r_1, r_2 : (0,\infty) \to [0,\infty)$ *with* $r_1(t)$, $r_2(t) \to 0$ *as* $t \to 0^+$ *such that*

$$|f(x,t) - a\cdot x - c_1| \le 10\epsilon x^{2\gamma} + r_1(t),$$
$$|g(x,t) - a\cdot x - c_2| \le 11\epsilon x^{2\gamma} + r_2(t),$$
$$|h(x,t) - a\cdot x - c_3| \le 11\epsilon x^{2\gamma} + r_2(t),$$

for all $x \in \mathbb{R}^n$, $t > 0$.

Proof. Let $x = y = 0$ in (3.5). Then by the triangle inequality we have

$$|g(0,t)| \le \epsilon\frac{(2\gamma)!}{\gamma!}\left(t^{|\gamma|} + s^{|\gamma|}\right) + |f(0,t+s) - h(0,s)|, \tag{3.6}$$

$$|h(0,s)| \le \epsilon\frac{(2\gamma)!}{\gamma!}\left(t^{|\gamma|} + s^{|\gamma|}\right) + |f(0,t+s) - g(0,t)|, \tag{3.7}$$

for all t, $s > 0$. Thus it follows from (3.6), (3.7) and the continuity of f

$$c_2 := \limsup_{t\to 0^+} g(0,t), \quad c_3 := \limsup_{s\to 0^+} h(0,s)$$

exist. Choose a sequence s_n, $n = 1, 2, \ldots$, of positive numbers which tends to 0 as $n \to \infty$ such that $h(0, s_n) \to c_3$ as $n \to \infty$. Putting $y = 0$, $s = s_n$ and letting $n \to \infty$ we have

$$|f(x,t) - g(x,t) - c_3| \le \epsilon H_{2\gamma}(x,t) \tag{3.8}$$

for all $x \in \mathbb{R}^n$, $t > 0$. Similarly we have

$$|f(y,s) - h(y,s) - c_2| \le \epsilon H_{2\gamma}(y,s) \tag{3.9}$$

for all $y \in \mathbb{R}^n$, $s > 0$. From (3.5), (3.8), (3.9) and the triangle inequality we have

$$|F(x+y,t+s) - F(x,t) - F(y,s)| \le 2\epsilon(H_{2\gamma}(x,t) + H_{2\gamma}(y,s)) \tag{3.10}$$

for all $x, y \in \mathbb{R}^n$, $t, s > 0$, where $F(x,t) = f(x,t) - c_2 - c_3$.

We first prove for $|\gamma| > 1$. For this case, we can follow the same approach as in [16, 9]. Indeed, replacing both x and y by $\frac{x}{2}$, both t and s by $\frac{t}{2}$ in (3.10) we have

$$|F(x,t) - 2F(2^{-1}x, 2^{-1}t)| \le 4\epsilon\, H_{2\gamma}(2^{-1}x, 2^{-1}t)$$

for all $x \in \mathbb{R}^n,\ t > 0$. Making use of the induction argument and triangle inequality we have

$$\begin{aligned}(3.11)\quad |F(x,t) - 2^m F(2^{-m}x, 2^{-m}t)| &\le 2\epsilon \sum_{j=1}^{m} 2^j H_{2\gamma}(2^{-j}x, 2^{-j}t)\\ &\le 2\epsilon\,(2\gamma)! \sum_{0\le\alpha\le\gamma} a_{m,\alpha} \frac{t^{|\alpha|} x^{2\gamma-2\alpha}}{\alpha!(2\gamma-2\alpha)!}\end{aligned}$$

for all $n \in \mathbb{N},\ x \in \mathbb{R}^n,\ t > 0$, where $a_{m,\alpha} = 2^{|\alpha|+1}(1-2^{(|\alpha|-|2\gamma|+1)m})/(2^{|2\gamma|}-2^{|\alpha|+1})$.

