Honam Mathematical J. **33** (2011), No. 4, pp. 453–465
http://dx.doi.org/10.5831/HMJ.2011.33.4.453

THE LATTICE OF ORDINARY SMOOTH TOPOLOGIES

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Abstract. Lim et al. [5] introduce the notion of ordinary smooth topologies by considering the gradation of openness[resp. closedness] of ordinary subsets of X. In this paper, we study a collection of all ordinary smooth topologies on X, say OST(X), in the sense of a lattice. And we prove that OST(X) is a complete lattice.

1. Introduction

In 1968, Chang [1] had first tried to fuzzify the concept of a topology on a set X by axiomatizing a collection of fuzzy sets in X. After that, Pu and Liu [8] and Lowen [6] had advanced the concept of fuzzy topology. Recently, a new concept of fuzzification of a topological space is considered by fuzzy topologists, the so-called smooth fuzzy topological space. The concept of smooth topological spaces was introduced by *Šostak*. After that some work has been done by Ramadan [9], Chattopadhyay et al. [2], Hazra et al [4], Chattopadhyay and Samanta [3], and Peeters [7]. They all have investigated the gradation of openness[resp. closedness] of fuzzy sets in a set X. Lim et al. [5] introduce the notion of ordinary smooth topologies by considering the gradation of openness[resp. closedness] of ordinary subsets of X.

In this paper, we study a collection of all ordinary smooth topologies on X, say OST(X), in the sense of a lattice. And we prove that OST(X) is a complete lattice.

Received August 17, 2011. Accepted September 16, 2011.
2000 Mathematics Subject Classification. 54A40.
Key words and phrases. Complete lattice, Ordinary smooth topology, Ordinary smooth cotopology, Ordinary smooth base, Ordinary smooth subbase.
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∗ This paper was supported by Wonkwang University in 2009.

2. Preliminaries

We give a partial order on OST(X) by taking the pointwise order with respect to the canonical order-induced by the order of the real numbers in I.

We list some definitions, one result and one example needed in the next section.

Definition 1. [5] *Let X be a nonempty set. Then a mapping $\tau : 2^X \to I$ is called an* ordinary smooth topology *on X or a* gradation of openness *of ordinary subsets of X if τ satisfies the following axioms:*

(OST_1) $\tau(\varnothing) = \tau(X) = 1$.

(OST_2) $\tau(A \cap B) \geq \tau(A) \wedge \tau(B)$ *for all* $A, B \in 2^X$.

(OST_3) $\tau(\cup_{\alpha\in\Gamma} A_\alpha) \geq \bigwedge_{\alpha\in\Gamma} \tau(A_\alpha)$ *for all* $\{A_\alpha\}_{\alpha\in\Gamma} \subset 2^X$.

The pair (X, τ) is called an ordinary smooth topological space. *We denote the set of all ordinary smooth topologies on X as OST(X).*

Definition 2. [5] *Let X be a nonempty set. Then a mapping $\mathfrak{F} : 2^X \to I$ is called an* ordinary smooth cotopology *on X or a* gradation of closedness *of ordinary subsets of X if $\mathfrak{F}$ satisfies the following axioms:*

(OST_1) $\mathfrak{F}(\varnothing) = \mathfrak{F}(X) = 1$.

(OST_2) $\mathfrak{F}(A \cup B) \geq \mathfrak{F}(A) \vee \mathfrak{F}(B)$ *for all* $A, B \in 2^X$.

(OST_3) $\mathfrak{F}(\cap_{\alpha\in\Gamma} A_\alpha) \geq \bigvee_{\alpha\in\Gamma} \mathfrak{F}(A_\alpha)$ *for all* $\{A_\alpha\}_{\alpha\in\Gamma} \subset 2^X$.

The pair $(X, \mathfrak{F})$ is called an ordinary smooth cotopological space. *We denote the set of all ordinary smooth cotopologies on X as OSCT(X).*

Result 3 ([5], Proposition 2.7)**.** *Let X be a nonempty set. We define two mappings $f : OST(X) \to OSCT(X)$ and $g : OSCT(X) \to OST(X)$ as follows, respectively:*

$$[f(\tau)](A) = \tau(A^c), \text{ for all } \tau \in OST(X) \text{ and } A \in 2^X$$

and

$$[g(\mathfrak{F})](A) = \mathfrak{F}(A^c), \text{ for all } \mathfrak{F} \in OSCT(X) \text{ and } A \in 2^X.$$

Then f and g are well-defined. Furthermore, $g \circ f = id_{OST(X)}$ and $f \circ g = id_{OSCT(X)}$. In fact, if we write $f(\tau) = \mathfrak{F}_\tau$ and $g(\mathfrak{F}) = \tau_{\mathfrak{F}}$, then $\tau_{\mathfrak{F}_\tau} = \tau$ and $\mathfrak{F}_{\tau_{\mathfrak{F}}} = \mathfrak{F}$.

Definition 4. *Let X be a nonempty set and $I^{(2^X)}$ denote the set of all mapping from 2^X into I. For t_1 and $t_2 \in I^{(2^X)}$, we say that t_1 is* finer *than t_2, or t_2 is* coarser *than t_1, denoted by $t_2 \leqslant t_1$, if $t_2(A) \leqslant t_1(A)$ for each $A \in 2^X$.*

From Definition 4, it can be easily seen that $(I^{(2^X)}, \leqslant)$ is a partially ordered set (in short, a poset). Since OST(X)[resp. OSCT(X)]$\subset I^{(2^X)}$, we can define an order relation "$\leqslant$" on OST(X)[resp. OSCT(X)], as the same way in $I^{(2^X)}$.

Furthermore, we can easily see that for any τ_1, τ_2 $\in$OST(X), τ_1 is finer than τ_2, i.e. $\tau_2 \leqslant \tau_1$ if and only if $\mathfrak{F}_{\tau_1}$ is finer than $\mathfrak{F}_{\tau_2}$, i.e. $\mathfrak{F}_{\tau_2} \leqslant \mathfrak{F}_{\tau_1}$, from Result 3.

Example 5. (*a*) *Let X be a nonempty set. We define the mapping $\tau_\varnothing : 2^X \to I$ as follows : For each $A \in 2^X$,*

$$\tau_\varnothing(A) = \begin{cases} 1, & \text{if } A = \varnothing \text{ or } A = X; \\ 0, & \text{otherwise.} \end{cases}$$

Then it is easily verified that $\tau_\varnothing \in OST(X)$(called the ordinary indiscrete smooth topology). In fact, $\tau_\varnothing$ is the coarsest ordinary smooth topology. Moreover, we can define the coarsest ordinary smooth cotopology on X, denoted by $\mathfrak{F}_\varnothing$.

(*b*) *Let X be a nonempty set. We define the mapping $\tau_X : 2^X \to I$ as follows : For each $A \in 2^X$, $\tau_X(A) = 1$. Then it can be easily see that $\tau_X \in OST(X)$(called the ordinary discrete smooth topology). In fact, τ_X is the finest ordinary smooth topology. Furthermore, we can define the finest ordinary smooth cotopology on X, denoted by $\mathfrak{F}_X$.*

(*c*) *Let $(X, \mathcal{T})$ be a classical topological space. We define the mapping $\tau_{\mathcal{T}} : 2^X \to I$ as follows : For each $A \in 2^X$,*

$$\tau_{\mathcal{T}}(A) = \begin{cases} 1, & \text{if } A \in \mathcal{T}; \\ 0, & \text{otherwise.} \end{cases}$$

Then it can be easily seen that $\tau_{\mathcal{T}} \in OST(X)$. Moreover, if $\mathcal{T}$ is the classical indiscrete topology, then $\tau_{\mathcal{T}} = \tau_\varnothing$, and if $\mathcal{T}$ is the classical discrete topology, then $\tau_{\mathcal{T}} = \tau_X$.

Remark 6. *Let X be a nonempty set. Then, from(b) and (c) in Example 5, it is clear that $\tau_\varnothing \leqslant \tau \leqslant \tau_X$ for each $\tau \in OST(X)$.*

The following is the immediate result of Definition 4 and Remark 6.

