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Some Characterization Results Based on Dynamic Survival and Failure Entropies

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Abstract

In this paper, we develop some characterization results in terms of survival entropy of the first order statistic. In addition, we generalize the cumulative entropy recently proposed by Di Crescenzo and Logobardi (2009) to a new measure of information (called the failure entropy) and study some properties of it and its dynamic version. Furthermore, power distribution is characterized based on dynamic failure entropy.

Keywords: First order statistic, power distribution, mean past life function, reversed Hazard function.

1. Introduction

Let X be a non-negative absolutely continuous random variable with probability density function f, distribution function F and survival function $\bar{F} = 1 - F$. The random variable X may be thought of as the random lifetime of a system or of a component. The basic measure of the uncertainty contained in random variable X is the Shannon entropy $H(X) = -\int_0^{+\infty} f(x)\log f(x)dx$ (Shannon, 1948). A generalization of $H(X)$ has been proposed by Rényi (1961) as

$$H_\alpha(X) = -\frac{1}{\alpha - 1}\log\int_{-\infty}^{+\infty} f^\alpha(x)dx, \quad \alpha > 0\ (\alpha \neq 1).$$

If a unit is known to have survived up to an age t, then $H(X)$ and $H_\alpha(X)$ is no longer useful in measuring the uncertainty about the remaining lifetime of the unit. Accordingly, Ebrahimi (1996) introduced the entropy of the residual lifetime $X_t = [X - t|X > t]$ as a dynamic measure of uncertainty.

Rao *et al.* (2004) defined a new measure of uncertainty based on the survival function of a random variable X, as

$$\mathcal{E}(X) = -\int_0^{+\infty} \bar{F}(x)\log\bar{F}(x)dx,$$

and called it cumulative residual entropy(CRE). This measure can be applied in reliability and image alignment. CRE is always non-negative; however $H(X)$ and $H_\alpha(X)$ are non-negative for discrete random variables and can be negative in a continuous case. For more properties of CRE, one can refer to Wang and Vemuri (2007) and Rao (2005). Asadi and Zohrevand (2007) proposed a dynamic form of CRE.

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Zografos and Nadarajah (2005) introduced the survival entropy(SE) of order α as

$$\mathcal{E}_\alpha(X) = -\frac{1}{\alpha-1}\log\int_0^{+\infty}\bar{F}^\alpha(x)dx. \tag{1.1}$$

SE parallels Renyi entropy of order α. Abbasnejad *et al.* (2010) proposed dynamic survival entropy(DSE) and showed how DSE are connected with the mean residual life function.

Di Crescenzo and Longobardi (2009) introduced a new measure of information similar to CRE, called it cumulative entropy(CE). CE is defined as

$$\mathcal{CE}(X) = -\int_0^{+\infty}F(x)\log F(x)dx.$$

Further, they proposed a dynamic version of it as $\mathcal{CE}(X,t) = -\int_0^t F(x)/F(t)\log\bar{F}(x)/\bar{F}(t)dx$, which is the CE of $_tX = [t - X|X \le t]$.

In analogy with Di Crescenzo and Longobardi (2009) and Zografos and Nadarajah (2005), we introduce the failure entropy of order α (FE) as

$$\mathcal{FE}_\alpha(X) = -\frac{1}{\alpha-1}\log\int_0^{+\infty}F^\alpha(x)dx.$$

Let $X_1,\ldots,X_n$ are independent and identically distributed(iid) observations from cdf $F(x)$ and pdf $f(x)$. The order statistics is defined by the arrangement of $X_1,\ldots,X_n$ from the smallest to the largest, denoted as $X_{1:n} \le X_{2:n} \le \cdots \le X_{n:n}$. These statistics have been used in a wide range of problems, that include robust statistical estimation, characterization of probability distributions and goodness of fit tests, entropy estimation, analysis of censored data and reliability theory; for more details see Arnold *et al.* (1992), David and Nagaraja (2003), and references therein.

The rest of the paper is organized as follows. Section 2 contains some characterization results based on SE and DSE of first order statistic $X_{1:n}$. In Section 3, we introduce the failure entropy and the dynamic failure entropy(DFE) and show that DFE uniquely determines the parent distribution. In Section 4, the power distribution is characterized in terms of DFE.

2. Characterization Based on SE and DSE of First Order Statistic

This section characterizes exponential and Weibull distributions in terms of SE and DSE of first order statistic.

Let $X_{1:n}$ be the first order statistic in a random sample $X_1,\ldots,X_n$ from a non-negative absolutely continuous random variable X with pdf f, cdf F and survival function $\bar{F}$, then the SE of $X_{1:n}$ is given by

$$\mathcal{E}_\alpha(X_{1:n}) = -\frac{1}{\alpha-1}\log\int_0^{+\infty}\bar{F}^\alpha_{1:n}(x)dx = -\frac{1}{\alpha-1}\log\int_0^{+\infty}\bar{F}^{n\alpha}(x)dx. \tag{2.1}$$

The next lemma compares SE of X and $X_{1:n}$.

Lemma 1. *The following statements hold:*

(a) $\mathcal{E}_\alpha(X_{1:n}) = \dfrac{n\alpha-1}{\alpha-1}\mathcal{E}_{n\alpha}(X).$

(b) $\mathcal{E}_\alpha(X_{1:n}) \geq (\leq)\mathcal{E}_\alpha(X), \quad$ *for* $\alpha > 1$ $(0 < \alpha < 1)$.

Example 1. Suppose X has a pareto distribution with cdf $F(x) = 1 - (\beta/x)^\theta$, $x \geq \beta$, $\theta > 0$, then, we obtain $\mathcal{E}_\alpha(X) = -1/(\alpha-1)\log[\beta/(\theta\alpha-1)]$, $\alpha > 1/\theta$ and $\mathcal{E}_\alpha(X_{1:n}) = -1/(\alpha-1)\log[\beta/(n\theta\alpha-1)]$. So we have $\mathcal{E}_\alpha(X_{1:n}) \geq (\leq)\mathcal{E}_\alpha(X)$, for $\alpha > 1$ $(0 < \alpha < 1)$. Let $\Delta = \mathcal{E}_\alpha(X_{1:n}) - \mathcal{E}_\alpha(X) = 1/(\alpha-1)\log[(n\theta\alpha-1)/(\theta\alpha-1)]$, which is an increasing function of n for $\alpha > \max\{1, 1/\theta\}$.

Note that using $u = \bar{F}(x)$, we have

$$\mathcal{E}_\alpha(X_{1:n}) = -\frac{1}{\alpha-1}\log\int_0^1 \frac{u^{n\alpha}}{f(F^{-1}(1-u))}du. \tag{2.2}$$

Example 2. If X has a Weibull distribution with cdf $F(x) = 1 - e^{-\theta x^\beta}$, $x > 0, \beta, \theta > 0$, then, we have $\mathcal{E}_\alpha(X) = -1/(\alpha-1)\log[\Gamma(1/\beta)/(\theta^{1/\beta}\alpha^{1/\beta}\beta)]$ and $\mathcal{E}_\alpha(X_{1:n}) = -1/(\alpha-1)\log[\Gamma(1/\beta)/(\theta^{1/\beta}n^{1/\beta}\alpha^{1/\beta}\beta)]$. Here $\Delta = \mathcal{E}_\alpha(X_{1:n}) - \mathcal{E}_\alpha(X) = (\log n)/\{\beta(\alpha-1)\}$, which is an increasing (decreasing) function of n for $\alpha > 1$ $(0 < \alpha < 1)$.

If $\alpha > 1$, then by Lemma 1, $\mathcal{E}_\alpha(X)$ is a lower bound for $\mathcal{E}_\alpha(X_{1:n})$. Another lower bound for $\mathcal{E}_\alpha(X_{1:n})$ is given in Abbasnejad *et al.* (2010) as $\mathcal{E}_\alpha(X_{1:n}) \geq -1/(\alpha-1)\log E(X_{1:n}) = -1/(\alpha-1)\log[\Gamma(1/\beta)/(\theta^{1/\beta}n^{1/\beta}\beta)]$. The difference between two lower bounds is $\mathcal{E}_\alpha(X) + 1/(\alpha-1)\log E(X_{1:n}) = 1/\{\beta(\alpha-1)\}\log\alpha/n < 0$, $\forall\ 1 < \alpha < n$. So, for $1 < \alpha < n$, the lower bound of Abbasnejad *et al.* (2010) is sharper.

Two different distributions may have equal survival entropy and a distribution cannot be determined by its SE. In subsequent theorems, we study conditions under which the SE of the first order statistic can uniquely determines the parent distribution.

We need following lemma, which is known as Müntz-Szasz Theorem (See, Kamps, 1998), in the following theorems.

Lemma 2. *For any increasing sequence of positive integers* $\{n_j, j \geq 1\}$*, the sequence of polynomials* $\{x^{n_j}\}$ *is complete on* $L(0,1)$*, if and only if,* $\sum_{j=1}^\infty n_j^{-1} = +\infty$.

In the sequel we assume that $\{n_j, j \geq 1\}$ is a strictly increasing sequence of positive integers.

Theorem 1. *Let X and Y be two random variables with absolutely continuous cdfs $F(x)$ and $G(y)$, pdfs $f(x)$ and $g(y)$, respectively. Then F and G belong to the same family of distributions, but for a change in location and scale if*

$$\mathcal{E}_\alpha(X_{1:n}) - \mathcal{E}_\alpha(X) = \mathcal{E}_\alpha(Y_{1:n}) - \mathcal{E}_\alpha(Y), \tag{2.3}$$

for all $n = n_j$, $j \geq 1$, *such that* $\sum_{j=1}^\infty n_j^{-1} = +\infty$.

Proof: The necessity is obvious. For the sufficiency part, using $u = \bar{F}^\alpha(x)$ in (1.1) and (2.1), we have

$$\mathcal{E}_\alpha(X_{1:n}) - \mathcal{E}_\alpha(X) = -\frac{1}{\alpha-1}\log\frac{\int_0^1\left[u^{n+\frac{1}{\alpha}-1}/f\left(F^{-1}\left(1-u^{\frac{1}{\alpha}}\right)\right)\right]du}{\int_0^1\left[u^{\frac{1}{\alpha}}/f\left(F^{-1}\left(1-u^{\frac{1}{\alpha}}\right)\right)\right]du}.$$

If (2.3) holds, then we get

$$\frac{\int_0^1\left[u^{n+\frac{1}{\alpha}-1}/f\left(F^{-1}\left(1-u^{\frac{1}{\alpha}}\right)\right)\right]du}{\int_0^1\left[u^{\frac{1}{\alpha}}/f\left(F^{-1}\left(1-u^{\frac{1}{\alpha}}\right)\right)\right]du} = \frac{\int_0^1\left[u^{n+\frac{1}{\alpha}-1}/g\left(G^{-1}\left(1-u^{\frac{1}{\alpha}}\right)\right)\right]du}{\int_0^1\left[u^{\frac{1}{\alpha}}/g\left(G^{-1}\left(1-u^{\frac{1}{\alpha}}\right)\right)\right]du}. \tag{2.4}$$

Let $c = \int_0^1 [u^{1/\alpha}/g(G^{-1}(1-u^{1/\alpha}))]du / \int_0^1 [u^{1/\alpha}/f(F^{-1}(1-u^{1/\alpha}))]du$, then (2.4) can be expressed as

$$\int_0^1 u^{n+\frac{1}{\alpha}-1}\left[\frac{1}{f\left(F^{-1}\left(1-u^{\frac{1}{\alpha}}\right)\right)} - \frac{1}{cg\left(G^{-1}\left(1-u^{\frac{1}{\alpha}}\right)\right)}\right]du = 0. \tag{2.5}$$

If (2.5) holds for $n = n_j$, $j \geq 1$, such that $\sum_{j=1}^{\infty} n_j^{-1} = +\infty$, then from Lemma 2 we conclude that $f(F^{-1}(w)) = cg(G^{-1}(w))$ for all $0 < w < 1$. Since $dF^{-1}(w)/dw = 1/f(F^{-1}(w))$, it then follows that $F^{-1}(w) = cG^{-1}(w) + d$. This means F and G belong to the same family of distributions, but for a change in location and scale. □

By Theorem 1, we get the following result that characterizes the exponential distribution.

Corollary 1. *The family of exponential distribution with survival function $\bar{F}(x) = e^{-a(\theta)x}$, $x > 0$ for some positive function $a(\theta)$, can be characterized by the condition*

$$\mathcal{E}_\alpha(X_{1:n}) - \mathcal{E}_\alpha(X) = \frac{1}{\alpha - 1}\log n,$$

for all $n = n_j$, $j \geq 1$, such that $\sum_{j=1}^{\infty} n_j^{-1} = +\infty$.

Similar result is obtained by Baratpour *et al.* (2008) based on Renyi entropy.

Theorem 2. *Let X and Y be two random variables with common support $[0, +\infty)$ and absolutely continuous cdfs $F(x)$ and $G(y)$, pdfs $f(x)$ and $g(y)$, respectively. Then F and G belong to the same family of distributions, but for a change in scale if*

$$\frac{e^{-(\alpha-1)\mathcal{E}_\alpha(X_{1:n})}}{E(X_{1:n})} = \frac{e^{-(\alpha-1)\mathcal{E}_\alpha(Y_{1:n})}}{E(Y_{1:n})},$$

for all $n = n_j$, $j \geq 1$, such that $\sum_{j=1}^{\infty} n_j^{-1} = +\infty$.

Proof: The necessity is trivial. For the sufficiency part, first note that, we can write

$$E(X_{1:n}) = \int_0^{+\infty} \bar{F}_{1:n}(x)dx = \int_0^{+\infty} \bar{F}^n(x)dx = \int_0^1 \frac{u^n}{f\left(F^{-1}(1-u)\right)}du,$$

where the last equality is obtained using $u = \bar{F}(x)$. By (2.2) we have

$$\frac{e^{-(\alpha-1)\mathcal{E}_\alpha(X_{1:n})}}{E(X_{1:n})} = \frac{\int_0^1 \left[u^{n\alpha}/f\left(F^{-1}(1-u)\right)\right]du}{\int_0^1 \left[u^n/f\left(F^{-1}(1-u)\right)\right]du}. \tag{2.6}$$

