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On the Hàjek-Rènyi-Type Inequality for Conditionally Associated Random Variables

Jeong-Yeol Choi[a], Hye-Young Seo[a], Jong-Il Baek[1,a]

[a]School of Mathematics and Informational Statistics, Wonkwang University

Abstract

Let $\{\Omega, \mathcal{F}, P\}$ be a probability space and $\{X_n | n \geq 1\}$ be a sequence of random variables defined on it. A finite sequence of random variables $\{X_i | 1 \leq i \leq n\}$ is a conditional associated given $\mathcal{F}$ if for any coordinate-wise nondecreasing functions f and g defined on R^n, $\mathrm{Cov}^{\mathcal{F}}(f(X_1, \ldots, X_n), g(X_1, \ldots, X_n)) \geq 0$ *a.s.* whenever the conditional covariance exists. We obtain the Hàjek-Rènyi-type inequality for conditional associated random variables. In addition, we establish the strong law of large numbers, the three series theorem, integrability of supremum, and a strong growth rate for $\mathcal{F}$-associated random variables.

Keywords: Associated random variables, conditional covariance, conditional associated random variables, Hàjek-Rènyi-type inequality.

1. Introduction

Let $\{\Omega, \mathcal{F}, P\}$ be a probability space and all random variables in this paper are defined on it unless specified otherwise. A finite sequence of random variables $\{X_i | 1 \leq i \leq n\}$is said to be associated if for any coordinate-wise nondecreasing functions f and g defined on R^n,

$$\mathrm{Cov}\,(f(X_1, \ldots, X_n), g(X_1, \ldots, X_n)) \geq 0.$$

Assuming that the covariance exists. An infinite sequence of random variables $\{X_n | n \geq 1\}$ is said to be associated if every finite subsequence is associated. Esary *et al.* (1967) introduced a concept of association. Lately significant efforts have been dedicated to prove reliability theory: Limit theorems and statistics applications for such random variables Birkel (1998) provided some inequalities that Shao and Yu (1996) later generalized. Ioannides and Roussas (1999) established some exponential type inequalities, Oliveira (2005) presented some extensions, and Yang and Chen (2007) made further extensions. Newman and Wright (1981) obtained an invariance principle, and Lin (1997) improved it. Wang and Zhang (2006) developed a nonclassical law of the iterated logarithm, among authors.

Let X and Y be random variables with $EX^2 < \infty$ and $EY^2 < \infty$. Let $\mathcal{F}$ be a sub-σ-algebra of $\mathcal{A}$. Prakasa Rao (2009) defined the notion of the conditional covariance of X and Y given $\mathcal{F}$($\mathcal{F}$-covariance) as

$$\mathrm{Cov}^{\mathcal{F}}(X, Y) = E^{\mathcal{F}}\left(\left(X - E^{\mathcal{F}}X\right)\left(Y - E^{\mathcal{F}}Y\right)\right),$$

where $E^{\mathcal{F}}Z$ denotes the conditional expectation of a random variable Z given $\mathcal{F}$.

In contrast to the ordinary concept of variance, conditional variance of X given $\mathcal{F}$ is defined as $\mathrm{Var}^{\mathcal{F}}X \equiv \mathrm{Cov}^{\mathcal{F}}(X, X)$. On the basis of the above definition of conditional covariance, Prakasa

[1] Corresponding author: Professor, School of Mathematics and Informational Statistics and Institute of Basic Natural Science, Wonkwang University, Ik-San, Chunbuk 570-749, South Korea. E-mail: jibaek@wku.ac.kr

Rao proposed a new kind of dependence called conditional association, which is an extension to the corresponding non-conditional case.

Definition 1. *A finite sequence of random variables $\{X_i|1 \leq i \leq n\}$ is said to be conditional associated given $\mathcal{F}$ ($\mathcal{F}$-associated) if for any two coordinate-wise nondecreasing functions f and g defined on R^n,*

$$Cov^{\mathcal{F}}\,(f(X_1, X_2, \ldots, X_n), g(X_1, X_2, \ldots, X_n)) \geq 0, \quad a.s.$$

whenever the conditional covariance exists. An infinite sequence of random variables $\{X_n|n \geq 1\}$ is said to be $\mathcal{F}$-associated if every finite subsequence is $\mathcal{F}$-associated. Yuan and Yang (2011) presented the relation between (positive) association and conditional association where the association does not imply the conditional association, and vice versa.

Hàjek-Rènyi (1955) proved the following important inequality. If $\{X_n, n \geq 1\}$ is sequence of independent random variables with mean zero and finite second moments, and $\{b_n, n \geq 1\}$ is a sequence of positive nondecreasing real numbers, then for any $\varepsilon > 0$,

$$P\left(\max_{m \leq k \leq n} \left|\frac{\sum_{j=1}^{k} X_j}{b_k}\right| > \varepsilon\right) \leq \varepsilon^{-2}\left(\sum_{j=m+1}^{n} \frac{EX_j^2}{b_j^2} + \frac{1}{b_m^2}\sum_{j=1}^{m} EX_j^2\right).$$

Many authors have studied the above inequality (Gan, 1997; Liu *et al.*, 1999; Cai, 2000; Prakasa Rao, 2002; Hu *et al.*, (2005); Qiu and Gan, 2005; Rao, 2002; Sung, 2008; *etc*). Recently, Yuan and Yang extended the Hàjek-Rènyi inequality to conditional associated random variables.

This paper develops a Hàjek-Rènyi-type inequality for $\mathcal{F}$-associated random variables and uses a proof method different from Yuan and Yang. Using this result, we obtain the strong law of large numbers, three series theorem, and integrability of supremum for $\mathcal{F}$-associated random variables. Finally, throughout this paper, c will represent positive constants where their value may change from one place to another.

2. Conditional Hàjek-Rènyi-Type Inequality

To prove the Hàjek-Rènyi-type inequality for $\mathcal{F}$-associated random variables, we need the following Lemma 1.

Lemma 1. *Let $\{X_i|1 \leq i \leq n\}$ be a sequence of $\mathcal{F}$-associated random variables with $E^{\mathcal{F}} X_k = 0$ and $E^{\mathcal{F}} X_k^2 < \infty$ for each k with $1 \leq k \leq n$. Then for an arbitrary $\mathcal{F}$-measurable random variables $\varepsilon > 0$ a.s.,*

$$P\left(\max_{1 \leq k \leq n} |S_k| \geq \varepsilon \,|\, \mathcal{F}\right) \leq cE^{\mathcal{F}} S_n^2,$$

where $S_n = \sum_{i=1}^{n} X_i$.

Proof: By applying proof of Newman and Wright (1982), we obtain that for $\varepsilon > 0$ *a.s.* is an arbitrary $\mathcal{F}$-measurable random variable,

$$\begin{aligned} P(\max(0, S_1, S_2, \ldots, S_n) \geq \varepsilon \,|\, \mathcal{F}) &\leq cE^{\mathcal{F}}\,(\max(0, S_1, \ldots, S_n))^2 \\ &\leq cE^{\mathcal{F}} S_n^2. \end{aligned}$$

Note that $-X_1, \ldots, -X_n$ are also $\mathcal{F}$-associated random variables, replacing random variables $X_1, \ldots, X_n$ by $-X_1, \ldots, -X_n$, we obtain that for $\varepsilon > 0$ *a.s.* is an arbitrary $\mathcal{F}$-measurable random variable,

$$P(\max(0, -S_1, -S_2, \ldots, -S_n) \geq \varepsilon \,|\, \mathcal{F}) \leq cE^{\mathcal{F}} S_n^2.$$

Hence,

$$\begin{aligned} P\left(\max_{1\leq k\leq n} |S_k| \geq \varepsilon \,|\, \mathcal{F}\right) &\leq P\left(\max(0, S_1, \ldots, S_n) \geq \frac{\varepsilon}{2} \,\Big|\, \mathcal{F}\right) + P\left(\max(0, -S_1, \ldots, -S_n) \geq \frac{\varepsilon}{2} \,\Big|\, \mathcal{F}\right) \\ &\leq cE^{\mathcal{F}} S_n^2. \end{aligned}$$

The proof is complete $\square$

First, we will give a Hàjek-Rènyi-type inequality for $\mathcal{F}$-associated random variables.