Proposition 7. *Let X be a nonempty set. Then $(OST(X), \leqslant)$ is a poset with the least element $\tau_\varnothing$ and the largest element τ_X.*

3. Bases and subbases of an ordinary smooth Topology

Any subset $\mathfrak{S}$ of 2^X, the collection of subsets of X, is contained in a classical topology that is created by first adding $\varnothing$ and X (if these are not in the subset yet), and then making it closed under the finite intersections and arbitrary unions. The topology we yield in this way is the coarsest one finer than $\mathfrak{S}$ with respect to the inclusion.

We would like to establish the ordinary smooth topology which is the counterpart of this construction.

Definition 8. *Let X be a nonempty set and let $t : 2^X \to I$ be a mapping. We define the mappings t^C, t^I, and $t^S : 2^X \to I$ as follows : For each $A \in 2^X$*

$$t^C(A) = \begin{cases} 1, & \text{if } A = \varnothing \text{ or } A{=}X\ ; \\ t(A), & \text{otherwise.} \end{cases}$$

$$t^I(A) = \bigvee_{n \in \mathbb{N}} \left\{ \bigwedge_{i=1}^{n} t(A_i) : A = \bigcap_{i=1}^{n} A_i \right\},$$

$$t^S(A) = \bigvee \left\{ \bigwedge_{\alpha \in \Gamma} t(A_\alpha : A = \bigcup_{\alpha \in \Gamma} A_\alpha \right\}.$$

The obvious thing to remark now is that the succession of the modification of these three mappings should define an ordinary smooth topology. However, in order to show this, we first need some lemmas.

Lemma 9. *Let $t : 2^X \to I$ be a mapping. Then we have the following:*

(a) If $t(\varnothing) = t(X) = 1$, then $t^C = t$.

(b) If $t(A \cap B) \geq t(A) \wedge t(B)$ for any $A, B \in 2^X$, then $t^I = t$.

(c) If $t(\cup_{\alpha \in \Gamma} A_\alpha) \geq \bigwedge_{\alpha \in \Gamma} t(A_\alpha)$ for each $\{A_\alpha\}_{\alpha \in \Gamma} \subset 2^X$, then $t^S = t$.

Proof. (a) It is obvious from the definition of t^C.

(b) Let $A \in 2^X$. Then, by the definition of t^I, it is obvious that $t^I(A) \geq t(A)$. Now, let $\varepsilon > 0$ be given. Then there exists $n \in \mathbb{N}$ and $\{A_i\}_{i=1}^n \subset 2^X$ such that $A = \cap_{i=1}^n A_i$ and $t^I(A) < \bigwedge_{i=1}^n t(A_i) + \varepsilon$. By the hypothesis and the induction,

$$t\left(\bigcap_{i=1}^n A_i\right) \geq \bigwedge_{i=1}^n t(A_i).$$

Thus $t(A) > t^I(A) - \varepsilon$. Since $\varepsilon > 0$ is arbitrary,

$$t(A) \geq t^I(A).$$

So $t^I(A) = t(A)$ for each $A \in 2^X$. Hence $t^I = t$.

(c) By the definition of t^S, it is clear that $t^S(a) \geq t(A)$ for each $A \in 2^X$. Let $A \in 2^X$, and let $\varepsilon > 0$ be given. Then, by the definition of t^S, there exists $\{A_\alpha\}_{\alpha\in\Gamma} \subset 2^X$ such that $A = \bigcup_{\alpha\in\Gamma} A_\alpha$ and $t^S(A) < \bigwedge_{\alpha\in\Gamma} t(A_\alpha) + \varepsilon$. By the hypothesis,

$$t(A) = t\left(\bigcup_{\alpha\in\Gamma} A_\alpha\right) \geq \bigwedge_{\alpha\in\Gamma} t(A_\alpha).$$

Thus $t(A) > t^S(A) - \varepsilon$. Since $\varepsilon > 0$ is arbitrary,

$$t(A) \geq t^S(A).$$

So $t^S(A) = t(A)$ for each $A \in 2^X$. Hence $t^S = t$. This completes the proof. □

Lemma 10. *The three operators* $\cdot^C$, $\cdot^I$, *and* $\cdot^S : I^{(2^X)} \to I^{(2^X)}$ *are idempotent, i.e., for any mapping* $t : 2^X \to I$, *we have the followings:*

(a) $t^{CC} = t^C$.