Replacing $x,\ t$ by $2^{-m}x,\ 2^{-m}t$, respectively in (3.11) and multiplying 2^m in the result it follows from $|\gamma| > 1$ that

$$A_m(x,t) := 2^m F(2^{-m}x, 2^{-m}t)$$

is a Cauchy sequence which converges locally uniformly. Now let

$$A(x,t) = \lim_{m\to\infty} A_m(x,t).$$

Letting $n \to \infty$ in (3.11) we have

$$(3.12)\qquad |F(x,t) - A(x,t)| \le 2\epsilon\,(2\gamma)! \sum_{0\le\alpha\le\gamma} a_\alpha \frac{t^{|\alpha|} x^{2\gamma-2\alpha}}{\alpha!(2\gamma-2\alpha)!}$$

for all $x, y \in \mathbb{R}^n,\ t, s > 0$, where $a_\alpha = 2^{|\alpha|+1}/(2^{|2\gamma|} - 2^{|\alpha|+1})$.

Replacing $x,\ y,\ t,\ s$ by $2^{-m}x,\ 2^{-m}y,\ 2^{-m}t,\ 2^{-m}s$ in (3.10), respectively, multiplying 2^m and letting $m \to \infty$ it follows immediately from the fact $|\gamma| > 1$ that

$$(3.13)\qquad A(x+y, t+s) - A(x,t) - A(y,s) = 0$$

for all $x, y \in \mathbb{R}^n,\ t, s > 0$. To prove the uniqueness of $A(x,t)$, let $B(x,t)$ be another function satisfying (3.12) and (3.13). Then it follows from (3.12), (3.13) and the triangle inequality that for all $n \in \mathbb{N}$,

$$(3.14)\qquad |A(x,t) - B(x,t)| \le n|A\left(\frac{x}{n}, \frac{t}{n}\right) - B\left(\frac{x}{n}, \frac{t}{n}\right)|$$

$$(3.15)\qquad \le 4\epsilon\,(2\gamma)!\, n^{1-|\gamma|} \sum_{0\le\alpha\le\gamma} a_\alpha \frac{t^{|\alpha|} x^{2\gamma-2\alpha}}{\alpha!(2\gamma-2\alpha)!}.$$

for all $x \in \mathbb{R}^n,\ t > 0$. Letting $n \to \infty$, we have $A(x,t) = B(x,t)$ for all $x \in \mathbb{R}^n,\ t > 0$. This proves the uniqueness.

Now it is well known that every continuous solution $A(x,t)$ of the Cauchy equation (3.13) has the form

$$A(x,t) = a \cdot x + bt$$

for some $a \in \mathbb{C}^n, b \in \mathbb{C}$. Thus we have

$$(3.16) \quad |f(x,t) - a \cdot x - bt - c_2 - c_3| \le 2\epsilon\,(2\gamma)! \sum_{0 \le \alpha \le \gamma} a_\alpha \frac{t^{|\alpha|} x^{2\gamma - 2\alpha}}{\alpha!(2\gamma - 2\alpha)!}$$

for all $x \in \mathbb{R}^n,\ t > 0$. Now it follows from (3.8),(3.9), (3.16) and the triangle inequality that

$$(3.17) \quad |g(x,t) - a \cdot x - bt - c_2| \le \epsilon(2\gamma)! \sum_{0 \le \alpha \le \gamma} \frac{(1 + 2a_\alpha) t^{|\alpha|} x^{2\gamma - 2\alpha}}{\alpha!(2\gamma - 2\alpha)!}$$

$$(3.18) \quad |h(x,t) - a \cdot x - bt - c_3| \le \epsilon(2\gamma)! \sum_{0 \le \alpha \le \gamma} \frac{(1 + 2a_\alpha) t^{|\alpha|} x^{2\gamma - 2\alpha}}{\alpha!(2\gamma - 2\alpha)!}.$$

for all $x \in \mathbb{R}^n,\ t > 0$, which gives the results for $|\gamma| > 1$.

We now prove for $|\gamma| = 1$. It follows from the inequality (3.10) and the continuity of F that

$$U(x) := \limsup_{t \to 0^+} F(x,t)$$

exists. From now on, we denote by

$$\Phi(x,y,t,s) := 2\epsilon(H_{2\gamma}(x,t) + H_{2\gamma}(y,s)).$$

In (3.10), letting $y = 0$ and $t \to 0^+$ so that $F(x,t) \to U(x)$ we have

$$(3.19) \quad |F(x,s) - U(x) - F(0,s)| \le \Phi(x,0,0,s).$$

From the inequality (3.10) and (3.19) we have

(3.20)