Theorem 1. *Let $\{X_n | n \geq 1\}$ be a sequence of $\mathcal{F}$-associated random variables with $E^{\mathcal{F}} X_i = 0$ and $E^{\mathcal{F}} X_i^2 < \infty$ a.s. for each $i \geq 1$ and let $\{b_n | n \geq 1\}$ be a sequence of positive nondecreasing real numbers. Then for an arbitrary $\mathcal{F}$-measurable random variables $\varepsilon > 0$ a.s. and positive integer $m \leq n$,*

$$\begin{aligned} &P\left(\max_{m\leq k\leq n} \frac{1}{b_k} \sum_{i=1}^{k} \left(X_i - E^{\mathcal{F}} X_i\right) \geq \varepsilon \,\Big|\, \mathcal{F}\right) \\ &\leq c\left(\sum_{i=1}^{m} \frac{Var^{\mathcal{F}} X_i}{b_m^2} + \sum_{1\leq i\neq k\leq m} \frac{Cov^{\mathcal{F}}(X_i, X_n)}{b_m^2} + \sum_{j=m+1}^{n} \frac{Var^{\mathcal{F}} X_j}{b_j^2} + \sum_{m+1\leq j\neq k\leq n} \frac{Cov^{\mathcal{F}}(X_j, X_k)}{b_j b_k}\right) \ a.s. \end{aligned}$$

Proof: Noting that $\{X_n | n \geq 1\}$ is an $\mathcal{F}$-associated random variables implies $\{(X_n - E^{\mathcal{F}} X_n) | n \geq 1\}$ is an also $\mathcal{F}$-associated random variables and that $\varepsilon > 0$ *a.s.* is an $\mathcal{F}$-measurable random variable, we obtain

$$\begin{aligned} &P\left(\max_{m\leq k\leq n} \left|\frac{1}{b_k}\left(\sum_{i=1}^{k} \left(X_i - E^{\mathcal{F}} X_i\right)\right)\right| \geq \varepsilon \,\Big|\, \mathcal{F}\right) \\ &= P\left(\max_{m\leq k\leq n} \left|\frac{1}{b_k}\left(\sum_{i=1}^{m} \left(X_i - E^{\mathcal{F}} X_i\right) + \sum_{i=1}^{k} \left(X_i - E^{\mathcal{F}} X_i\right) - \sum_{i=1}^{m} \left(X_i - E^{\mathcal{F}} X_i\right)\right)\right| \geq \varepsilon \,\Big|\, \mathcal{F}\right) \\ &\leq P\left(\max_{m\leq k\leq n} \left|\frac{1}{b_k}\left(\sum_{i=1}^{m} \left(X_i - E^{\mathcal{F}} X_i\right)\right)\right| \geq \frac{\varepsilon}{2} \,\Big|\, \mathcal{F}\right) + P\left(\max_{m\leq k\leq n} \left|\frac{1}{b_k}\left(\sum_{i=1}^{k} \left(X_i - E^{\mathcal{F}} X_i\right) - \sum_{i=1}^{m} \left(X_i - E^{\mathcal{F}} X_i\right)\right)\right| \geq \frac{\varepsilon}{2} \,\Big|\, \mathcal{F}\right) \\ &=: I_1 + I_2. \end{aligned}$$

As to I_1, using Lemma 1, we obtain

$$\begin{aligned} I_1 &= P\left(\max_{m\leq k\leq n} \left|\frac{1}{b_k}\left(\sum_{i=1}^{m} \left(X_i - E^{\mathcal{F}} X_i\right)\right)\right| \geq \frac{\varepsilon}{2} \,\Big|\, \mathcal{F}\right) \\ &\leq P\left(\frac{1}{b_m} \left|\sum_{i=1}^{m} \left(X_i - E^{\mathcal{F}} X_i\right)\right| \geq \frac{\varepsilon}{2} \,\Big|\, \mathcal{F}\right) \\ &\leq P\left(\frac{1}{b_m} \max_{1\leq k\leq m} \left|\sum_{i=1}^{k} \left(X_i - E^{\mathcal{F}} X_i\right)\right| \geq \frac{\varepsilon}{2} \,\Big|\, \mathcal{F}\right) \end{aligned}$$

$$
\begin{aligned}
&\le \frac{c}{b_m^2} E^{\mathcal{F}} \left(\sum_{i=1}^{m} X_i - E^{\mathcal{F}} X_i \right)^2 \\
&= \frac{c}{b_m^2} \mathrm{Var}^{\mathcal{F}} \left(\sum_{i=1}^{m} X_i \right) \\
&= c \left(\frac{1}{b_m^2} \sum_{i=1}^{m} \mathrm{Var}^{\mathcal{F}} X_i + \frac{1}{b_m^2} \sum_{1 \le i \ne k}^{m} \mathrm{Cov}^{\mathcal{F}} \left(X_i, X_k \right) \right).
\end{aligned}
$$