(b) $t^{II} = t^I$.

(c) $t^{SS} = t^S$.

Proof. (a) From the definition of t^C, it is clear.

(b) By the definition of t^I, it is clear that $t^I(A) \geq t(A)$ for each $A \in 2^X$. Thus $t^{II}(A) \geq t^I(A)$ for each $A \in 2^X$. Now let $A \in 2^X$

and let $\varepsilon > 0$. Then by the definition of t^{II}, there exists $n \in \mathbb{N}$ and $\{A_i\}_{i=1}^{n} \subset 2^X$ such that

$$A = \bigcap_{i=1}^{n} A_i$$

and

$$t^{II}(A) < \bigwedge_{i=1}^{n} t^{I}(A_i) + \frac{\varepsilon}{2}. \tag{1}$$

By the definition of $t^I(A_i)$, for each $i \in \{1, 2, ..., n\}$, there exists $m(i) \in \mathbb{N}$ and $\{A_i\}_{i=1}^{m(i)} \subset 2^X$ such that

$$A_i = \bigcap_{j=1}^{m(i)} A_{ij}$$

and

$$t^{II}(A) < \bigwedge_{i=1}^{n} \bigwedge_{j=1}^{m(i)} t(A_{ij}) + \frac{\varepsilon}{2}. \tag{2}$$

By (1) and (2),

$$t^{II}(A) < \bigwedge_{i=1}^{n} \bigwedge_{j=1}^{m(i)} t(A_{ij}) + \varepsilon.$$

If we reindex the collection $\{A_i\}_{(i,j)=(1,1)}^{(n,m(i))} \subset 2^X$, we might as well write that $\{A_i\}_{(i,j)=(1,1)}^{(n,m(i))} = \{B_k\}_{k=1}^{M} \subset 2^X$ such that

$$A = \bigcap_{k=1}^{M} B_k$$

and

$$t^{II}(A) < \bigwedge_{k=1}^{M} t(B_k) + \varepsilon. \tag{3}$$

By the definition of t^I, and the induction

$$t^{I}(A) \geq \bigwedge_{k=1}^{M} t(B_k). \tag{4}$$

By (3) and (4),

$$t^{II}(A) < t^{I}(A) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary,

$$t^{II}(A) \leq t^{I}(A).$$

So $t^{II}(A) = t^{I}(A)$ for each $A \in 2^X$. Hence $t^{II} = t^{I}$.

(c) It is obvious that $t^{SS}(A) \geq t^{S}(A)$ for each $A \in 2^X$. Let $A \in 2^X$ and let $\varepsilon > 0$. Then by the definition of t^{SS}, there exists $\{A_\alpha\}_{\alpha\in\Gamma} \subset 2^X$ such that

$$A = \bigcup_{\alpha\in\Gamma} A_\alpha$$

and

$$t^{SS}(A) < \bigwedge_{\alpha\in\Gamma} t^{S}(A_\alpha) + \frac{\varepsilon}{2}. \tag{5}$$

By the definition of $t^{S}(A_\alpha)$, for each $\alpha \in \Gamma$, there exists $\{A_{\alpha\beta}\}_{\beta\in\Gamma_\alpha} \subset 2^X$ such that

$$A_\alpha = \bigcup_{\beta\in\Gamma_\alpha} A_{\alpha\beta}$$

and

$$t^{S}(A_\alpha) < \bigwedge_{\beta\in\Gamma_\alpha} t(A_{\alpha\beta}) + \frac{\varepsilon}{2}. \tag{6}$$

By (5) and (6),

$$t^{SS}(A) < \bigwedge_{\alpha\in\Gamma} \bigwedge_{\beta\in\Gamma_\alpha} t(A_{\alpha\beta}) + \varepsilon.$$

