

On $N(\kappa)$ -Contact Metric Manifolds Satisfying Certain Curvature Conditions

AVIK DE

Department of Pure Mathematics, University of Calcutta, 35, B. C. Road, Kolkata-700019, West Bengal, India
e-mail: de.math@gmail.com

JAE-BOK JUN*

Department of Mathematics, College of Natural Science, Kook-Min University, Seoul, 136-702, Korea
e-mail: jbjun@kookmin.ac.kr

ABSTRACT. We consider pseudo-symmetric and Ricci generalized pseudo-symmetric $N(\kappa)$ -contact metric manifolds. We also consider $N(\kappa)$ -contact metric manifolds satisfying the condition $S \cdot R = 0$ where R and S denote the curvature tensor and the Ricci tensor respectively. Finally we give some examples.

1. Introduction

An n -dimensional Riemannian manifold (M, g) is called *locally symmetric* if the condition $\nabla R = 0$ holds on M , where ∇ denotes the Levi-Civita connection and R is the corresponding curvature tensor of M . The class of locally symmetric manifolds is a natural generalization of the class of manifolds of constant curvature. As a generalization of locally symmetric spaces, many geometers have considered semi-symmetric spaces and in turn their generalizations. A Riemannian manifold M is said to be *semi-symmetric* [14] if its curvature tensor R satisfies $R \cdot R = 0$, where $R(X, Y)$ acts on R as a derivation. Several studies have been done in contact geometry related to semi-symmetry condition and its generalizations. In [15], S. Tanno showed that a semi-symmetric K -contact manifold M^{2n+1} is locally isometric to the unit sphere $S^{2n+1}(1)$. In [13], D. Perrone studied contact metric manifolds satisfying $R(\xi, X) \cdot R = 0$ and under additional assumptions, it was shown that the manifold is either a Sasakian manifold of constant curvature 1 or $R(X, \xi)\xi = 0$. In [10], it was proved that a Sasakian manifold M^{2n+1} satisfying $R(\xi, X) \cdot C = 0$ is

* Corresponding Author.

Received April 9, 2010; revised June 2, 2011; accepted September 26, 2011.

2010 Mathematics Subject Classification: 53C25, 53D10.

Key words and phrases: $N(\kappa)$ -contact metric manifold, Sasakian manifold, pseudo-symmetric manifold, Ricci generalized pseudo-symmetric manifold.

locally isometric to $S^{2n+1}(1)$, where C is the Weyl conformal curvature tensor. As a generalization of this result, in [4], it was proved that if ξ belongs to the κ -nullity distribution and if $R(\xi, X) \cdot C = 0$, then the contact metric manifold M^{2n+1} is locally isometric to $S^{2n+1}(1)$ or to $E^{n+1} \times S^n(4)$. In [7], the concircular curvature tensor of $N(\kappa)$ -contact metric manifold have been studied. These circumstances motivate us to study some pseudo-symmetry type conditions for a contact metric manifold, which are another generalizations of semi-symmetry type conditions. The paper is organized as follows: In section 2, we give a brief introduction for $N(\kappa)$ -contact metric manifolds. In section 3, pseudo-symmetry type manifolds are introduced. In section 4, pseudo-symmetric and Ricci generalized pseudo-symmetric $N(\kappa)$ -contact metric manifolds are studied. In section 5, we consider a $N(\kappa)$ -contact metric manifold satisfying the condition $S \cdot R = 0$. Finally we give some examples.

2. $N(\kappa)$ -contact metric manifolds

A $(2n+1)$ -dimensional manifold M is said to admit an *almost contact structure* if it admits a tensor field φ of type $(1,1)$, a vector field ξ and a 1-form η satisfying

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \varphi\xi = 0, \quad \eta \circ \varphi = 0.$$

An almost contact structure is said to be *normal* if the induced almost complex structure J on the product manifold $M \times \mathbb{R}$ defined by

$$J(X, f \frac{d}{dt}) = \left(\varphi X - f\xi, \eta(X) \frac{d}{dt} \right)$$

is integrable, where X is tangent to M , t is the coordinate of \mathbb{R} and f is a smooth function on $M \times \mathbb{R}$. Let g be a compatible Riemannian metric with almost contact structure, that is,

$$(2.1) \quad g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y),$$

then M becomes an almost contact metric manifold equipped with an almost contact metric structure (φ, ξ, η, g) . From (2.1), it can be easily seen that

$$g(X, \varphi Y) = -g(\varphi X, Y)$$

and

$$g(X, \xi) = \eta(X)$$

for all vector fields X and Y . An almost contact metric structure becomes a contact metric structure if

$$g(X, \varphi Y) = d\eta(X, Y),$$

for vector fields X and Y . The 1-form η is then a contact form and ξ is its characteristic vector field. We call the normal contact metric manifold as a *Sasakian manifold*. A contact metric manifold is Sasakian if and only if

$$(\nabla_X \varphi)Y = g(X, Y)\xi - \eta(Y)X, \quad X, Y \in TM,$$

where ∇ is Levi-Civita connection of the Riemannian metric g .

The κ -nullity distribution $N(\kappa)$ of a Riemannian manifold M is defined by

$$N(\kappa) : p \rightarrow N_p(\kappa) = \{Z \in T_pM : R(X, Y)Z = \kappa(g(Y, Z)X - g(X, Z)Y)\},$$

κ being a constant. If the characteristic vector field $\xi \in N(\kappa)$, then we call a contact metric manifold an $N(\kappa)$ -contact metric manifold. If $\kappa = 1$, then an $N(\kappa)$ -contact metric manifold is Sasakian and if $\kappa = 0$, then $N(\kappa)$ -contact metric manifold is locally isometric to the product $E^{n+1} \times S^n(4)$ for $n > 1$ and flat for $n = 1$. If $\kappa < 1$, the scalar curvature is $r = 2n(2n - 2 + \kappa)$ [8].

In a $N(\kappa)$ -contact metric manifold,

$$(2.2) \quad R(X, Y)\xi = \kappa(\eta(Y)X - \eta(X)Y),$$

$$(2.3) \quad R(\xi, X)Y = \kappa(g(X, Y)\xi - \eta(Y)X)$$

and

$$(2.4) \quad S(X, \xi) = 2n\kappa\eta(X), \quad Q\xi = 2n\kappa\xi$$

hold [8], where Q is the Ricci operator defined by $S(X, Y) = g(QX, Y)$.

We also recall the notion of a \mathcal{D} -homothetic deformation. For a given contact metric structure (ϕ, ξ, η, g) , this is, the structure defined by

$$\eta = a\eta, \quad \bar{\xi} = \frac{1}{a}\xi, \quad \bar{\phi} = \phi, \quad \bar{g} = ag + a(a - 1)\eta \oplus \eta,$$

where a is a positive constant. While such a change preserves the state of being contact metric K -contact, Sasakian or strongly pseudo-convex CR -manifold, it destroys a condition like $R(X, Y)\xi = 0$ or $R(X, Y)\xi = k(\eta(Y)X - \eta(X)Y)$. However the form of the (k, μ) -nullity condition is preserved under a \mathcal{D} -homothetic deformation with

$$\bar{k} = \frac{k + a^2 - 1}{a^2}, \quad \bar{\mu} = \frac{\mu + 2a - 2}{a}.$$

Given a non-Sasakian (k, μ) -manifold M , E.Boeckx [9] introduced an invariant

$$I_M = \frac{1 - \frac{\mu}{2}}{\sqrt{1 - k}}$$

and showed that for two non-Sasakian (k, μ) -manifolds $M_i(\phi_i, \xi_i, \eta_i, g_i), i = 1, 2$, we have $IM_1 = IM_2$ if and only if up to a \mathcal{D} -homothetic deformation, the two

manifolds are locally isometric as contact metric manifolds. Thus we know all non-Sasakian (k, μ) -manifolds locally as soon as we have for every odd dimension $2n+1$ and for every possible value of the invariant I , one (k, μ) -manifold $M(\phi, \xi, \eta, g)$ with $I_M = I$. For $I > 1$ such examples may be found from the standard contact metric structure on the tangent sphere bundle of a manifold of contact curvature c where we have $I = \frac{1+c}{|1-c|}$. Boeckx also gives a Lie algebra construction for any odd dimension and value of $I \leq -1$.

Using this invariant, we now construct an example of a $(2n+1)$ -dimension $N(1 - \frac{1}{n})$ -contact metric manifold, $n > 1$.

Example 2.1. Since the Boeckx invariant for a $(1 - \frac{1}{n}, 0)$ -manifold is $\sqrt{n} > -1$, we consider the tangent sphere bundle of an $(n+1)$ -dimensional manifold of constant curvature c so chosen that the resulting \mathcal{D} -homothetic deformation will be a $(1 - \frac{1}{n}, 0)$ -manifold. That is, for $k = c(2-c)$ and $\mu = -2c$ we solve

$$1 - \frac{1}{n} = \frac{k + a^2 - 1}{a^2} \quad 0 = \frac{\mu + 2a - 2}{a}$$

for a and c . The result is

$$c = \frac{(\sqrt{n} \pm 1)^2}{n-1}, \quad a = 1 + c$$

and taking c and a to be these values we obtain an $N(1 - \frac{1}{n})$ -contact metric manifold.

The above example will be used in Theorem 5.1.

3. Pseudo-symmetry type manifolds

Let (M, g) be an $n(\geq 3)$ -dimensional differentiable manifold of class C^∞ . We define tensors $R \cdot R$ and $\bar{Q}(g, R)$ by

$$(3.1) \quad \begin{aligned} & (R(X, Y) \cdot R)(X_1, X_2, X_3) = R(X, Y)R(X_1, X_2)X_3 \\ & -R(R(X, Y)X_1, X_2)X_3 - R(X_1, R(X, Y)X_2)X_3 - R(X_1, X_2)R(X, Y)X_3 \end{aligned}$$

and

$$\begin{aligned} \bar{Q}(g, R)(X_1, X_2, X_3; X, Y) &= (X \wedge Y)R(X_1, X_2)X_3 \\ &-R((X \wedge Y)X_1, X_2)X_3 - R(X_1, (X \wedge Y)X_2)X_3 - R(X_1, X_2)(X \wedge Y)X_3, \end{aligned}$$

respectively, where $X_1, X_2, X_3, X, Y \in TM$ and $X \wedge Y$ is an endomorphism [11] defined by

$$(3.2) \quad (X \wedge Y)Z = g(Y, Z)X - g(X, Z)Y.$$

If the tensors $R \cdot R$ and $\bar{Q}(g, R)$ are linearly dependent, then M is called *pseudo-symmetric* which is introduced by R. Deszcz [11] as a generalization of the semi-symmetry. This is equivalent to

$$R \cdot R = L_R \bar{Q}(g, R)$$

holding on the set $U_R = \{x \in M : \bar{Q}(g, R) \neq 0 \text{ at } x\}$, where L_R is some function on U_R [11]. In particular, if L_R is constant, M is called a *pseudo-symmetric manifold of constant type* [3]. A pseudo-symmetric manifold is said to be *proper* if it is not semi-symmetric. Every semi-symmetric manifold is pseudo-symmetric, but the converse statement is not true. It is trivial that if M is locally symmetric, then it is semi-symmetric.

If the tensors $R \cdot R$ and $\bar{Q}(S, R)$ are linearly dependent, then M is called *Ricci generalized pseudo-symmetric* [11]. This is equivalent to

$$R \cdot R = LQ(S, R)$$

holding on the set $U = \{x \in M : Q(S, R) \neq 0 \text{ at } x\}$, where L is some function on U . The tensors $Q(S, R)$ and $X \wedge_S Y$ are defined [11] by

$$(3.3) \quad Q(S, R)(X_1, X_2, X_3; X, Y) = (X \wedge_S Y)R(X_1, X_2)X_3$$

$$-R((X \wedge_S Y)X_1, X_2)X_3 - R(X_1, (X \wedge_S Y)X_2)X_3 - R(X_1, X_2)(X \wedge_S Y)X_3$$

and

$$(3.4) \quad (X \wedge_S Y)Z = S(Y, Z)X - S(X, Z)Y,$$

respectively. Moreover for a non-flat Riemannian manifold (M, g) , the tensor $S \cdot R$ is defined by

$$(3.5) \quad (S \cdot R)(X_1, X_2, X_3, X_4) = -R(QX_1, X_2, X_3, X_4) - R(X_1, QX_2, X_3, X_4)$$

$$- R(X_1, X_2, QX_3, X_4) - R(X_1, X_2, X_3, QX_4),$$

where $X_1, X_2, X_3, X_4, X, Y, Z \in TM$. Semi-Riemannian manifolds satisfying the condition $S \cdot R = 0$ were investigated in [1] and [2].

4. Pseudo-symmetric and Ricci generalized pseudo-symmetric $N(\kappa)$ -contact metric manifolds

We know from [6] that a contact metric manifold of constant curvature is necessarily a Sasakian manifold of constant curvature $+1$ or is 3-dimensional and flat, and a contact metric manifold M^{2n+1} satisfying $R(X, Y)\xi = 0$ is locally isometric to $E^{n+1} \times S^n(4)$ for $n > 1$ and flat in dimension 3.

Now, we begin with the following:

Theorem 4.1. *Let M be a non-flat $(2n + 1)$ -dimensional $N(\kappa)$ -contact metric manifold. If M is a proper pseudo-symmetric manifold, then the manifold is a pseudosymmetric manifold of constant type.*

Proof. We assume that M is a proper pseudo-symmetric $N(\kappa)$ -contact metric manifold. From (2.3) and (3.2), since

$$R(\xi, X)Y = \kappa(\xi \wedge X)Y,$$

it is easy to see that

$$R(\xi, X) \cdot R = \kappa(\xi \wedge X) \cdot R,$$

which implies that the pseudo-symmetry function $L_R = \kappa$. Hence the manifold is a pseudo-symmetric manifold of constant type. This completes the proof. \square

Let us suppose that the $N(\kappa)$ -contact metric manifold is semi-symmetric, that is,

$$(R(U, X) \cdot R)(Y, Z)W = 0,$$

which implies

$$(4.1) \quad (R(\xi, X) \cdot R)(Y, Z)W = 0.$$

From (4.1) we can write

$$(4.2) \quad \begin{aligned} R(\xi, X)R(Y, Z)W &- R(R(\xi, X)Y, Z)W - R(Y, R(\xi, X)Z)W \\ &- R(Y, Z)R(\xi, X)W = 0. \end{aligned}$$

Then using (2.3), the equation (4.2) can be written as

$$(4.3) \quad \begin{aligned} \kappa[R(Y, Z, W, X)\xi &- \eta(R(Y, Z)W)X - g(X, Y)R(\xi, Z)W + \eta(Y)R(X, Z)W \\ &- g(X, Z)R(Y, \xi)W + \eta(Z)R(Y, X)W - g(X, W)R(Y, Z)\xi \\ &+ \eta(W)R(Y, Z)X] = 0. \end{aligned}$$

Hence taking the inner product of (4.3) with ξ we get

$$(4.4) \quad \begin{aligned} \kappa[R(Y, Z, W, X) &- \eta(R(Y, Z)W)\eta(X) - g(X, Y)\eta(R(\xi, Z)W) + \eta(Y)\eta(R(X, Z)W) \\ &- g(X, Z)\eta(R(Y, \xi)W) + \eta(Z)\eta(R(Y, X)W) + \eta(W)\eta(R(Y, Z)X)] \\ &= 0. \end{aligned}$$

So in view of (2.2) and (2.3), the equation (4.4) turns into the form

$$\kappa[R(Y, Z, W, X) + \kappa(g(X, Z)g(Y, W) - g(X, Y)g(Z, W))] = 0,$$

which gives either $\kappa = 0$, or $R(Y, Z, W, X) = \kappa(g(X, Y)g(Z, W) - g(X, Z)g(Y, W))$.

For $\kappa = 0$, we get the manifold is locally isometric to $E^{n+1} \times S^n(4)$ for $n > 1$ [6]. On the other hand, if $R(Y, Z, W, X) = \kappa(g(X, Y)g(Z, W) - g(X, Z)g(Y, W))$, then M is a space of constant curvature κ . So from [6], it is necessarily a Sasakian manifold of constant curvature $+1$ for $n > 1$.

Thus we have the following:

Corollary 4.1. *Let M be a semisymmetric $N(\kappa)$ -contact metric manifold, then the manifold is either locally isometric to $E^{n+1} \times S^n(4)$ for $n > 1$, or a Sasakian manifold of constant curvature $+1$.*

Theorem 4.2. *Let M be a non-flat $(2n + 1)$ -dimensional $N(\kappa)$ -contact metric manifold. Then M satisfies the condition $R(\xi, X) \cdot R = L(\xi \wedge_S X) \cdot R$ if and only if it is a Sasakian manifold of constant curvature $+1$ or locally isometric to $E^{n+1} \times S^n(4)$ for $n > 1$.*

Proof. By the use of (3.3), we can write

$$\begin{aligned}
 (4.5) \quad ((\xi \wedge_S X) \cdot R)(Y, Z, W) &= \bar{Q}(S, R)(Y, Z, W; \xi, X) \\
 &= (\xi \wedge_S X)R(Y, Z)W \\
 &\quad - R((\xi \wedge_S X)Y, Z)W - R(Y, (\xi \wedge_S X)Z)W \\
 &\quad - R(Y, Z)(\xi \wedge_S X)W.
 \end{aligned}$$

So we get by using (3.4) from the above that

$$\begin{aligned}
 (4.6) \quad \bar{Q}(S, R)(Y, Z, W; \xi, X) &= S(R(Y, Z)W, X)\xi - S(\xi, R(Y, Z)W)X \\
 &\quad - S(X, Y)R(\xi, Z)W + S(\xi, Y)R(X, Z)W \\
 &\quad - S(X, Z)R(Y, \xi)W + S(\xi, Z)R(Y, X)W \\
 &\quad - S(X, W)R(Y, Z)\xi + S(\xi, W)R(Y, Z)X.
 \end{aligned}$$

Taking the inner product of (4.6) with ξ , we have

$$\begin{aligned}
 (4.7) \quad g(\bar{Q}(S, R)(Y, Z, W; \xi, X), \xi) &= S(R(Y, Z)W, X) - S(\xi, R(Y, Z)W)\eta(X) \\
 &\quad - S(X, Y)\eta(R(\xi, Z)W) + S(\xi, Y)\eta(R(X, Z)W) \\
 &\quad - S(X, Z)\eta(R(Y, \xi)W) + S(\xi, Z)\eta(R(Y, X)W) \\
 &\quad + S(\xi, W)\eta(R(Y, Z)X).
 \end{aligned}$$

Hence making use of (2.2), (2.3) and (2.4) in the last equation (4.7), we obtain

$$\begin{aligned}
 &g(((\xi \wedge_S X) \cdot R)(Y, Z, W), \xi) = g(\bar{Q}(S, R)(Y, Z, W; \xi, X), \xi) \\
 &= S(R(Y, Z)W, X) + 2n\kappa^2[g(X, Z)\eta(Y)\eta(W) - g(X, Y)\eta(Z)\eta(W)] \\
 &\quad - \kappa\{g(Z, W)S(X, Y) - S(X, Y)\eta(Z)\eta(W) \\
 &\quad - g(Y, W)S(X, Z) + S(X, Z)\eta(Y)\eta(W)\}.
 \end{aligned}$$

Also we can deduce

$$\begin{aligned}
 (4.8) \quad g(R(\xi, X) \cdot R)(Y, Z, W), \xi &= k[R(Y, Z, W, X) + k(g(X, Z)g(Y, W) - g(X, Y)g(Z, W))].
 \end{aligned}$$

Since the condition $R(\xi, X) \cdot R = L(\xi \wedge_S X) \cdot R$ holds on M , by (4.5) and (4.8), we can write as

$$(4.9) \quad \begin{aligned} &\kappa[R(Y, Z, W, X) + \kappa(g(X, Z)g(Y, W) - g(X, Y)g(Z, W))] \\ &= L\{2n\kappa^2[g(X, Z)\eta(Y)\eta(W) - g(X, Y)\eta(Z)\eta(W)] \\ &+ S(R(Y, Z)W, X) - \kappa[g(Z, W)S(X, Y) - S(X, Y)\eta(Z)\eta(W) \\ &- g(Y, W)S(X, Z) + S(X, Z)\eta(Y)\eta(W)]\}. \end{aligned}$$

Taking $Y = \xi$ in (4.9) and using (2.3) and (2.4), we have

$$L\{2n\kappa(1 - \kappa)g(Z, W)\eta(X) - S(X, Z)\eta(W) + 2n\kappa^2g(X, Z)\eta(W)\} = 0.$$

Hence either $L = 0$ or

$$(4.10) \quad 2n\kappa(1 - \kappa)g(Z, W)\eta(X) - S(X, Z)\eta(W) + 2n\kappa^2g(X, Z)\eta(W) = 0.$$

If $L = 0$, then $R(\xi, X) \cdot R = 0$ holds on M . Then by the proof of Corollary 4.1, it is a Sasakian manifold of constant curvature $+1$ or locally isometric to $E^{n+1} \times S^n(4)$ for $n > 1$.

Next, if the condition (4.10) holds on M , then contracting (4.10) over Z and W we get

$$4n^2\kappa(1 - \kappa) = 0,$$

which gives us either $\kappa = 0$ or $\kappa = 1$. Hence in the case of $\kappa = 0$, M is locally isometric to the product $E^{n+1} \times S^n(4)$ for $n > 1$ and in the case of $\kappa = 1$, M is a Sasakian manifold of constant curvature 1 .

The converse statement is trivial. This proves the theorem. □

5. $N(\kappa)$ -contact metric manifolds satisfying the condition $S \cdot R = 0$

In this section we prove the following:

Theorem 5.1. *Let M be a $(2n + 1)$ -dimensional non-flat and non-Sasakian $N(\kappa)$ -contact metric manifold. If M satisfies the condition $S \cdot R = 0$, then M is either locally isometric to $E^{n+1} \times S^n(4)$ for $n > 1$ or the manifold is locally isometric to the Example 2.1.*

Proof. From (3.5) we know

$$(5.1) \quad \begin{aligned} (S \cdot R)(X, Y, Z, W) &= -R(QX, Y, Z, W) - R(X, QY, Z, W) \\ &- R(X, Y, QZ, W) - R(X, Y, Z, QW). \end{aligned}$$

Now assume that the condition $S \cdot R = 0$ holds on M . Then taking $X = \xi$ in (5.1), we get

$$(5.2) \quad \begin{aligned} &R(Q\xi, Y, Z, W) + R(\xi, QY, Z, W) \\ &+ R(\xi, Y, QZ, W) + R(\xi, Y, Z, QW) = 0. \end{aligned}$$

So by the use of (2.4), the equation (5.2) turns into the form

$$(5.3) \quad \begin{aligned} & 2n\kappa R(\xi, Y, Z, W) + R(\xi, QY, