

Central Limit Theorem of the Cross Variation Related to Fractional Brownian Sheet

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Abstract

By using Malliavin calculus, we study a central limit theorem of the cross variation related to fractional Brownian sheet with Hurst parameter $H = (H_1, H_2)$ such that $1/4 < H_1 < 1/2$ and $1/4 < H_2 < 1/2$.

Keywords: Malliavin calculus, fractional Brownian sheet, central limit theorem, cross variation, multiple stochastic integral.

1. Introduction

Let M be a two-parameter continuous martingale bounded in L^2 and zero on the axes. Then M^2 has the following Doob-Mayer decomposition.

$$M_{st}^2 = 2 \int_0^s \int_0^t M_z dM_z + 2\tilde{M}_{st} + \langle M_{s\cdot} \rangle_t + \langle M_{\cdot t} \rangle_s - \langle M \rangle_{st},$$

where $\langle M \rangle$ is the quadratic variation of two-parameter martingale and \tilde{M} is a martingale obtained by the L^1 limit of the sequence

$$\sum_{(s_i, t_j) \in \Pi_z} [M_{s_{i+1}, t_j} - M_{s_i, t_j}] [M_{s_i, t_{j+1}} - M_{s_i, t_j}].$$

Here Π_z is a partition of the rectangle $[0, z] \subseteq [0, 1]^2$ with $0 = s_0 < s_1 < \dots < s_p < 1$ and $0 = t_0 < t_1 < \dots < t_q < 1$. For two martingales \tilde{M} and M , the cross variation $\langle \tilde{M}, M \rangle$ is needed for the stochastic calculus of M .

In this paper, we study the asymptotic behavior of the cross variations corresponding to the fractional Brownian sheet (fBs) $B^H = (B_z^H)$, $z \in [0, 1]^2$, with Hurst parameters $H = (H_1, H_2)$, $H_i \in (0, 1)$, $i = 1, 2$. We state our main result in the following theorem.

Theorem 1. Let $B^H = (B_z^H, z \in [0, 1]^2)$ be fBs with Hurst parameter $H = (H_1, H_2)$. If $1/4 < H_1 < 1/2$ and $1/4 < H_2 < 1/2$, then we have

$$\begin{aligned} Q_n = n^{2(H_1+H_2)-1} \sum_{k,l=1}^n & \left[\left(B_{\frac{k+1}{n}, \frac{l}{n}}^H - B_{\frac{k}{n}, \frac{l}{n}}^H \right) \left(B_{\frac{k}{n}, \frac{l+1}{n}}^H - B_{\frac{k}{n}, \frac{l}{n}}^H \right) \times \left(B_{\frac{k+1}{n}, \frac{l+1}{n}}^H - B_{\frac{k+1}{n}, \frac{l}{n}}^H - B_{\frac{k}{n}, \frac{l+1}{n}}^H + B_{\frac{k}{n}, \frac{l}{n}}^H \right) \right. \\ & \left. + \frac{1}{2n^{2(H_1+H_2)+1}} \left(\int_0^1 B_{s_1,1}^H ds_1 + \int_0^1 B_{1,s_2}^H ds_2 \right) \right] \xrightarrow{\mathcal{L}} \mathbb{N}(0, \sigma^2(H)). \end{aligned} \quad (1.1)$$

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Here the notation $\xrightarrow{\mathcal{L}}$ denote the convergence in distribution, and $\sigma^2(H) > 0$ denote a constant depending only on the Hurst parameters $H = (H_1, H_2)$, where $\sigma^2(H)$ is given by

$$\sigma^2(H) = \frac{1}{4^3} \prod_{r=1}^2 \frac{1}{2(4H_r + 1)} \sum_{p=-\infty}^{\infty} \left| |p+1|^{2H_r} + |p-1|^{2H_r} - 2|p|^{2H_r} \right|^2.$$

The main tool is the result on the convergence of multiple stochastic integrals worked by Nualart and Ortiz-Latorre in Nualart and Ortiz-Latorre (2008) based on Malliavin calculus.