$$\begin{aligned}
|U(x+y) - U(x) - U(y)| \le & |F(x+y,t+s) - F(x,t) - F(y,s)| \\
& + |-F(x+y,t+s) + U(x+y) + F(0,t+s)| \\
& + |F(x,t) - U(x) - F(0,t)| \\
& + |F(y,s) - U(y) - F(0,s)| \\
& + |-F(0,t+s) + F(0,t) + F(0,s)| \\
\le & \Phi(x,y,t,s) + \Phi(x+y,0,0,t+s) \\
& + \Phi(x,0,0,t) + \Phi(y,0,0,s) + \Phi(0,0,t,s)
\end{aligned}$$

for all $x, y \in \mathbb{R}^n$, $t, s > 0$. Since the left hand side of (3.20) is independent of t and s we have

(3.21)
$$\begin{aligned}|U(x+y) - U(x) - U(y)| \le &\Phi(x, y, 0, 0) + \Phi(x+y, 0, 0, 0)\\ &+ \Phi(x, 0, 0, 0) + \Phi(y, 0, 0, 0) + \Phi(0, 0, 0, 0)\\ =&2\epsilon(2x^{2\gamma} + 2y^{2\gamma} + (x+y)^{2\gamma})\end{aligned}$$

for all $x, y \in \mathbb{R}^n$. Following the same approach as in [9, 10] we obtain that there exists a unique function $L : \mathbb{R}^n \to \mathbb{C}$ such that

$$L(x+y) - L(x) - L(y) = 0, \tag{3.22}$$

$$|U(x) - L(x)| \le 8\epsilon x^{2\gamma} \tag{3.23}$$

for all $x, y \in \mathbb{R}^n$. Also $L(x)$ is given by

$$L(x) = \lim_{m\to\infty} 2^m U(2^{-m}x) \tag{3.24}$$

locally uniformly. It follows from (3.24) and the continuity of $f(x, t)$ that L is continuous. Thus the solutions of (3.22) are given by $L(x) = a \cdot x$. From (3.19), (3.23) we have

(3.25)
$$|f(x, t) - a \cdot x - c_2 - c_3| \le 10\epsilon x^{2\gamma} + 4\epsilon t + |F(0, t)| := 10\epsilon x^{2\gamma} + r_1(t)$$

for all $x \in \mathbb{R}^n$, $t > 0$. Now from (3.8), (3.9) and (3.25) we have

(3.26)
$$|g(x, t) - a \cdot x - c_2| \le 11\epsilon x^{2\gamma} + 6\epsilon t + |F(0, t)| := 11\epsilon x^{2\gamma} + r_2(t),$$

(3.27)
$$|h(x, t) - a \cdot x - c_3| \le 11\epsilon x^{2\gamma} + 6\epsilon t + |F(0, t)| := 11\epsilon x^{2\gamma} + r_2(t)$$

for all $x \in \mathbb{R}^n$, $t > 0$. Now it remain to show that $\lim_{t\to 0^+} |F(0, t)| = 0$. Putting $x = y = 0$ in (3.10) and using the triangle inequality we have

$$|F(0, t)| \le |F(0, t+s) - F(0, s)| + 4\epsilon(t+s) \tag{3.28}$$

for all $t, s > 0$. By the continuity of F we have

$$\limsup_{t\to 0^+} |F(0, t)| \le 4\epsilon s$$

for all $s > 0$, which implies that $\lim_{t\to 0^+} |F(0, t)| = 0$. This completes the proof. □

Now, for $p = 1, 2, \ldots$, we denote by

$$\mathcal{H}_{2p}(x, t) = [|\xi|^{2p} * E_t(\xi)](x, t).$$

Since $|x|^{2p} = \sum_{|\gamma|=p} \frac{p!}{\gamma!} x^{2\gamma}$ we have

$$\mathcal{H}_{2p}(x,t) = \sum_{|\gamma|=p} \frac{p!}{\gamma!} H_{2\gamma}(x,t).$$

Now, in view of the proof of Lemma 3.2 we also obtain the following.

Lemma 3.3. *Let f, g, $h : \mathbb{R}^n \times (0,\infty) \to \mathbb{C}$ be continuous functions satisfying the inequality*

$$|f(x+y,t+s) - g(x,t) - h(y,s)| \le \epsilon(\mathcal{H}_{2p}(x,t) + \mathcal{H}_{2p}(y,s)) \tag{3.29}$$

for all $x, y \in \mathbb{R}^n$, $t, s > 0$. Then, for $p \ge 2$, there exist a unique $a \in \mathbb{C}^n$, a unique $b \in \mathbb{C}$ and complex constants c_1, c_2 and c_3 such that

$$|f(x,t) - a\cdot x - bt - c_1| \le \epsilon \sum_{|\gamma|=p} \frac{p!}{\gamma!}\psi_{1,\gamma}(x,t),$$
$$|g(x,t) - a\cdot x - bt - c_2| \le \epsilon \sum_{|\gamma|=p} \frac{p!}{\gamma!}\psi_{2,\gamma}(x,t),$$
$$|h(x,t) - a\cdot x - bt - c_3| \le \epsilon \sum_{|\gamma|=p} \frac{p!}{\gamma!}\psi_{2,\gamma}(x,t),$$

where $\psi_{1,\gamma}, \psi_{2,\gamma}$ are given in Lemma 3.2, and for $p = 1$, there exist a unique $a \in \mathbb{C}^n$, c_1, $c_2, c_3 \in \mathbb{C}$ and $s_1, s_2 : (0,\infty) \to [0,\infty)$ with $s_1(t)$, $s_2(t) \to 0$ as $t \to 0^+$ such that

$$|f(x,t) - a\cdot x - c_1| \le 10\epsilon|x|^2 + s_1(t),$$
$$|g(x,t) - a\cdot x - c_2| \le 11\epsilon|x|^2 + s_2(t),$$
$$|h(x,t) - a\cdot x - c_3| \le 11\epsilon|x|^2 + s_2(t),$$

for all $x \in \mathbb{R}^n$, $t > 0$.

Theorem 3.4. *Let $u, v, w \in \mathcal{S}'$ satisfy the inequality*

$$\|u \circ A - v \circ P_1 - w \circ P_2\| \le \epsilon(x^{2\gamma} + y^{2\gamma}) \tag{3.30}$$

for some $|\gamma| \geq 1$. *Then for* $|\gamma| \geq 2$, *there exist a unique* $a \in \mathbb{C}^n$ *and complex constants* c_1, c_2, c_3 *such that*

$$\begin{aligned}
\|u - a \cdot x - c_1\| &\leq \frac{4\epsilon}{4^{|\gamma|} - 2} x^{2\gamma}, \\
\|v - a \cdot x - c_2\| &\leq \frac{\epsilon(4^{|\gamma|} + 2)}{4^{|\gamma|} - 2} x^{2\gamma}, \\
\|w - a \cdot x - c_2\| &\leq \frac{\epsilon(4^{|\gamma|} + 2)}{4^{|\gamma|} - 2} x^{2\gamma},
\end{aligned}$$

and for $|\gamma| = 1$, *there exist a unique* $a \in \mathbb{C}^n$ *and complex constants* c_1, c_2, c_3 *such that*

$$\begin{aligned}
\|u - a \cdot x - c_1\| &\leq 10\epsilon x^{2\gamma}, \\
\|v - a \cdot x - c_2\| &\leq 11\epsilon x^{2\gamma}, \\
\|w - a \cdot x - c_3\| &\leq 11\epsilon x^{2\gamma}.
\end{aligned}$$

Proof. Convolving in each side of (3.30) the tensor product $E_t(x)E_s(y)$ of n-dimensional heat kernels we have in view of (2.3), (2.4), (2.5) and the semigroup property (3.3),

$$\begin{aligned}
[(u \circ A) * (E_t(\xi)E_s(\eta))](x,\, y) &= \langle u_\xi,\, \int E_t(x - \xi + \eta)E_s(y - \eta)\, d\eta \rangle \\
&= \langle u_\xi,\, (E_t * E_s)(x + y - \xi) \rangle \\
&= \tilde{u}(x + y, t + s).
\end{aligned}$$

Similarly we have

$$\begin{aligned}
&[(v \circ P_1) * (E_t(\xi)E_s(\eta))](x,\, y) = \tilde{v}(x, t), \\
&[(w \circ P_2) * (E_t(\xi)E_s(\eta))](x,\, y) = \tilde{w}(y, s),
\end{aligned}$$

where $\tilde{u}(x,t), \tilde{v}(x,t), \tilde{w}(x,t)$ are the Gauss transform of u, v, w, respectively.

Thus the inequality (3.30) is converted to the stability problem of quadratic–additive type functional equation:

$$|\tilde{u}(x + y, t + s) - \tilde{v}(x, t) - \tilde{w}(y, s)| \leq \epsilon(H_{2\gamma}(x, t) + H_{2\gamma}(y, s))$$

for $x,\, y \in \mathbb{R}^n$, $t, s > 0$.

By Lemma 3.2 for $|\gamma| > 1$, there exist a unique $a \in \mathbb{C}^n$, a unique $b \in \mathbb{C}$ and complex constants c_1, c_2 and c_3 such that

$$|\tilde{u}(x,t) - a \cdot x - bt - c_1| \le \epsilon\psi_{1,\gamma}(x,t), \tag{3.31}$$

$$|\tilde{v}(x,t) - a \cdot x - bt - c_2| \le \epsilon\psi_{2,\gamma}(x,t), \tag{3.32}$$

$$|\tilde{w}(x,t) - a \cdot x - bt - c_3| \le \epsilon\psi_{2,\gamma}(x,t). \tag{3.33}$$

Multiplying the test functions $\varphi \in \mathcal{S}$ in (3.31), (3.32) and (3.33), integrating the result and letting $t \to 0^+$ we get the result for $|\gamma| \ge 2$.

Using Lemma 3.2 for $|\gamma| = 1$, there exist a unique $a \in \mathbb{C}^n$, c_1, $c_2, c_3 \in \mathbb{C}$ and $r_1, r_2 : (0,\infty) \to [0,\infty)$ with $r_1(t)$, $r_2(t) \to 0$ as $t \to 0^+$ such that

$$|\tilde{u}(x,t) - a \cdot x - c_1| \le 10\epsilon x^{2\gamma} + r_1(t),$$
$$|\tilde{v}(x,t) - a \cdot x - c_2| \le 11\epsilon x^{2\gamma} + r_2(t),$$
$$|\tilde{w}(x,t) - a \cdot x - c_3| \le 11\epsilon x^{2\gamma} + r_2(t),$$

for all $x \in \mathbb{R}^n$, $t > 0$. Similarly as in the proof for $|\gamma| > 1$, letting $t \to 0^+$ in the above inequalities we get the results for $|\gamma| = 1$. This completes the proof.

□

Theorem 3.5. *Let $u, v, w \in \mathcal{S}'$ satisfy the inequality*

$$\|u \circ A - v \circ P_1 - w \circ P_2\| \le \epsilon(|x|^{2p} + |y|^{2p}) \tag{3.34}$$

for some integer $p \ge 1$. Then, for $p \ge 2$, there exist a unique $a \in \mathbb{C}^n$ and complex constants c_1, c_2, c_3 such that

$$\|u - a \cdot x - c_1\| \le \frac{4\epsilon}{4^p - 2}|x|^{2p},$$
$$\|v - a \cdot x - c_2\| \le \frac{\epsilon(4^p + 2)}{4^p - 2}|x|^{2p},$$
$$\|w - a \cdot x - c_2\| \le \frac{\epsilon(4^p + 2)}{4^p - 2}|x|^{2p},$$

and for $p = 1$, there exist a unique $a \in \mathbb{C}^n$ and complex constants c_1, c_2, c_3 such that

$$\|u - a \cdot x - c_1\| \le 10\epsilon|x|^2,$$
$$\|v - a \cdot x - c_2\| \le 11\epsilon|x|^2,$$
$$\|w - a \cdot x - c_3\| \le 11\epsilon|x|^2.$$

Proof. Convolving in each side of (3.34) the tensor product $E_t(x)E_s(y)$ of n-dimensional heat kernels as a function of x, y the inequality (3.34) is converted to the following inequality

$$|\tilde{u}(x+y,t+s)-\tilde{v}(x,t)-\tilde{w}(y,s)|\le \epsilon(\mathcal{H}_{2p}(x,t)+\mathcal{H}_{2p}(y,s))$$

for all $x,y\in\mathbb{R}^n$, $t,s>0$.