Next, noting that

$$
\max_{m \le k \le n} \frac{\left| \sum_{i=1}^{k} \left(X_i - E^{\mathcal{F}} X_i \right) - \sum_{i=1}^{m} \left(X_i - E^{\mathcal{F}} X_i \right) \right|}{b_k} = \max_{1 \le k \le n-m} \frac{\left| \sum_{j=1}^{k} \left(X_{m+j} - E^{\mathcal{F}} X_{m+j} \right) \right|}{b_{m+k}},
$$

and using Lemma 1, we have

$$
\begin{aligned}
I_2 &= P \left(\max_{m \le k \le n} \frac{\left| \sum_{i=1}^{k} \left(X_i - E^{\mathcal{F}} X_i \right) - \sum_{i=1}^{m} \left(X_i - E^{\mathcal{F}} X_i \right) \right|}{b_k} \ge \frac{\varepsilon}{2} \middle| \mathcal{F} \right) \\
&= P \left(\max_{1 \le k \le n-m} \frac{\left| \sum_{j=1}^{k} \left(X_{m+j} - E^{\mathcal{F}} X_{m+j} \right) \right|}{b_{m+k}} \ge \frac{\varepsilon}{2} \middle| \mathcal{F} \right) \\
&\le \frac{c E^{\mathcal{F}} \left(\sum_{j=1}^{n-m} \left(X_{m+j} - E^{\mathcal{F}} X_{m+j} \right) \right)^2}{b_{m+j}^2} \\
&= \frac{c E^{\mathcal{F}} \left(\sum_{j=m+1}^{n} \left(X_j - E^{\mathcal{F}} X_j \right) \right)^2}{b_j^2} \\
&= \frac{c \, \mathrm{Var}^{\mathcal{F}} \left(\sum_{j=m+1}^{n} X_j \right)}{b_j^2} \\
&= c \left(\sum_{j=m+1}^{n} \frac{\mathrm{Var}^{\mathcal{F}} X_j}{b_j^2} + \sum_{m+1 \le j \ne k}^{n} \frac{\mathrm{Cov}^{\mathcal{F}} \left(X_j, X_n \right)}{b_j b_k} \right).
\end{aligned}
$$

The proof is complete. □

As corollary to Theorem 1, we obtain the following results.

Corollary 1. *In Theorem 1, if $b_k = k$, $k = 1, 2, \ldots, m$, then we obtain the following inequality.*

$$
\begin{aligned}
&P \left(\max_{m \le k \le n} \left| \frac{\sum_{i=1}^{k} \left(X_m - E^{\mathcal{F}} X_i \right)}{k} \right| \ge \varepsilon \middle| \mathcal{F} \right) \\
&\le c \left(\sum_{i=1}^{m} \frac{Var^{\mathcal{F}} X_i}{m^2} + \sum_{1 \le j \ne k}^{n} \frac{Cov^{\mathcal{F}} \left(X_j, X_k \right)}{m^2} + \sum_{j=m+1}^{m} \frac{Var^{\mathcal{F}} X_j}{j^2} + \sum_{m+1 \le j \ne k} \frac{Cov^{\mathcal{F}} \left(X_j, X_k \right)}{jk} \right) \ a.s.
\end{aligned}
$$

3. Almost Sure Convergence for $\mathcal{F}$-Associated Random Variables

In this section, we will give the strong law of large numbers, the three series theorem, integrability of supremum, and a strong growth rate for $\mathcal{F}$-associated random variables by using the results which we have obtained in Section 2.