Recently in several works, the asymptotic behavior on the weighted power variations of a fractional Brownian motion has been studied by using Malliavin calculus (See Nourdin, 2008; Nourdin and Nualart, 2008; Nourdin *et al.*, 2010). For the two-parameter processes, a central limit theorem has been obtained in Réveillac (2009a) for the weighted quadratic variations of a standard Brownian sheet. In addition, Réveillac in Réveillac (2009b) proved a central limit theorem for the finite-dimensional laws of the weighted quadratic variations of fBs. In Park *et al.* (2011), author consider central limit theorem and Berry-Essen bounds for the cross variation of this type with respect to standard Brownian sheet. To the best of our knowledge, our studies on the cross variation of this type with respect to fBs are a first attempt. Hence we consider the non-weighted cross variation of fBs in the simplest possible variation.

2. Preliminaries

Now we recall some basic facts about Malliavin calculus for Gaussian processes. For a more detailed reference, see Nualart (2006). Suppose that \mathbb{H} is a real separable Hilbert space with scalar product denoted by $\langle \cdot, \cdot \rangle_{\mathbb{H}}$. Let $B = (B(h), h \in \mathbb{H})$ be an isonormal Gaussian process, that is a centered Gaussian family of random variables such that $E(B(h)B(g)) = \langle h, g \rangle_{\mathbb{H}}$. In particular, if B is fBs B^H with Hurst parameter $H = (H_1, H_2)$, then the scalar product is given by

$$\langle \mathbf{1}_{[0,a]}, \mathbf{1}_{[0,b]} \rangle_{\mathbb{H}} = \frac{1}{4} \prod_{i=1}^2 (a_i^{2H_i} + b_i^{2H_i} - |a_i - b_i|^{2H_i}), \quad \text{for } a, b \in [0, 1]^2. \quad (2.1)$$

Kim *et al.* in Kim *et al.* (2008) have developed the theory of stochastic calculus for fBs B^H . For every $n \geq 1$, let \mathcal{H}_n be the n^{th} Wiener chaos of B^H , that is the closed linear subspace of $\mathbb{L}^2(\Omega)$ generated by $\{H_n(B^H(h)) : h \in \mathbb{H}, \|h\|_{\mathbb{H}} = 1\}$, where H_n is the n^{th} Hermite polynomial. We define a linear isometric mapping $I_n : \mathbb{H}^{\odot n} \rightarrow \mathcal{H}_n$ by $I_n(h^{\otimes n}) = n! H_n(B^H(h))$, where $\mathbb{H}^{\odot n}$ is the symmetric tensor product.

In this paper we will only use multiple stochastic integrals with respect to a fBs $B^H = (B_z^H, z \in [0, 1]^2)$, and in this case the scalar product in \mathbb{H} is defined by (2.1). We will use this notation \mathbb{H} throughout this paper.

If $f \in \mathbb{H}^{\odot p}$, the Malliavin derivative of the multiple stochastic integrals is given by

$$D_z I_n(f_n) = n I_{n-1}(f_n(\cdot, z)), \quad \text{for } z \in [0, 1]^2.$$

Let $\{e_l, l \geq 1\}$ be a complete orthonormal system in \mathbb{H} . If $f \in \mathbb{H}^{\odot p}$ and $g \in \mathbb{H}^{\odot q}$, the contraction $f \otimes_r g$, $1 \leq r \leq p \wedge q$, is the element of $\mathbb{H}^{\otimes(p+q-2r)}$ defined by

$$f \otimes_r g = \sum_{l_1, \dots, l_r=1}^{\infty} \langle f, e_{l_1} \otimes \dots \otimes e_{l_r} \rangle_{\mathbb{H}^{\otimes r}} \langle g, e_{l_1} \otimes \dots \otimes e_{l_r} \rangle_{\mathbb{H}^{\otimes r}}. \quad (2.2)$$

Notice that the tensor product $f \otimes g$ and the contraction $f \otimes_r g$, $1 \leq r \leq p \wedge q$, are not necessarily symmetric even though f and g are symmetric. We will denote their symmetrizations by $f \tilde{\otimes} g$ and $f \tilde{\otimes}_r g$, respectively. The following formula for the product of the multiple stochastic integrals will be frequently used to prove the main result in this paper.

Proposition 1. *Let $f \in \mathbb{H}^{\odot p}$ and $g \in \mathbb{H}^{\odot q}$ be two symmetric functions. Then*

$$I_p(f)I_q(g) = \sum_{r=0}^{p \wedge q} r! \binom{p}{r} \binom{q}{r} I_{p+q-2r}(f \otimes_r g). \quad (2.3)$$

From Proposition 1, we have

$$\mathbb{E}[I_p(f)I_q(g)] = \begin{cases} 0, & \text{if } p \neq q, \\ p! \langle \tilde{f}, \tilde{g} \rangle_{\mathbb{H}^{\odot p}}, & \text{if } p = q, \end{cases} \quad (2.4)$$

where \tilde{f} denotes the symmetrization of f .

3. Variations and Computations of Expectation

Let us set $a_n(H) = n^{2(H_1+H_2)-1}$ and

$$\begin{aligned} G_n = a_n(H) \sum_{k,l=0}^{n-1} & \left[\left(B_{\frac{k+1}{n}, \frac{l}{n}}^H - B_{\frac{k}{n}, \frac{l}{n}}^H \right) \left(B_{\frac{k}{n}, \frac{l+1}{n}}^H - B_{\frac{k}{n}, \frac{l}{n}}^H \right) \times \left(B_{\frac{k+1}{n}, \frac{l+1}{n}}^H - B_{\frac{k+1}{n}, \frac{l}{n}}^H - B_{\frac{k}{n}, \frac{l+1}{n}}^H + B_{\frac{k}{n}, \frac{l}{n}}^H \right) \right. \\ & \left. + \frac{1}{2n^2 a_n(H)} \left(B_{\frac{k}{n}, 1}^H + B_{1, \frac{l}{n}}^H \right) \right]. \end{aligned}$$

For simplicity, we write

$$\epsilon_{\frac{k}{n}, \frac{l}{n}}^{(1)} = \mathbf{1}_{[\frac{k}{n}, \frac{k+1}{n}] \times [0, \frac{l}{n}]}, \quad \epsilon_{\frac{k}{n}, \frac{l}{n}}^{(2)} = \mathbf{1}_{[0, \frac{k}{n}] \times [\frac{l}{n}, \frac{l+1}{n}]} \quad \text{and} \quad \epsilon_{\frac{k}{n}, \frac{l}{n}}^{(3)} = \mathbf{1}_{[\frac{k}{n}, \frac{k+1}{n}] \times [\frac{l}{n}, \frac{l+1}{n}]}.$$

The multiplication formula in Proposition 1 yields the decomposition $G_n = G_{1,n} + G_{2,n}$, where the sequences $G_{1,n}$ and $G_{2,n}$ are given by

$$\begin{aligned} G_{1,n} &= a_n(H) \sum_{k,l=0}^{n-1} I_3 \left(\epsilon_{\frac{k}{n}, \frac{l}{n}}^{(1)} \tilde{\otimes} \epsilon_{\frac{k}{n}, \frac{l}{n}}^{(2)} \tilde{\otimes} \epsilon_{\frac{k}{n}, \frac{l}{n}}^{(3)} \right) \\ G_{2,n} &= a_n(H) \sum_{k,l=0}^{n-1} \left[\frac{1}{2n^{2(H_1+H_2)}} \left((l+1)^{2H_2} - l^{2H_2} \right) I_1 \left(\epsilon_{\frac{k}{n}, \frac{l}{n}}^{(2)} \right) \right. \\ & \quad \left. + \frac{1}{2n^{2(H_1+H_2)}} \left((k+1)^{2H_1} - k^{2H_1} \right) I_1 \left(\epsilon_{\frac{k}{n}, \frac{l}{n}}^{(1)} \right) + \left\langle \epsilon_{\frac{k}{n}, \frac{l}{n}}^{(1)}, \epsilon_{\frac{k}{n}, \frac{l}{n}}^{(2)} \right\rangle_{\mathbb{H}} I_1 \left(\epsilon_{\frac{k}{n}, \frac{l}{n}}^{(3)} \right) \right]. \end{aligned}$$

By the isometric formula (2.4) of the symmetric functions for multiple stochastic integral, the L^2 -norm of $G_{1,n}$ is given by $E[G_{1,n}^2] = J_1(n; H) + R_1(n; H)$, where

$$J_1(n; H) = a_n^2(H) \sum_{i,j,k,l=0}^{n-1} \left\langle \epsilon_{\frac{i}{n}, \frac{j}{n}}^{(1)}, \epsilon_{\frac{k}{n}, \frac{l}{n}}^{(1)} \right\rangle_{\mathbb{H}} \left\langle \epsilon_{\frac{i}{n}, \frac{j}{n}}^{(2)}, \epsilon_{\frac{k}{n}, \frac{l}{n}}^{(2)} \right\rangle_{\mathbb{H}} \left\langle \epsilon_{\frac{i}{n}, \frac{j}{n}}^{(3)}, \epsilon_{\frac{k}{n}, \frac{l}{n}}^{(3)} \right\rangle_{\mathbb{H}},$$

and the remaining term $R_1(n; H)$ consists of the four sums having the summands of the following forms: for $a = b, c \neq d, e \neq f$ or $a \neq b, c \neq d, e \neq f$

$$\left\langle \epsilon_{\frac{i}{n}, \frac{j}{n}}^{(a)}, \epsilon_{\frac{k}{n}, \frac{l}{n}}^{(b)} \right\rangle_{\mathbb{H}} \left\langle \epsilon_{\frac{i}{n}, \frac{j}{n}}^{(c)}, \epsilon_{\frac{k}{n}, \frac{l}{n}}^{(d)} \right\rangle_{\mathbb{H}} \left\langle \epsilon_{\frac{i}{n}, \frac{j}{n}}^{(e)}, \epsilon_{\frac{k}{n}, \frac{l}{n}}^{(f)} \right\rangle_{\mathbb{H}}.$$

For simplicity, we introduce the functions that will be used throughout this paper: for $r = 1, 2$,

$$f_r(x, y) = x^{2H_r} + y^{2H_r} - |x - y|^{2H_r},$$

$$\rho_r(x - y) = |x - y + 1|^{2H_r} + |x - y - 1|^{2H_r} - 2|x - y|^{2H_r}.$$

By the limit of Riemann sums, we have

$$\begin{aligned} J_1(n; H) &= \frac{1}{4^3} \prod_{r=1}^2 \left(\sum_{i,k=0}^{n-1} f_r\left(\frac{i}{n}, \frac{k}{n}\right) \left(\frac{1}{n}\right) \rho_r^2(i - k) \right) \\ &= \frac{1}{4^3} \prod_{r=1}^2 \left(\sum_{p=-\infty}^{\infty} \rho_r^2(p) \sum_{i=0 \vee (-p)}^{(n-1) \wedge (n-1-p)} f_r\left(\frac{i}{n}, \frac{i+p}{n}\right) \frac{1}{n} \right) \\ &\rightarrow \frac{1}{4^3} \prod_{r=1}^2 \frac{1}{2(4H_r + 1)} \sum_{p=-\infty}^{\infty} \rho_r^2(p), \quad \text{for } 0 < H_1, H_2 < \frac{3}{4}. \end{aligned} \quad (3.1)$$

By the similar argument as for $J_1(n; H)$ and the following inequality

$$\left| \left\langle \epsilon_{\frac{i}{n}, \frac{j}{n}}^{(a)}, \epsilon_{\frac{k}{n}, \frac{l}{n}}^{(b)} \right\rangle_{\mathbb{H}} \right| \leq C \times \begin{cases} \frac{1}{n^{2(H_1+H_2)}}, & \text{if } a = 1, b = 2, \\ \frac{|\rho_1(i - k)|}{n^{2(H_1+H_2)}}, & \text{if } a = 1, b = 3, \\ \frac{|\rho_2(j - l)|}{n^{2(H_1+H_2)}}, & \text{if } a = 2, b = 3, \end{cases} \quad (3.2)$$

we can easily show that $R_{1,n}(H)$ converges to zero.

Now we consider the sequence $G_{2,n}$. By the mean value theorem, the L^2 -norm of the first term in $G_{2,n}$ can be bounded by

$$Cn^{2H_2-1} \vartheta_1(n; H) \sum_{p=-\infty}^{\infty} |\rho_2(p)| \vartheta_2(n; p, H), \quad (3.3)$$

where the sequences $\vartheta_1(n; H)$ and $\vartheta_2(n; p, H)$ satisfy

$$\begin{aligned} \vartheta_1(n; H) &= \sum_{i,k=0}^{n-1} \left| f_1\left(\frac{i}{n}, \frac{k}{n}\right) \right| \left(\frac{1}{n^2}\right) \rightarrow \int_0^1 \int_0^1 |f_1(x, y)| dx dy, \\ \vartheta_2(n; p, H) &= \sum_{i=0 \vee (-p)}^{(n-1) \wedge (n-1-p)} \left(\frac{j(j+p)}{n^2} \right)^{2H_2-1} \left(\frac{1}{n}\right) \rightarrow \frac{1}{4H_2-1}, \quad \text{if } \frac{1}{4} < H_2 < \frac{1}{2}. \end{aligned}$$

Since $1/4 < H_2 < 1/2$, the sequence (3.3) converges to zero. By interchanging the role of H_1 and H_2 in the first term of $G_{2,n}$, we can show that the L^2 -norm of the second term in the sequence $G_{2,n}$ converges to zero if $1/4 < H_1 < 1/2$. By a similar estimate as for $R_1(n; H)$, we can also prove that the L^2 -norm of the third term in $G_{2,n}$ converges to zero if $0 < H_1, H_2 < 1/2$.

4. Proof of Main Theorem

For the proof of main theorem, we need the following theorem (Theorem 4 in Nualart and Ortiz-Latorre (2008) or see Nualart and Peccati (2005)).

Theorem 2. Let $\{F_n = I_k(f_n), n \geq 1\}$, $f_n \in \mathbb{H}^{\otimes k}$ for every $n \geq 1$, be a sequence of square integrable random variables in the k^{th} Wiener chaos such that

$$\mathbb{E}[F_n^2] = \|f_n\|_{\mathbb{H}^{\otimes k}}^2 \rightarrow 1 \quad \text{as } n \rightarrow \infty.$$

Then the followings are equivalent.

- (i) The sequence $\{F_n, n \geq 1\}$ converges to a normal distribution $\mathbb{N}(0, 1)$.
- (ii) $\lim_{n \rightarrow \infty} \mathbb{E}[F_n^4] = 3$.
- (iii) For all $1 \leq l \leq k-1$, $\lim_{n \rightarrow \infty} \|f_n \otimes_l f_n\|_{\mathbb{H}^{\otimes 2(k-l)}} = 0$.
- (iv) $\|DF_n\|_{\mathbb{H}}^2 \rightarrow k$ in $L^2(\Omega)$, where D is the Malliavin derivative with respect to a Brownian sheet $B = \{B_z, z \in [0, 1]^2\}$.

By (i) and (iv) in Theorem 2, we will show the following lemma in order to prove that $G_{1,n} \xrightarrow{\mathcal{L}} \mathbb{N}(0, \sigma^2(H))$.