By Lemma 3.3 for $p\ge 2$, there exist a unique $a\in\mathbb{C}^n$, a unique $b\in\mathbb{C}$ and complex constants c_1, c_2 and c_3 such that

$$|\tilde{u}(x,t)-a\cdot x-bt-c_1|\le \epsilon\sum_{|\gamma|=p}\frac{p!}{\gamma!}\psi_{1,\gamma}(x,t), \tag{3.35}$$

$$|\tilde{v}(x,t)-a\cdot x-bt-c_2|\le \epsilon\sum_{|\gamma|=p}\frac{p!}{\gamma!}\psi_{2,\gamma}(x,t), \tag{3.36}$$

$$|\tilde{w}(x,t)-a\cdot x-bt-c_3|\le \epsilon\sum_{|\gamma|=p}\frac{p!}{\gamma!}\psi_{2,\gamma}(x,t). \tag{3.37}$$

Letting $t\to 0^+$ in (3.35)$\sim$ (3.37) we have

$$\|u-a\cdot x-c_1\|\le\sum_{|\gamma|=p}\frac{p!}{\gamma!}\left(\frac{4\epsilon}{4^{|\gamma|}-2}x^{2\gamma}\right)=\frac{4\epsilon}{4^p-2}|x|^{2p},$$

$$\|v-a\cdot x-c_2\|\le\sum_{|\gamma|=p}\frac{p!}{\gamma!}\left(\frac{\epsilon(4^{|\gamma|}+2)}{4^{|\gamma|}-2}x^{2\gamma}\right)=\frac{\epsilon(4^p+2)}{4^p-2}|x|^{2p},$$

$$\|w-a\cdot x-c_2\|\le\sum_{|\gamma|=p}\frac{p!}{\gamma!}\left(\frac{\epsilon(4^{|\gamma|}+2)}{4^{|\gamma|}-2}x^{2\gamma}\right)=\frac{\epsilon(4^p+2)}{4^p-2}|x|^{2p}.$$

Finally, by Lemma 3.3 for $p=1$, there exist a unique $a\in\mathbb{C}^n$, c_1, $c_2,c_3\in\mathbb{C}$ and $s_1,s_2:(0,\infty)\to[0,\infty)$ with $s_1(t)$, $s_2(t)\to 0$ as $t\to 0^+$ such that

$$|\tilde{u}(x,t)-a\cdot x-c_1|\le 10\epsilon|x|^2+s_1(t), \tag{3.38}$$

$$|\tilde{v}(x,t)-a\cdot x-c_2|\le 11\epsilon|x|^2+s_2(t), \tag{3.39}$$

$$|\tilde{w}(x,t)-a\cdot x-c_3|\le 11\epsilon|x|^2+s_2(t), \tag{3.40}$$

for all $x\in\mathbb{R}^n$, $t>0$. Letting $t\to 0^+$ in (3.38)$\sim$ (3.40) we have the result for $p=1$. This completes the proof.

□

4. Stability in $\mathcal{D}'$

In this section, we prove that all the previous results hold for the case of distributions. It is well known that the following topological inclusions hold:

$$C_c^\infty \hookrightarrow \mathcal{S}, \qquad \mathcal{S}' \hookrightarrow \mathcal{D}'.$$

We denote by $\delta(x)$ the function on $\mathbb{R}^n$,

$$\delta(x) = \begin{cases} A\exp\left(-\frac{1}{\sqrt{1-|x|^2}}\right), & |x| < 1 \\ 0, & |x| \geq 1, \end{cases}$$

where

$$A = \left(\int_{|x|<1} \exp\left(-\frac{1}{\sqrt{1-|x|^2}}\right) dx\right)^{-1}.$$

It is easy to see that $\delta(x)$ an infinitely differentiable function with support $\{x : |x| \leq 1\}$. In the space of distributions the function $\delta_t(x) := t^{-n}\delta(x/t)$, $t > 0$, acts a similar role as the heat kernel $E_t(x)$ employed in the space of tempered distributions. To prove the previous results in the space of distributions it suffices to show the following.

Theorem 4.1. *Let $u, v, w \in \mathcal{D}'$ satisfy the inequality*

$$\|u \circ A - v \circ P_1 - w \circ P_2\| \leq \epsilon(|x|^{2p} + |y|^{2p}) \tag{4.1}$$

for some integer $p \geq 1$. Then $u, v, w \in \mathcal{S}'$.

Proof. We denote by

$$\Psi(x, y, t, s) = \epsilon(|\xi|^{2p} * \delta_t(\xi))(x) + \epsilon(|\eta|^{2p} * \delta_s(\eta))(y).$$

Convolving $\delta_t(x)\delta_s(y)$ in each side of (4.1) the inequality (4.1) is converted to the following stability problem

$$|(u * \delta_t * \delta_s)(x + y) - (v * \delta_t)(x) - (w * \delta_s)(y)| \leq \Psi(x, y, t, s) \tag{4.2}$$

for x, $y \in \mathbb{R}^n$, $t, s > 0$. From (4.2) it is easy to see that

$$g(x) := \limsup_{t\to 0^+} (v * \delta_t)(x),$$

$$h(x) := \limsup_{t\to 0^+} (w * \delta_t)(x)$$

exist. In (4.2), letting $y = 0$ and $s \to 0^+$ so that $(w * \delta_s)(0) \to h(0)$ we have

$$|(u * \delta_t)(x) - (v * \delta_t)(x) - h(0)| \leq \Psi(x, 0, t, 0^+). \tag{4.3}$$

Similarly we have

$$|(u*\delta_s)(y)-(w*\delta_s)(y)-g(0)|\le\Psi(0,y,0^+,s). \tag{4.4}$$

From (4.2), (4.3) and (4.4) we have

$$\begin{aligned}|(u*\delta_t*\delta_s)(x+y)-(u*\delta_t)(x)-(u*\delta_s)(y)+g(0)+h(0)| &\le \Psi(x,y,t,s)\\ &+\Psi(x,0,t,0^+)\\ &+\Psi(0,y,0^+,s).\end{aligned} \tag{4.5}$$

In (4.5), putting $y=0$ we have

$$\begin{aligned}|(u*\delta_t*\delta_s)(x)-(u*\delta_t)(x)-(u*\delta_s)(0)+g(0)+h(0)| &\le \Psi(x,0,t,s)\\ &+\Psi(x,0,t,0^+)\\ &+\Psi(0,0,0^+,s).\end{aligned} \tag{4.6}$$

It follows from (4.6) that

$$f(x):=\limsup_{t\to0^+}(u*\delta_t)(x)$$

exists. In (4.6), letting $t\to0^+$ so that $(u*\delta_t)(x)\to f(x)$ we have

$$\begin{aligned}|(u*\delta_s)(x)-f(x)-(u*\delta_s)(0)+g(0)+h(0)| &\le\Psi(x,0,0^+,s)\\ &+\Psi(x,0,0^+,0^+)\\ &+\Psi(0,0,0^+,s).\end{aligned} \tag{4.7}$$

Letting $s\to0^+$ in (4.7) so that $(u*\delta_s)(0)\to f(0)$ we have

$$\|u-f(x)-f(0)+g(0)+h(0)\|\le 2\epsilon|x|^{2p} \tag{4.8}$$

On the other hand, let

$$D(x,y,t,s)=(u*\delta_t*\delta_s)(x+y)-(u*\delta_t)(x)-(u*\delta_s)(y)+g(0)+h(0).$$

Then we have

(4.9)

$$
\begin{aligned}
|f(x+y)-f(x)-f(y)+g(0)+h(0)| \le\, & |D(x,y,t,s)| + |-D(x+y,0,t,s)\\
& + |-D(x+y,0,0^+,t)||\\
& + |D(x,0,0^+,t)| + |D(y,0,0^+,s)|\\
\le\, & \Psi(x,y,t,s) + \Psi(x,0,t,0^+) + \Psi(0,y,0^+,s)\\
& + \Psi(x+y,0,t,s) + \Psi(x+y,0,t,0^+)\\
& + \Psi(0,0,0^+,s) + \Psi(x+y,0,0^+,t)\\
& + \Psi(x+y,0,0^+,0^+) + \Psi(0,0,0^+,t)\\
& + \Psi(x,0,0^+,t) + \Psi(x,0,0^+,0^+)\\
& + \Psi(0,0,0^+,t) + \Psi(y,0,0^+,s)\\
& + \Psi(y,0,0^+,0^+) + \Psi(0,0,0^+,s)
\end{aligned}
$$

for all $x, y \in \mathbb{R}^n$, $t, s > 0$. Letting $t, s \to 0^+$ in the above inequality we have

$$(4.10)\quad |f(x+y)-f(x)-f(y)+g(0)+h(0)| \le 4\epsilon(|x|^{2p}+|y|^{2p}+|x+y|^{2p}).$$

By the results in [9, 10], there exists a unique function A satisfying

$$(4.11)\qquad A(x+y) = A(x) + A(y)$$

such that

$$(4.12)\qquad |f(x) - A(x) - g(0) - h(0))| \le \frac{4\epsilon(4^p+2)}{4^p-2}|x|^{2p}.$$

It is easy to see that A is a Lebesegue measurable function. Thus the solution A of the Cauchy functional equation (4.10) has the form $A(x) = a \cdot x$ for some $a \in \mathbb{C}^n$. Now, from (4.8) and (4.11) we have

$$(4.13)\qquad \|u - a\cdot x - f(0)\| \le K|x|^{2p}$$

where $K = \frac{2\epsilon(3\cdot 4^p+2)}{4^p-2}$. It follows from (4.12) that u is a locally integrable function satisfying

$$|u(x)| \le |a\cdot x| + |f(0)| + K|x|^{2p}.$$

Thus $u \in \mathcal{S}'$ and that $v, w \in \mathcal{S}'$ in view of (4.3). This completes the proof. □

As a consequence of the Theorem 3.5 and Theorem 4.1, we have the following.

Theorem 4.2. *Let $u, v, w \in \mathcal{D}'$ satisfy the inequality*

$$\|u \circ A - v \circ P_1 - w \circ P_2\| \le \epsilon(|x|^{2p} + |y|^{2p}) \tag{4.14}$$

for some integer $p \ge 1$. Then, for $p \ge 2$, there exist a unique $a \in \mathbb{C}^n$ and complex constants c_1, c_2, c_3 such that

$$\|u - a \cdot x - c_1\| \le \frac{4\epsilon}{4^p - 2}|x|^{2p},$$
$$\|v - a \cdot x - c_2\| \le \frac{\epsilon(4^p + 2)}{4^p - 2}|x|^{2p},$$
$$\|w - a \cdot x - c_2\| \le \frac{\epsilon(4^p + 2)}{4^p - 2}|x|^{2p},$$

and for $p = 1$, there exist a unique $a \in \mathbb{C}^n$ and complex constants c_1, c_2, c_3 such that

$$\|u - a \cdot x - c_1\| \le 10\epsilon|x|^2,$$
$$\|v - a \cdot x - c_2\| \le 11\epsilon|x|^2,$$
$$\|w - a \cdot x - c_3\| \le 11\epsilon|x|^2.$$

Since every locally integrable function $f(x)$ can be view as a distribution via the equation

$$\langle f, \varphi \rangle = \int f(x)\varphi(x)dx,$$

we have the following stability theorem for locally integrable functions in almost everywhere sense.

Theorem 4.3. *Let Ω_1, $\Omega_2 \subset \mathbb{R}^n$ such that $m(\mathbb{R}^n \setminus \Omega_1) = m(\mathbb{R}^n \setminus \Omega_2) = 0$ and let $f, g, h : \mathbb{R}^n \to \mathbb{C}$ locally integrable functions satisfying the inequality*

$$|f(x + y) - g(x) - h(y)| \le \epsilon(|x|^{2p} + |y|^{2p}) \tag{4.15}$$

for all $x \in \Omega_1$, $y \in \Omega_2$. Then there exist a unique $a \in \mathbb{C}^n$, complex constants c_1, c_2, c_3 and $\Omega \subset \mathbb{R}^n$ with $m(\mathbb{R}^n \setminus \Omega) = 0$ such that for $p \ge 2$,

$$|f(x) - a \cdot x - c_1| \le \frac{4\epsilon}{4^p - 2}|x|^{2p},$$
$$|g(x) - a \cdot x - c_2| \le \frac{\epsilon(4^p + 2)}{4^p - 2}|x|^{2p},$$
$$|h(x) - a \cdot x - c_2| \le \frac{\epsilon(4^p + 2)}{4^p - 2}|x|^{2p},$$

and for $p = 1$,

$$\|u - a \cdot x - c_1\| \le 10\epsilon |x|^2,$$
$$\|v - a \cdot x - c_2\| \le 11\epsilon |x|^2,$$
$$\|w - a \cdot x - c_3\| \le 11\epsilon |x|^2,$$

for all $x \in \Omega$.

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