Theorem 2. *Let $\{X_n | n \geq 1\}$ be a sequence of $\mathcal{F}$-associated random variables such that*

$$\sum_{j=1}^{\infty} Var^{\mathcal{F}} X_j + \sum_{1 \leq j \neq k}^{\infty} Cov^{\mathcal{F}}(X_j, X_k) < \infty \ \ a.s.$$

Then, conditionally on $\mathcal{F}$,

$$\sum_{j=1}^{n} \left(X_j - E^{\mathcal{F}} X_j\right) \to 0 \ \ a.s. \quad as\ n \to \infty.$$

Proof: Without loss of generality, we may assume that $E^{\mathcal{F}} X_j = 0$ for all $j \geq 1$, and let $\varepsilon > 0$ *a.s.* be an $\mathcal{F}$-measurable random variable. Then by Lemma 1 and Theorem 1,

$$\begin{aligned}
& P\left(\max_{k,m \geq n} \left| \sum_{j=1}^{k} \left(X_j - E^{\mathcal{F}} X_j\right) - \sum_{j=1}^{m} \left(X_j - E^{\mathcal{F}} X_j\right) \right| \geq \varepsilon \,\middle|\, \mathcal{F}\right) \\
& \leq P\left(\max_{k \geq n} \left| \sum_{j=1}^{k} \left(X_j - E^{\mathcal{F}} X_j\right) - \sum_{j=1}^{m} \left(X_j - E^{\mathcal{F}} X_j\right) \right| \geq \frac{\varepsilon}{2} \,\middle|\, \mathcal{F}\right) \\
& \quad + P\left(\max_{m \geq n} \left| \sum_{j=1}^{m} \left(X_j - E^{\mathcal{F}} X_j\right) - \sum_{j=1}^{n} \left(X_j - E^{\mathcal{F}} X_j\right) \right| \geq \frac{\varepsilon}{2} \,\middle|\, \mathcal{F}\right) \\
& \leq c \lim_{N \to \infty} P\left(\max_{n \leq k \leq N} \left| \sum_{j=1}^{k} \left(X_j - E^{\mathcal{F}} X_j\right) - \sum_{j=1}^{n} \left(X_j - E^{\mathcal{F}} X_j\right) \right| \geq \frac{\varepsilon}{2} \,\middle|\, \mathcal{F}\right) \\
& \leq c \lim_{N \to \infty} E^{\mathcal{F}}\left(\max_{n \leq k \leq N} \left| \sum_{j=1}^{k} \left(X_j - E^{\mathcal{F}} X_j\right) - \sum_{j=1}^{n} \left(X_j - E^{\mathcal{F}} X_j\right) \right|^2\right) \\
& \leq c \left(\sum_{j=n}^{\infty} \mathrm{Var}^{\mathcal{F}} X_j + \sum_{n \leq j \neq k}^{\infty} \mathrm{Cov}^{\mathcal{F}}\left(X_j, X_k\right)\right) < \infty \ a.s.
\end{aligned}$$

Hence, the sequence of $\mathcal{F}$-associated random variables $\{\sum_{j=1}^{n}(X_j - E^{\mathcal{F}} X_j) | n \geq 1\}$ is a Cauchy which implies that $\sum_{j=1}^{n}(X_j - E^{\mathcal{F}} X_j) \to 0$ *a.s. as* $n \to \infty$.

The proof is complete. □

To prove Theorem 3, we need the following conditional version of Borel-cantelli lemma that is proved by Majerak *et al.* (2005).

Lemma 2. *Majerak et al. (2005) Let $\{\Omega, \mathcal{F}, P\}$ be a probability space and let $\mathcal{F}$ be a sub-σ-algebra of $\mathcal{A}$. Then the following results hold.*

(i) *Let* $\{A_n|n \geq 1\}$ *be a sequence of events such that* $\sum_{n=1}^{\infty} P(A_n) < \infty$. *Then* $\sum_{n=1}^{\infty} P(A_n|\mathcal{F}) < \infty$ *a.s.*

(ii) *Let* $\{A_n|n \geq 1\}$ *be a sequence of events and let* $A = \{\omega| \sum_{n=1}^{\infty} P(A_n|\mathcal{F}) < \infty\}$ *with* $P(A) < 1$. *Then, only finitely many events from the sequence* $\{A_n \cap A, n \geq 1\}$ *hold with probability one, namely* $P(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} (A_k \bigcap A)) = 0$.