Lemma 1. If $1/4 < H_1 < 1/2$ and $1/4 < H_2 < 1/2$, we have that as $n \rightarrow \infty$

$$\|DG_{1,n}\|_{\mathbb{H}}^2 \rightarrow 3\sigma^2(H) \quad \text{in } L^2(\Omega),$$

where D is Malliavin derivative corresponding to fBs B^H .

Proof: By the rule of Malliavin derivative of the multiple stochastic integrals, the derivative of $G_{1,n}$ is

$$D_z G_{1,n} = a_n(H) \sum_{k,l=1}^n \left[I_2 \left(\epsilon_{\frac{k}{n}, \frac{l}{n}}^{(2)} \tilde{\otimes} \epsilon_{\frac{k}{n}, \frac{l}{n}}^{(3)} \right) \epsilon_{\frac{k}{n}, \frac{l}{n}}^{(1)}(z) + I_2 \left(\epsilon_{\frac{k}{n}, \frac{l}{n}}^{(1)} \tilde{\otimes} \epsilon_{\frac{k}{n}, \frac{l}{n}}^{(3)} \right) \epsilon_{\frac{k}{n}, \frac{l}{n}}^{(2)}(z) + I_2 \left(\epsilon_{\frac{k}{n}, \frac{l}{n}}^{(1)} \tilde{\otimes} \epsilon_{\frac{k}{n}, \frac{l}{n}}^{(2)} \right) \epsilon_{\frac{k}{n}, \frac{l}{n}}^{(3)}(z) \right].$$

By the formula for the product of the multiple stochastic integrals in Proposition 1, the norm $\|D \bullet G_{1,n}\|_{\mathbb{H}}^2$ can be expressed as

$$\|D \bullet G_{1,n}\|_{\mathbb{H}}^2 = \mathbb{J}_4(n; H) + \mathbb{J}_2(n; H) + \mathbb{J}_0(n; H),$$

where $\mathbb{J}_r(n; H) \in \mathcal{H}_r$. By using the same arguments as for $G_{1,n}$, the constant term $\mathbb{J}_0(n; H)$ can be written as $\mathbb{J}_0(n; H) = 3J_1(n; H) + R_2(n; H)$, where $R_2(n; H)$ converges to zero. Hence we have, by (3.1), that $\lim_{n \rightarrow \infty} \mathbb{J}_0(n; H) = 3\sigma^2(H)$. Similarly as for $E[G_{1,n}^2]$, we can write $E[\mathbb{J}_4^2(n; H)] = J_2(n; H) + R_3(n; H)$, where $R_3(n; H)$ converges to zero and

$$\begin{aligned} J_2(n; H) &= a_n^4(H) \sum \left\langle \epsilon_{\frac{i}{n}, \frac{j}{n}}^{(1)}, \epsilon_{\frac{k}{n}, \frac{l}{n}}^{(1)} \right\rangle_{\mathbb{H}} \left\langle \epsilon_{\frac{i'}{n}, \frac{j'}{n}}^{(1)}, \epsilon_{\frac{k'}{n}, \frac{l'}{n}}^{(1)} \right\rangle_{\mathbb{H}} \left\langle \epsilon_{\frac{i}{n}, \frac{j}{n}}^{(2)}, \epsilon_{\frac{i'}{n}, \frac{j'}{n}}^{(2)} \right\rangle_{\mathbb{H}} \\ &\quad \times \left\langle \epsilon_{\frac{k}{n}, \frac{l}{n}}^{(2)}, \epsilon_{\frac{k'}{n}, \frac{l'}{n}}^{(2)} \right\rangle_{\mathbb{H}} \left\langle \epsilon_{\frac{i}{n}, \frac{j}{n}}^{(3)}, \epsilon_{\frac{i'}{n}, \frac{j'}{n}}^{(3)} \right\rangle_{\mathbb{H}} \left\langle \epsilon_{\frac{k}{n}, \frac{l}{n}}^{(3)}, \epsilon_{\frac{k'}{n}, \frac{l'}{n}}^{(3)} \right\rangle_{\mathbb{H}}. \end{aligned}$$

By a similar argument as for $J_1(n; H)$, it is clear that

$$\begin{aligned} J_2(n; H) &\leq \frac{C}{n} \sum_{p,q,r=-\infty}^{\infty} |\rho_1(p)\rho_1(q)\rho_1(r)| \sum_{k \in \mathbf{D}_{p,q,r,n}} \left| f_1\left(\frac{k+r}{n}, \frac{k+p+r}{n}\right) \right| \times \left| f_1\left(\frac{k}{n}, \frac{k+q}{n}\right) \right| \left(\frac{1}{n} \right) \\ &\quad \times \sum_{p,q=-\infty}^{\infty} \rho_2^2(p)\rho_2^2(q) \sum_{\substack{0 \vee (-p) \leq j \leq (n-1)-p \wedge (n-1) \\ 0 \vee (-q) \leq l \leq (n-1)-q \wedge (n-1)}} \left| f_2\left(\frac{j}{n}, \frac{l}{n}\right) \right| \\ &\quad \times \left| f_2\left(\frac{j+p}{n}, \frac{l+q}{n}\right) \right| \left(\frac{1}{n^2} \right) \rightarrow 0 \quad \text{as } n \rightarrow \infty, \end{aligned}$$

where the set $\mathbf{D}_{p,q,r,n}$ is given by

$$\begin{aligned} \mathbf{D}_{p,q,r,n} = \{i \in \mathbb{Z} : (0 \vee (-p)) \vee ((0 \vee (-q)) - r) \leq i \leq \\ (n-1) \wedge (n-1-p) \wedge (((n-1) \wedge (n-1-q)) - r)\}. \end{aligned}$$

Thus we have that $\lim_{n \rightarrow \infty} E[\mathbb{J}_4^2(n; H)] = 0$. By a similar estimate as for $E[\mathbb{J}_4^2(n; H)]$, we can also prove that $\lim_{n \rightarrow \infty} E[\mathbb{J}_2^2(n; H)] = 0$. Combining the above results, we obtain

$$\begin{aligned} &E\left[\left(\|DG_{1,n}\|_{\mathbb{H}}^2 - 3\sigma^2(H)\right)^2\right] \\ &\leq 3\left\{E\left[\mathbb{J}_4^2(n; H)\right] + E\left[\mathbb{J}_2^2(n; H)\right] + \left(\mathbb{J}_0(n; H) - 3\sigma^2(H)\right)^2\right\} \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Hence the proof of this lemma is complete. \square

Applying Theorem 2, we have, from Lemma 1, that $G_n \xrightarrow{\mathcal{L}} \mathbb{N}(0, \sigma^2(H))$. The sequence $\{Q_n\}$ in Theorem 1 can be written as

$$Q_n = G_n + \frac{1}{2} \left\{ \left(\int_0^1 B_{s_1,1}^H ds_1 + \int_0^1 B_{1,s_2}^H ds_2 \right) - \left(\sum_{k=1}^n B_{\frac{k}{n},1}^H \frac{1}{n} + \sum_{l=1}^n B_{1,\frac{l}{n}}^H \frac{1}{n} \right) \right\}. \quad (4.1)$$

Note that in $L^2(\Omega)$

$$\sum_{k=0}^{n-1} B_{\frac{k}{n},1}^H \left(\frac{1}{n} \right) \rightarrow \int_0^1 B_{s_1,1}^H ds_1 \quad \text{and} \quad \sum_{l=0}^{n-1} B_{1,\frac{l}{n}}^H \left(\frac{1}{n} \right) \rightarrow \int_0^1 B_{1,s_2}^H ds_2. \quad (4.2)$$

Therefore, from (4.1) and (4.2), it follows that Theorem 1 holds.

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