If we reindex the collection $\{A_{\alpha\beta}\}_{(\alpha,\beta)\in\Gamma\times\Gamma_\alpha} \subset 2^X$, we might as well write that $\{A_{\alpha\beta}\}_{(\alpha,\beta)\in\Gamma\times\Gamma_\alpha} = \{B_k\}_{k\in K} \subset 2^X$ such that

$$A = \bigcap_{k\in K} B_k$$

and

$$t^{SS}(A) < \bigwedge_{k\in K} t(B_k) + \varepsilon.$$

By the definition of t^{S},

$$t^{S}(A) \geq \bigwedge_{k\in K} t(B_k).$$

Thus $t^{SS}(A) < t^{S}(A) + \varepsilon$. Since $\varepsilon > 0$ is arbitrary,

$$t^{SS}(A) \leq t^{S}(A)$$

for each $A \in 2^X$. Hence $t^{SS}(A) = t^S(A)$ for each $A \in 2^X$. Therefore

$$t^{SS} = t^S.$$

This completes the proof. □

Corollary 11. *The implications from Lemma 9 may be reserved, i.e., for each mapping* $t : 2^X \to I$, *we have the following :*

(a) $t^C(\phi) = t^C(X) = 1$.

(b) $t^I(A \cap B) \geq t^I(A) \wedge t^I(B)$, *for any* $A, B \in 2^X$.

(c) $t^S(\cup_{\alpha\in\Gamma} A_\alpha) \geq \bigwedge_{\alpha\in\Gamma} t^S(A_\alpha)$, *for each* $\{A_\alpha\}_{\alpha\in\Gamma} \subset 2^X$.

Proof. (a) It is obvious from the definition of t^C.

(b) Let $A, B \in 2^X$. Then, by the definition of t^{II},

$$t^{II}(A \cap B) \geq t^I(A) \wedge t^I(B).$$

By Lemma 10 (b), we have

$$t^{II}(A \cap B) = t^I(A \cap B).$$

So $t^I(A \cap B) \geq t^I(A) \wedge t^I(B)$.

(c) It can be proved analogously. □

The following lemma is the immediate result of Definition 8.

Lemma 12. *The three operators* $\cdot^C$, $\cdot^I$, *and* $\cdot^S : I^{(2^X)} \to I^{(2^X)}$ *are increasing, i.e for any two mappings* $t_1, t_2 : 2^X \to I$ *with* $t_2 \leq t_1$, *we have the followings:*

(a) $t_2^C \leq t_1^C$.

(b) $t_2^I \leq t_1^I$.

(c) $t_2^S \leq t_1^S$.

Now we show that every mapping $t : 2^X \to I$ can be extended to an ordinary smooth topology on X, just as in the case of classical topologies in $T(X)$, where $T(X)$ denotes the set of all classical topologies on X.

Theorem 13. [Ordinary smooth topological subbase theorem] *Let $t : 2^X \to I$ be any mapping, and let $t^{CIS} : 2^X \to I$ be the mapping defined by $t^{CIS} = \left((t^C)^I\right)^S$. Then $t^{CIS} \in OST(X)$.*

Proof. (OST_1) By Corollary 11 (a), we have $t^C(\varnothing) = t^C(X) = 1$. Thus,

$$\begin{aligned}(t^C)^I(\varnothing) &= \bigvee_{n\in\mathbb{N}} \left\{ \bigwedge_{i=1}^{n} t^C(A_i) : \varnothing = \bigwedge_{i=1}^{n} A_i \right\} \\ &\geq t^C(\varnothing) \\ &= 1.\end{aligned}$$

Similarly, we have that $(t^C)^I(X) = 1$. So

$$\begin{aligned}t^{CIS}(\varnothing) &= \bigvee \left\{ \bigwedge_{\alpha\in\Gamma} (t^C)^I(A_\alpha) : \varnothing = \bigcap_{\alpha\in\Gamma} A_\alpha \right\} \\ &\geq (t^C)^I(\varnothing) \\ &= 1.\end{aligned}$$

Similarly, we have that $t^{CIS}(X) = 1$. Hence $t^{CIS}(\varnothing) = t^{CIS}(X) = 1$.

(OST_3) Let $\{A_\alpha\}_{\alpha\in\Gamma} \subset 2^X$. Then, by Corollary 11 (c), we have

$$\begin{aligned}t^{CIS}\left(\bigcup_{\alpha\in\Gamma} A_\alpha\right) &= (t^{CI})^S\left(\bigvee_{\alpha\in\Gamma} A_\alpha\right) \\ &\geq \bigwedge_{\alpha\in\Gamma} (t^{CI})^S(A_\alpha) \\ &= \bigwedge_{\alpha\in\Gamma} t^{CIS}(A_\alpha).\end{aligned}$$

(OST_2) Let $A_1, A_2 \in 2^X$. Then, by the definitions, we have

$$t^{CIS}(A_1 \cap A_2) = \bigvee \left\{ \bigwedge_{\alpha\in\Gamma} (t^{CI})(C_\alpha) : A_1 \cap A_2 = \bigcup_{\alpha\in\Gamma} C_\alpha \right\},$$

$$t^{CIS}(A_1) = \bigvee \left\{ \bigwedge_{\alpha\in\Gamma_1} (t^{CI})(A_{1\alpha}) : A_1 = \bigcup_{\alpha\in\Gamma_1} A_{1\alpha} \right\},$$

and

$$t^{CIS}(A_2) = \bigvee \left\{ \bigwedge_{\alpha\in\Gamma_2} (t^{CI})(A_{2\alpha}) : A_2 = \bigcup_{\alpha\in\Gamma_2} A_{2\alpha} \right\}.$$

Let $\varepsilon > 0$ be given. Then there exists $\{A_{1\alpha}\}_{\alpha\in\Gamma_1} \subset 2^X$ and $\{A_{2\alpha}\}_{\alpha\in\Gamma_2} \subset 2^X$ such that

$$A_1 = \bigcup_{\alpha\in\Gamma} A_{1\alpha}, \quad t^{CIS}(A_1) < \bigwedge_{\alpha\in\Gamma_1} t^{CI}(A_{1\alpha}) + \varepsilon,$$

and

$$A_2 = \bigcup_{\alpha\in\Gamma} A_{2\alpha}, \quad t^{CIS}(A_2) < \bigwedge_{\alpha\in\Gamma_2} t^{CI}(A_{2\alpha}) + \varepsilon.$$

For each $(\alpha, \beta) \in \Gamma_1 \times \Gamma_2$, let $A_{\alpha\beta} = A_{1\alpha} \cap A_{2\beta}$. Then,

$$\bigcup_{(\alpha,\beta)\in\Gamma_1\times\Gamma_2} A_{\alpha\beta} = \left(\bigcup_{\alpha\in\Gamma_1} A_{1\alpha} \right) \cap \left(\bigcup_{\beta\in\Gamma_2} A_{2\beta} \right) = A_1 \cap A_2.$$

Thus,

$$\begin{aligned} \bigwedge_{(\alpha,\beta)\in\Gamma_1\times\Gamma_2} t^{CI} A_{\alpha\beta} &\geq \left(\bigwedge_{\alpha\in\Gamma_1} t^{CI} A_{1\alpha} \right) \wedge \left(\bigwedge_{\beta\in\Gamma_2} t^{CI} A_{2\beta} \right) \\ &> t^{CIS}(A_1) \wedge t^{CIS}(A_2) - \varepsilon. \end{aligned}$$

Since $t^{CIS}(A_1 \cap A_2)$ is defined as the supremum over such infima, we have

$$t^{CIS}(A_1) \wedge t^{CIS}(A_2) - \varepsilon < t^{CIS}(A_1 \cap A_2).$$

Since $\varepsilon > 0$ is arbitrary, we obtain that

$$t^{CIS}(A_1) \wedge t^{CIS}(A_2) \leq t^{CIS}(A_1 \cap A_2).$$

Hence, $t^{CIS} \in OST(X)$. □

Definition 14. *Let (X, τ) be an ordinary smooth topological space.*

(a) *A mapping $\mathfrak{B} : 2^X \to I$ is called an* ordinary smooth base *for τ if $\mathfrak{B}^S = \tau$.*

(b) *A mapping $\mathfrak{S} : 2^X \to I$ is called an* ordinary smooth subbase *for τ if $\mathfrak{S}^{CIS} = \tau$.*

Proposition 15. *Let X be a nonempty set and let $\{\tau_\alpha\}_{\alpha\in\Gamma} \subset OST(X)$. Then there exists a unique ordinary smooth topology τ that is coarser than τ_α for each $\alpha \in \Gamma$ and the finest one with this property.*

Proof. Let $\tau = \bigcap_{\alpha\in\Gamma} \tau_\alpha$, and let us define the mapping $\tau : 2^X \to I$ as follows : For each $A \in 2^X$,

$$\tau(A) = \bigwedge_{\alpha\in\Gamma} \tau_\alpha(A).$$

(OST_1) $\tau(\varnothing) = \bigwedge_{\alpha\in\Gamma} \tau_\alpha(\varnothing) = \bigwedge_{\alpha\in\Gamma} 1 = 1$. Similarly, we have that $\tau(X) = 1$.

(OST_2) Let $A, B \in 2^X$. Then,

$$\begin{aligned}
\tau(A\cap B) &= \bigwedge_{\alpha\in\Gamma} \tau_\alpha(A\cap B) \\
&\geq \bigwedge_{\alpha\in\Gamma} [\tau_\alpha(A) \wedge \tau_\alpha(B)] \qquad \text{Since } \tau_\alpha \in OST(X) \\
&= \left(\bigwedge_{\alpha\in\Gamma} \tau_\alpha(A)\right) \wedge \left(\bigwedge_{\alpha\in\Gamma} \tau_\alpha(B)\right) \\
&= \tau(A) \wedge \tau(B).
\end{aligned}$$

(OST_3) Let $\{A_j\}_{j\in J} \subset 2^X$. Then,

$$\begin{aligned}
\tau(\cup_{j\in J} A_j) &= \bigwedge_{\alpha\in\Gamma} \tau_\alpha(\cup_{j\in J} A_j) \\
&\geq \bigwedge_{\alpha\in\Gamma} \bigwedge_{j\in J} \tau_\alpha(A_j) \\
&= \bigwedge_{j\in J} \left(\bigwedge_{\alpha\in\Gamma} \tau_\alpha(A_j)\right) \\
&= \bigwedge_{j\in J} \tau(A_j).
\end{aligned}$$

So $\tau = \bigcap_{\alpha\in\Gamma} \in OST(X)$. It can be easily seen that $\tau = \inf\{A_\alpha : \alpha \in \Gamma\}$. This completes the proof. □

The union of topologies need not be a topology in general. However, it can always be considered as a subbase for a unique topology, finer than all topologies involved, and the coarsest with that property. Now we have a description of ordinary smooth bases and ordinary smooth subbases for ordinary smooth topologies. The following is easily verified.

Theorem 16. *Let X be a nonempty set. Then $(OST(X), \leqslant)$ is a complete lattice with the least element $\tau_{\varnothing}$ and the largest element τ_X.*

Proof. From Proposition 7, (OST(X), $\leqslant$) is a poset with the least element τ_{ϕ} and the largest element τ_X. By Proposition 15, it is obvious that $\bigcap_{\alpha\in\Gamma}\tau_\alpha = \bigwedge_{\alpha\in\Gamma}\tau_\alpha \in$OST(X), for each $\{\tau_\alpha\}_{\alpha\in\Gamma} \subset$ OST(X). By Theorem 13, $(\bigcup_{\alpha\in\Gamma}\tau_\alpha)^{CIS} = \bigvee_{\alpha\in\Gamma}\tau_\alpha \in$ OST(X), for each $\{\tau_\alpha\}_{\alpha\in\Gamma} \subset$ OST(X). Hence (OST(X), $\leqslant$) is a complete lattice. □

Proposition 17. *Let X be a nonempty set and let $\{\tau_\alpha\}_{\alpha\in\Gamma} \subset$ OST(X). Then there exists a unique ordinary smooth topology τ on X that is finer than τ_α for each $\alpha \in \Gamma$ and is the coarsest with that property. In fact, the mapping $\mathfrak{B} : 2^X \to I$ is given by*

$$\mathfrak{B}(A) = \left(\bigcup_{\alpha\in\Gamma}\tau_\alpha\right)(A) = \bigvee_{\alpha\in\Gamma}\tau_\alpha(A) \ \textit{for each}\ A \in 2^X$$

and the mapping $\tau : 2^X \to I$ is given by $\tau = \mathfrak{B}^{CIS}$.

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