If (2.6) holds, then we get

$$\frac{\int_0^1 \left[u^{n\alpha}/f\left(F^{-1}(1-u)\right)\right]du}{\int_0^1 \left[u^n/f\left(F^{-1}(1-u)\right)\right]du} = \frac{\int_0^1 \left[u^{n\alpha}/g\left(G^{-1}(1-u)\right)\right]du}{\int_0^1 \left[u^n/g\left(G^{-1}(1-u)\right)\right]du}. \tag{2.7}$$

Let $c = \int_0^1 [u^n/g(G^{-1}(1-u))]du / \int_0^1 [u^n/f(F^{-1}(1-u))]du$, then (2.7) can be expressed as

$$\int_0^1 u^{n\alpha}\left[\frac{1}{f\left(F^{-1}(1-u)\right)} - \frac{1}{cg\left(G^{-1}(1-u)\right)}\right]du = 0. \tag{2.8}$$

If (2.8) holds for $n = n_j$, $j \geq 1$, such that $\sum_{j=1}^{\infty} n_j^{-1} = +\infty$, then from Lemma 2 we have $f(F^{-1}(w)) = cg(G^{-1}(w))$ for all $0 < w < 1$, and similar to Theorem 1, it follows that $F^{-1}(w) = cG^{-1}(w) + d$. Since X and Y have a common support $[0, +\infty)$, we can conclude that $d = 0$, which means F and G belong to the same family of distributions, but for a change in scale. □

Corollary 2. *The family of Weibull distribution with survival function $\bar{F}(x) = e^{-\theta x^{\beta}}$, $x > 0$, $\theta, \beta > 0$, can be characterized by the condition $e^{-(\alpha-1)\mathcal{E}_{\alpha}(X_{1:n})}/E(X_{1:n}) = \alpha^{-1/\beta}$. Note that, for Weibull distribution $e^{-(\alpha-1)\mathcal{E}_{\alpha}(X_{1:n})}/E(X_{1:n}) = e^{-(\alpha-1)\mathcal{E}_{\alpha}(X)}/E(X)$ is constant and does not depend on n.*

Baratpour (2010) showed that F belongs to Weibull family if $\mathrm{CRE}(X_{1:n})/E(X_{1:n})$ is constant and does not depend on n. In addition, Gertsbakh and Kagan (1999) and Zheng (2001) obtained related characterizations of the Weibull family based on the properties of the Fisher information under types I and II censoring.

In the following theorem, we show that the parent distribution can be characterized by DSE of $X_{1:n}$.

Theorem 3. *Under the assumptions of Theorem 1, F and G belong to the same location family of distributions if*

$$\mathcal{E}_{\alpha}(X_{1:n}) = \mathcal{E}_{\alpha}(Y_{1:n}), \quad \forall\, n_j \geq 1,$$

such that $\sum_{j=1}^{\infty} n_j^{-1} = +\infty$.

Proof: The necessity is trivial. For the sufficiency part, if for two cdfs F and G, $\mathcal{E}_{\alpha}(X_{1:n}) = \mathcal{E}_{\alpha}(Y_{1:n})$, we have

$$\int_0^1 \frac{u^{n\alpha}}{f\,(F^{-1}(1-u))} du = \int_0^1 \frac{u^{n\alpha}}{f\,(F^{-1}(1-u))} du,$$

or equivalently,

$$\int_0^1 u^{n\alpha} \left[\frac{1}{f\,(F^{-1}(1-u))} - \frac{1}{g\,(G^{-1}(1-u))} \right] du = 0. \tag{2.9}$$

If (2.9) holds for $n = n_j$, $j \geq 1$, such that $\sum_{j=1}^{\infty} n_j^{-1} = +\infty$, then from Lemma 2 we conclude that $f(F^{-1}(w)) = g(G^{-1}(w))$ for all $0 < w < 1$, and the result follows similar to Theorem 1. □

Let X be the life time of a component under the condition that the component has survived to age t. In such a case, we need to compute the entropy of the residual lifetime $X_t = [X - t | X > t]$. Various dynamic information measures have been proposed to describe the uncertainty of X_t. Ebrahimi (1996) considered the Shannon entropy of X_t. Asadi *et al.* (2005), Abraham and Sankaran (2005) studied the dynamic Renyi entropy of order α and its properties. Asadi and Zohrevand (2007) proposed the dynamic cumulative residual entropy. Abbasnejad *et al.* (2010) introduced the dynamic survival entropy of order α. This is defined as

$$\mathcal{E}_{\alpha}(X, t) = -\frac{1}{\alpha - 1} \log \int_t^{+\infty} \left[\frac{\bar{F}(x)}{\bar{F}(t)} \right]^{\alpha} dx.$$

It is obvious that $\mathcal{E}_{\alpha}(X, t) = \mathcal{E}_{\alpha}(X)$, for $t = 0$.

In the following theorem, we show that the equality of the dynamic survival entropy in first order statistics can determine uniquely the parent distribution.

Theorem 4. *Under the assumptions of Theorem 1, F and G belong to the same family of distributions, but for a change in location and scale if*

$$\mathcal{E}_\alpha(X_{1:n}, t) = \mathcal{E}_\alpha(Y_{1:n}, t), \quad \forall\, n_j \geq 1, \tag{2.10}$$

such that $\sum_{j=1}^{\infty} n_j^{-1} = +\infty$.

Proof: The necessity is trivial. For the sufficiency part, if (2.10) holds, then by Theorem 3 we can conclude that $X|X > t$ and $Y|Y > t$ have a same distribution but for a change in location parameter. This means $f_t(x) = g_t(x + c)$, where f_t and g_t are pdfs of $X|X > t$ and $Y|Y > t$, respectively. Thus, $f(x)/\bar{F}(t) = g(x + c)/\bar{G}(t)$ or equivalently, $f(x) = \bar{F}(t)/\bar{G}(t)\, g(x + c)$, which means F and G belong to the same family of distributions, but for a change in location and scale. □

Remark 1. In a series system consisting of n components with the lifetimes $X_1, \ldots, X_n$, which $X_1, \ldots, X_n$ are continuous and iid random variables, $X_{1:n}$ describes the lifetime of the system. Thus, under the assumptions of Theorems 3 and 4, two series system A and B have the same lifetime distributions, but for a change in location and scale if one of the following statements holds:

(a) $\mathcal{E}_\alpha(X_{1:n}) = \mathcal{E}_\alpha(Y_{1:n})$,

(b) $\mathcal{E}_\alpha(X_{1:n}, t) = \mathcal{E}_\alpha(Y_{1:n}, t)$.

3. Failure Entropy of Order α

In this section, we propose an information measure that is analogous to $\mathcal{E}_\alpha(X)$. For a non-negative random variable X, we define the failure entropy of order α as

$$\mathcal{FE}_\alpha(X) = -\frac{1}{\alpha - 1} \log \int_0^{+\infty} F^\alpha(x)dx, \quad \forall\, \alpha > 0\ (\alpha \neq 1). \tag{3.1}$$

Example 3.

(a) Suppose X has a uniform distribution on (a, b). Then, $\mathcal{FE}_\alpha(X) = -1/(\alpha - 1)\log\{(b - a)/(\alpha + 1)\}$.

(b) Let X have a pareto distribution with cdf $F(x) = 1 - (\beta/x)^\theta$, $x \geq \beta, \theta > 0$, then, $\mathcal{FE}_\alpha(X) = -1/(\alpha - 1)\log[\Gamma(\alpha + 1)\Gamma(2 + 1/\theta)/\Gamma(\alpha + 1/\theta) \cdot \theta/\beta]$.

In the following lemma, we show that $\mathcal{FE}_\alpha$ is a shift-independent measure.

Lemma 3. *If* $Y = aX + b$, *with* $a > 0$ *and* $b \geq 0$, *then*

$$\mathcal{FE}_\alpha(Y) = -\frac{1}{\alpha - 1}\log a + \mathcal{E}_\alpha(X).$$

Proof: The result follows by noting that $F_{aX+b}(x) = F_X((x - b)/a)$, $x \in R$ and (3.1). □

The two dimensional version of (3.1) can be defined as

$$\mathcal{FE}_\alpha(X, Y) = -\frac{1}{\alpha - 1} \log \int_0^{+\infty}\int_0^{+\infty} F^\alpha(x, y)dxdy, \quad \forall\, \alpha > 0\ (\alpha \neq 1).$$

It is easily follows that if X and Y are independent, then

$$\mathcal{FE}_\alpha(X, Y) = \mathcal{FE}_\alpha(X) + \mathcal{FE}_\alpha(Y).$$

Note that the above property holds for Shannon, Renyi and survival entropy.

It is reasonable to presume that in many situations uncertainty is related to the past. In such situations, one usually works with the conditional random variable ${}_tX = [t - X|X \leq t]$ which is usually known as inactivity time (see the Nanda *et al.*, 2003). For instance, if at time t, a system which is not monitored continuously, is found to be failed; then the uncertainty of the system life relies on the past. Based on this idea, various measures have been proposed in the literature. For more details, one can refer to Di Crescenzo and Longobardi (2002, 2004), Nanda and Paul (2006) and Di Crescenzo and Longobardi (2009).

In analogy with (3.1), we define the dynamic failure entropy of order α that computes the uncertainty related to the past. DFE is defined as

$$\mathcal{FE}_\alpha(X, t) = -\frac{1}{\alpha - 1} \log \int_0^t \left[\frac{F(x)}{F(t)}\right]^\alpha dx, \quad \forall\, \alpha > 0\ (\alpha \neq 1), \tag{3.2}$$

which is the FE of ${}_tX = [t - X|X \leq t]$.

Lemma 4. *If $Y = aX + b$, with $a > 0$ and $b \geq 0$, then*

$$\mathcal{FE}_\alpha(Y, t) = -\frac{1}{\alpha - 1} \log a + \mathcal{FE}_\alpha\left(X, \frac{t-b}{a}\right).$$

Example 4.

(a) Suppose X has a uniform distribution on (a, b). Then, $\mathcal{FE}_\alpha(X, t) = -1/(\alpha - 1) \log\{(t - a)/(\alpha + 1)\}$.

(b) Let X have a pareto distribution, then, $\mathcal{FE}_\alpha(X, t) = \alpha/(\alpha - 1) \log[1 - (\beta/t)^\theta] - 1/(\alpha - 1) \log B(\alpha + 1, 1/\theta + 2, 1 - F(t))$, where the $B(\cdot, \cdot, \cdot)$ is the incomplete beta function.

Theorem 5. *Let X and Y be two absolutely continuous random variables such that $X \overset{DSE}{\leq} Y$, that is $\mathcal{FE}_\alpha(X, t) \leq \mathcal{FE}_\alpha(Y, t)$, for all $t \geq 0$. Define $Z_1 = a_1X + b_1$ and $Z_2 = a_2Y + b_2$, where $a_1, a_2 > 0$ and $b_1, b_2 \geq 0$ are constants. If $a_1 \geq a_2$ and $b_1 \geq b_2$, then $Z_1 \overset{DSE}{\leq} Z_2$, $\forall\, \alpha > 1$, if $\mathcal{FE}_\alpha(X, t)$ or $\mathcal{FE}_\alpha(Y, t)$ is increasing in $t > b_1$.*

Proof: Since $(t - b_1)/a_1 \leq (t - b_2)/a_2$ and $\mathcal{FE}_\alpha(X, t)$ is increasing in t, we have

$$\mathcal{FE}_\alpha\left(X, \frac{t-b_1}{a_1}\right) \leq \mathcal{FE}_\alpha\left(X, \frac{t-b_2}{a_2}\right) \leq \mathcal{FE}_\alpha\left(Y, \frac{t-b_2}{a_2}\right).$$

So for $\alpha > 1$

$$\mathcal{FE}_\alpha(Z_1, t) = \mathcal{FE}_\alpha\left(X, \frac{t-b_1}{a_1}\right) - \frac{\log a_1}{\alpha - 1} \leq \mathcal{FE}_\alpha(Z_2, t) = \mathcal{FE}_\alpha\left(Y, \frac{t-b_2}{a_2}\right) - \frac{\log a_2}{\alpha - 1}.$$

□

Corollary 3. *Let X and Y be two absolutely continuous random variables such that $X \overset{DSE}{\leq} Y$. Define $X_1 = aX + b$ and $Y_1 = aY + b$, where $a > 0$ and $b \geq 0$ are constants. Then for $t > b$, $X_1 \overset{DSE}{\leq} Y_1$, $\forall\, \alpha > 0\, (\alpha \neq 1)$.*

Definition 1. *A distribution function $F(x)$ is said to have decreasing (increasing) dynamic Failure entropy (DDFE (IDFE)) if $\mathcal{FE}_\alpha(X,t)$ is decreasing (increasing) in $t \geq 0$.*

In the next theorem we give sufficient conditions for a function $\phi(X)$ of a random variable X to have more (less) DFE than X itself.

Theorem 6.

(a) If ϕ is a non-negative function on $[0,+\infty)$ with $\phi'(x) \geq 1$ and X is IDFE, then for $\alpha > 1$, $\mathcal{FE}_\alpha(\phi(X),t) \leq \mathcal{FE}_\alpha(X,t)$.

(b) If ϕ is a non-negative increasing function on $[0,+\infty)$ with $\phi(0) = 0$ and $\phi'(x) \leq 1$ and X is DDFE, then for $\alpha > 1$, $\mathcal{FE}_\alpha(\phi(X),t) \geq \mathcal{FE}_\alpha(X,t)$.

Proof:

(a) The past failure entropy of $\phi(X)$ is

$$\mathcal{FE}_\alpha(\phi(X),t) = -\frac{1}{\alpha-1}\log\int_0^t \left[\frac{F_\phi(x)}{F_\phi(t)}\right]^\alpha dx,$$

where F_ϕ is the distribution function of $\phi(X)$. Using $u = \phi^{-1}(x)$ we have

$$\mathcal{FE}_\alpha(\phi(X),t) = -\frac{1}{\alpha-1}\log\int_0^{\phi^{-1}(t)} \left[\frac{F(u)}{F(\phi^{-1}(t))}\right]^\alpha \phi'(u)du. \tag{3.3}$$

Thus, we have for $\alpha > 1$, $\mathcal{FE}_\alpha(\phi(X),t) \leq \mathcal{FE}_\alpha(X,\phi^{-1}(t))$. Also, note that if $\phi'(x) \geq 1$, $x \geq 0$, then $\phi(x) - \phi(0) \geq x$. So $x \geq \phi^{-1}(x)$ for non-negative increasing ϕ. Thus the result follows.

(b) The proof is similar to that of (a). □

In many cases of practical interest is important to know whether the DDFE (IDFE) property of X is inherited by a transformation of X. The next theorem provides a partial answer.

Theorem 7.

(a) If X is IDFE, and if ϕ is monotone and convex (concave), then $\phi(X)$ is also IDFE for $\alpha > 1$ $(0 < \alpha < 1)$.

(b) If X is DDFE, and if ϕ is monotone and concave (convex), then $\phi(X)$ is also DDFE for $\alpha > 1$ $(0 < \alpha < 1)$.

Proof:

(a) Differentiating both sides of (3.3) we get

$$\begin{aligned}\frac{\partial}{\partial t}\mathcal{FE}_\alpha(\phi(X),t) &= \frac{\alpha}{\alpha-1}\cdot\frac{1}{\phi'(\phi^{-1}(t))}\cdot\frac{f(\phi^{-1}(t))}{F(\phi^{-1}(t))} - \frac{1}{\alpha-1}\cdot\frac{1}{\int_0^{\phi^{-1}(t)}\left[\frac{F(u)}{F(\phi^{-1}(t))}\right]^\alpha \phi'(u)du}\\ &= \frac{\alpha}{\alpha-1}\cdot\frac{r(\phi^{-1}(t))}{\phi'(\phi^{-1}(t))} - \frac{1}{\alpha-1}\cdot\frac{1}{\int_0^{\phi^{-1}(t)}\left[\frac{F(u)}{F(\phi^{-1}(t))}\right]^\alpha \phi'(u)du}.\end{aligned}$$

Let $\alpha > 1$. $\phi'(x)$ is an increasing function because $\phi(x)$ is a convex function. So $\phi'(u) \le \phi'(\phi^{-1}(t))$, $\forall\ 0 < u < \phi^{-1}(t)$ and hence,

$$\frac{\partial}{\partial t}\mathcal{FE}_\alpha(\phi(X),t) \ge \frac{\alpha}{\alpha-1}\cdot\frac{r(\phi^{-1}(t))}{\phi'(\phi^{-1}(t))} - \frac{1}{\alpha-1}\cdot\frac{1}{\int_0^{\phi^{-1}(t)}\left[\frac{F(u)}{F(\phi^{-1}(t))}\right]^\alpha \phi'(\phi^{-1}(t))du}$$
$$= \frac{\partial}{\partial t}\mathcal{FE}_\alpha(X,\phi^{-1}(t)) \ge 0.$$

A similar result follows for $0 < \alpha < 1$.

(b) The proof is similar to that of (a). □

Example 5. Let X have the uniform distribution on $(0,1)$. By Example 4, it is obvious that X is DDFE (IDFE) for $\alpha > 1\,(0 < \alpha < 1)$. If $Y = \phi_1(X) = -(\log X)/\theta$, $(\theta > 0)$, then Y has the exponential distribution with mean $1/\theta$. The decreasing function $\phi_1(x)$ is convex and hence, exponential distribution is IDFE (DDFE) for $\alpha > 1$ $(0 < \alpha < 1)$. Let $Z = Y^{1/\beta}\,(\beta > 0)$. Then Z is distributed as Weibull distribution with survival function $\bar{G}(t) = \exp(-\theta t^\beta)$ $t > 0$. The increasing function $\phi_2(y) = y^{1/\beta}$ $y > 0, \beta > 0$ is convex (concave) if $0 < \beta < 1$ $(\beta > 1)$. By Theorem 7, it follows that, Weibull distribution is IDFE for $\alpha > 1,\ 0 < \beta < 1$ and $0 < \alpha < 1,\ \beta > 1$ and it is DDFE for $\alpha > 1,\ \beta > 1$ and $0 < \alpha < 1,\ 0 < \beta < 1$.

The mean past life function(MPL) and reversed hazard function(RH) play important roles in reliability to model and analyze the data. For a continuous random variable X the reversed hazard function is defined as $\eta_F(t) = f(t)/F(t)$, for t such that $F(t) > 0$. The MPL of X is defined as $m_F(t) = E(t - X|X \le t) = \int_0^t F(x)dx/F(t)$.

The next theorem is related to DFE and RH ordering.

Theorem 8. *Let X and Y be two non-negative absolutely continuous random variables with distribution functions $F(t)$ and $G(t)$, and RH functions $\eta_F(t)$ and $\eta_G(t)$, respectively. If $X \overset{rh}{\le} Y$, that is $\eta_F(t) \le \eta_G(t)$ for all $t \ge 0$, then*

$$\mathcal{FE}_\alpha(X;t) \le (\ge)\mathcal{FE}_\alpha(Y;t), \quad \forall\ \alpha > 1\ (0 < \alpha < 1).$$

Proof: The result follows immediately using the fact $F(t)G(x) \le F(x)G(t)$ for all $x \le t$. □

Example 6. The relation between the reversed hazard functions of $X_{n:n} = \max(X_1,\ldots,X_n)$ and X is given by $\eta_{X_{n:n}}(t) = n\eta_X(t)$, $t > 0$. So $\eta_X(t) \le \eta_{X_{n:n}}(t)$ or $X \overset{rh}{\le} X_{n:n}$, then, $\mathcal{FE}_\alpha(X;t) \le (\ge)\mathcal{FE}_\alpha(X_{n:n};t)$, $\forall\ \alpha > 1$ $(0 < \alpha < 1)$.

Theorem 9. *Let X be a non-negative absolutely continuous random variable with distribution function F. Then $\mathcal{FE}_\alpha(X;t)$ uniquely determines $F(t)$.*

Proof: By (3.2) we have

$$\int_0^t F^\alpha(x)dx = e^{-(\alpha-1)\mathcal{FE}_\alpha(X,t)}F^\alpha(x).$$

Differentiating both sides of the above expression with respect to t, we get

$$F^{\alpha}(t)\left[1 - e^{-(\alpha-1)\mathcal{FE}_{\alpha}(X,t)}\eta_F(t) + (\alpha-1)\frac{\partial}{\partial t}\mathcal{FE}_{\alpha}(X,t)e^{-(\alpha-1)\mathcal{FE}_{\alpha}(X,t)}\right] = 0. \tag{3.4}$$

For a fixed $t > 0$, $\eta_F(t)$ is a solution of

$$g(x) = F^{\alpha}(t)\left[1 - e^{-(\alpha-1)\mathcal{FE}_{\alpha}(X,t)}x + (\alpha-1)\frac{\partial}{\partial t}\mathcal{FE}_{\alpha}(X,t)e^{-(\alpha-1)\mathcal{FE}_{\alpha}(X,t)}\right] = 0.$$

$g(x)$ is a decreasing function, thus $g(x) = 0$ has a unique and particular solution. From (3.4) we see that $\eta_F(t)$ is a solution of $g(x) = 0$. Hence, $\eta_F(t)$ is a unique solution of $g(x) = 0$. Thus, we get $\mathcal{FE}_{\alpha}(X,t)$ uniquely determines $\eta_F(t)$, which again uniquely determines $F(t)$. □

Remark 2. Similar result given in Section 2 holds for last order statistic ($X_{n:n}$) and FE and DFE.

4. Characterization Based on DFE

Let X have the power distribution with cdf

$$F(x) = \left(\frac{x}{\beta}\right)^{\theta}, \quad 0 < x < \beta,\ \beta, \theta > 0. \tag{4.1}$$

In the following theorem we show that the power distribution can be characterized in terms of DFE.

Theorem 10. *Let X be a random variable RH rate $\eta_F(t)$. Then*

$$\mathcal{E}_{\alpha}(X;t) = c + \frac{1}{\alpha-1}\log \eta_F(t), \tag{4.2}$$

where c is a constant if F is power distribution defined as (4.1).

Proof: If X has a Power distribution in the form of (4.1), it can be easily shown that (4.2) holds with $c = 1/(\alpha-1)\log[(\alpha\theta+1)/\theta]$. Conversely, let (4.2) hold. Then we have

$$\int_0^t F^{\alpha}(x)dx = a\frac{F^{\alpha+1}(t)}{f(t)},$$

where $a = e^{-(\alpha-1)c}$. Differentiating both sides with respect to t, we get

$$\frac{1-a(\alpha+1)}{a} = -\frac{f'(t)}{F(t)}\cdot\frac{1}{\eta_F^2(t)}.$$

Using the relation $\eta'_F(t) = f'(t)/F(t) - \eta_F^2(t)$ it follows that $\eta'_F(t)/\eta_F^2(t) = (a\alpha-1)/a$. Integrating both sides, we obtain that $\eta_F(t) = 1/(a_1 t)$, where $a_1 = (a\alpha-1)/a$. This is the reversed hazard rate of power distribution and the result follows. □

Also we can have this characterization in terms of MPL.

Theorem 11. *Let X be a random variable with distribution function F and MPL $m_F(t)$. Then*

$$\mathcal{E}_{\alpha}(X;t) = d - \frac{1}{\alpha-1}\log m_F(t), \tag{4.3}$$

where d is a constant if F is power distribution with density (4.1).

Proof: The MPL of power distribution is $m_F(t) = t/(\theta + 1)$. Taking $d = 1/(\alpha - 1)\log\{(\alpha\theta+1)/(\theta+1)\}$ gives the if part of theorem. For the only if part of the theorem, if (4.3) holds, we have

$$\int_0^t F^\alpha(x)dx = bm_F(t)F^\alpha(t),$$

where $b = e^{-(\alpha-1)d}$. Differentiating both sides with respect to t, and using the relation $\eta_F(t) = (1 - m'_F(t))/m_F(t)$, we get $m'_F(t) = (1 - b\alpha)/\{b(\alpha - 1)\}$ = constant, that is the MPL function of X is linear which is the desired result. □

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