Theorem 3. *(The three series theorem). Let* $\{X_n|n \geq 1\}$ *be a sequence of* $\mathcal{F}$*-associated random variables such that*

(1) $\sum_{j=1}^{\infty} Var^{\mathcal{F}} X_j^c + \sum_{1 \leq j \neq k}^{\infty} Cov^{\mathcal{F}}(X_j^c, X_k^c) < \infty$ *a.s.*

(2) $\sum_{j=1}^{\infty} E^{\mathcal{F}} X_j^c < \infty$ *a.s.*

(3) $\sum_{j=1}^{\infty} P(|X_j| \geq c|\mathcal{F}) < \infty$ *for some constant* $c > 0$.

Then $\sum_{j=1}^{n} X_j \to 0$ *a.s. as* $n \to \infty$.

Proof: Let $X_j^c = X_j I(|X_j| \leq c)$. Then, by Theorem 2 and (1), it follows that for condition on $\mathcal{F}$,

$$\sum_{j=1}^{n} \left(X_j^c - E^{\mathcal{F}} X_j^c\right) \to 0 \ \ a.s. \quad as \ \ n \to \infty,$$

and from (2), $\sum_{j=1}^{n} X_j^c \to 0$ *a.s. as*.

Which together with (3), implies that $n \to \infty$.

$$\sum_{j=1}^{\infty} P\left(X_j \neq X_j^c \,|\, \mathcal{F}\right) = \sum_{j=1}^{\infty} P\left(|X_j| \geq c \,|\, \mathcal{F}\right) < \infty.$$

Hence, by Lemma 2, we obtain that

$$\sum_{j=1}^{n} X_j \to 0 \ \ a.s. \quad as \ \ n \to \infty.$$

The proof is complete. □

Theorem 4. *Let* $\{X_n|n \geq 1\}$ *be a sequence of* $\mathcal{F}$*-associated random variables such that*

$$\sum_{j=1}^{\infty} \frac{Var^{\mathcal{F}} X_j}{b_j^2} + \sum_{1 \leq j \neq k}^{\infty} \frac{Cov^{\mathcal{F}}(X_j, X_k)}{b_j b_k} < \infty \ a.s.$$

and let $\{b_n|n \geq 1\}$ *be a sequence of positive nondecreasing real numbers. Then,*

(a) *For* $0 < r < 2$ *and conditionally on* $\mathcal{F}$,

$$E^{\mathcal{F}} \sup_{n \geq 1} \left(\frac{\sum_{i=1}^{n} \left|\left(X_i - E^{\mathcal{F}} X_i\right)\right|}{b_n} \right)^r < \infty \ a.s.$$

(b) *If* $0 < b_n \to \infty$, *then* $\sum_{i=1}^{n} (X_i - E^{\mathcal{F}} X_i)/b_n \to 0$ *a.s. as* $n \to \infty$.

Proof: Proof of (a). Noting that

$$E^{\mathcal{F}} \sup_{n\geq 1}\left(\frac{\sum_{i=1}^{n}\left|\left(X_i - E^{\mathcal{F}}X_i\right)\right|}{b_n}\right)^r < \infty \ a.s.$$

$$\Leftrightarrow \int_1^{\infty} P\left(\sup_{n\geq 1}\frac{\left|\sum_{j=1}^{n}\left(X_j - E^{\mathcal{F}}X_j\right)\right|}{b_n} \geq t^{\frac{1}{r}} \middle| \mathcal{F}\right) dt < \infty \ a.s.$$

by Lemma 1 and Theorem 1, we obtain that

$$\begin{aligned}
&\int_1^{\infty} P\left(\sup_{n\geq 1}\frac{\left|\sum_{j=1}^{n}\left(X_j - E^{\mathcal{F}}X_j\right)\right|}{b_n} \geq t^{\frac{1}{r}} \middle| \mathcal{F}\right) dt \\
&\leq c\int_1^{\infty} t^{-\frac{2}{r}}\left(\sum_{j=1}^{\infty}\frac{\mathrm{Var}^{\mathcal{F}}X_j}{b_j^2} + \sum_{1<j\neq k}\frac{\mathrm{Cov}^{\mathcal{F}}(X_j, X_k)}{b_j b_k}\right)dt \\
&= c\left(\sum_{j=1}^{\infty}\frac{\mathrm{Var}^{\mathcal{F}}X_j}{b_j^2} + \sum_{1\leq j\neq k}^{\infty}\frac{\mathrm{Cov}^{\mathcal{F}}\left(X_j, X_k\right)}{b_j b_k}\right)\int_1^{\infty} t^{-\frac{2}{r}} < \infty \ a.s.
\end{aligned}$$

Proof of (b). For $\varepsilon > 0$ $a.s.$ is an $\mathcal{F}$-measurable random variable, by Lemma 1 and Theorem 1,

$$\begin{aligned}
&P\left(\max_{m\leq k\leq n}\frac{\left|\sum_{j=1}^{k}X_j - E^{\mathcal{F}}X_j\right|}{b_k} \geq \varepsilon \middle| \mathcal{F}\right) \\
&\leq c\left(\frac{E^{\mathcal{F}}\left(\sum_{j=1}^{m}\left(X_j - E^{\mathcal{F}}X_j\right)\right)^2}{b_m^2} + \frac{E^{\mathcal{F}}\left(\sum_{j=m+1}^{n}\left(X_j - E^{\mathcal{F}}X_j\right)\right)^2}{b_j^2}\right) \\
&= c\left(\frac{\sum_{j=1}^{m}\mathrm{Var}^{\mathcal{F}}X_j}{b_m^2} + \sum_{1\leq j\neq k}\frac{\mathrm{Cov}^{\mathcal{F}}\left(X_j, X_k\right)}{b_m^2} + \sum_{j=m+1}^{n}\frac{\mathrm{Var}^{\mathcal{F}}X_j}{b_j^2} + \sum_{m+1\leq j\neq k}\frac{\mathrm{Cov}^{\mathcal{F}}\left(X_j, X_k\right)}{b_j b_k}\right).
\end{aligned}$$

But for $\varepsilon > 0$ $a.s.$ is an $\mathcal{F}$-measurable random variable,

$$\begin{aligned}
&P\left(\bigcup_{n=m}\max_{m\leq k\leq n}\frac{\left|\sum_{j=1}^{k}\left(X_j - E^{\mathcal{F}}X_j\right)\right|}{b_k} \geq \varepsilon \middle| \mathcal{F}\right) \\
&= \lim_{n\to\infty} P\left(\max_{m\leq k\leq n}\frac{\left|\sum_{j=1}^{k}\left(X_j - E^{\mathcal{F}}X_j\right)\right|}{b_k} \geq \varepsilon \middle| \mathcal{F}\right) \\
&\leq c\left(\frac{E^{\mathcal{F}}\left(\sum_{j=1}^{m}\left(X_j - E^{\mathcal{F}}X_j\right)\right)^2}{b_m^2} + \frac{E^{\mathcal{F}}\left(\sum_{j=m+1}^{n}\left(X_j - E^{\mathcal{F}}X_j\right)\right)^2}{b_j^2}\right) \\
&= c\left(\frac{\sum_{j=1}^{m}\mathrm{Var}^{\mathcal{F}}X_j}{b_m^2} + \sum_{1\leq j\neq k}\frac{\mathrm{Cov}^{\mathcal{F}}\left(X_j, X_k\right)}{b_m^2} + \sum_{j=m+1}^{n}\frac{\mathrm{Var}^{\mathcal{F}}X_j}{b_j^2} + \sum_{m+1\leq j\neq k}\frac{\mathrm{Cov}^{\mathcal{F}}\left(X_i, X_k\right)}{b_j b_k}\right).
\end{aligned} \tag{3.1}$$

By the kronecker Lemma and $\sum_{j=1}^{\infty} \mathrm{Var}^{\mathcal{F}} X_j b_j^2 + \sum_{1\le j\ne k} \mathrm{Cov}^{\mathcal{F}}(X_j, X_k)/(b_j b_k) < \infty$ *a.s.*, we obtain that

$$\sum_{j=1}^{m} \frac{\mathrm{Var}^{\mathcal{F}} X_j}{b_m^2} + \sum_{1\le j\ne k}^{m} \frac{\mathrm{Cov}^{\mathcal{F}}(X_j, X_k)}{b_m^2} \to 0 \quad as\ m \to \infty. \tag{3.2}$$

Hence, by (3.1) and (3.2), we obtain that

$$\lim_{n\to\infty} P\left(\sup_{k\ge n} \frac{\sum_{j=1}^{k} \left(X_j - E^{\mathcal{F}} X_j\right)}{b_k} \ge \varepsilon \,\middle|\, \mathcal{F} \right) = 0,$$

$$i.e,\ \frac{\sum_{j=1}^{n} \left(X_j - E^{\mathcal{F}} X_j\right)}{b_n} \to 0\ a.s. \quad as\ n \to \infty.$$

The proof is complete. □

Remark 1. Taking $b_n = 1$, we can obtain the result

$$\sum_{j=1}^{n} \left(X_j - E^{\mathcal{F}} X_j\right) \to 0\ a.s. \quad as\ n \to \infty.$$

Finally, we obtain the almost sure convergence of weighted sums of sequence of $\mathcal{F}$-associated random variables.

Theorem 5. *Assume that $\{X_n | n \ge 1\}$ be a sequence of $\mathcal{F}$-associated random variables satisfying*

$$\sum_{j=1}^{\infty} Var^{\mathcal{F}} X_j + \sum_{1\le j\ne k} Cov^{\mathcal{F}}(X_j, X_k) < \infty\ a.s.$$

and let $\{a_{ni} | 1 \le i \le n, n \ge 1\}$ be a sequence of real numbers such that $a_{ni} = 0, i > n, \sup_{n\ge1} \sum_{i=1}^{n} |a_{ni}| < \infty$ and let $\{b_n | n \ge 1\}$ be a sequence of positive nondecreasing real numbers such that $0 < b_n \to \infty$. Then

$$\frac{\sum_{i=1}^{n} a_{ni} X_i}{b_n} \to 0\ a.s. \quad as\ n \to \infty.$$

Proof: Note that $\{a_{ni} X_i | 1 \le i \le n, n \ge 1\}$ is a sequence of $\mathcal{F}$-associated random variables and let

$$S_k = \sum_{i=1}^{k} \frac{X_i}{b_k}, \qquad c_{ni} = \frac{b_i}{b_n}(a_{ni} - a_{ni+1}), \quad \text{for } 1 \le i \le n-1,$$

and $c_{nn} = a_{nn}$. Then

$$\sum_{i=1}^{n} \frac{a_{ni} X_i}{b_n} = \sum_{i=1}^{n} c_{ni} S_i, \qquad \sum_{i=1}^{n} |c_{ni}| \le 2 \sup_{n\ge1} \sum_{i=1}^{n} |a_{ni}| \tag{3.3}$$

$$\text{and } \lim_{n\to\infty} |c_{ni}| = 0, \quad \text{for every fixed 1.} \tag{3.4}$$

From (3.3) and (3.4), we obtain for every sequence real numbers d_n with $d_n \to 0$ as $n \to \infty$, $\sum_{j=1}^{n} c_{nj} d_j \to 0$ as $n \to \infty$.

Hence, by Theorem 4(b), (3.3) and (3.4), we obtain the result of Theorem 5.

The proof is complete. □

Corollary 2. *Let $\{X_n | n \geq 1\}$ be a sequence of $\mathcal{F}$-associated random variables $E^{\mathcal{F}} X_k = 0$ and $\sum_{j=1}^{\infty} Var^{\mathcal{F}} X_j + \sum_{1 \leq j \neq k} Cov^{\mathcal{F}}(X_j, X_k) < \infty$ a.s., and $\{a_{ni} | 1 \leq i \leq n, n \geq 1\}$ be a sequence of real numbers with $a_{ni} = 0, i > n, \sup_{n \geq 1} \sum_{i=1}^{n} |a_{ni}| < \infty$. Then for $0 < t < 1$,*

$$\sum_{i=1}^{n} \frac{a_{ni} X_i}{n^{\frac{1}{t}}} \to 0 \; a.s. \quad as\; n \to \infty.$$

Proof: Note that $\{a_{ni} X_i | 1 \leq i \leq n, n \geq 1\}$ is a sequence of $\mathcal{F}$-associated random variables. Then taking $b_n = n^{1/t}$, from Theorem 5, we obtain the result of Corollary 2. □

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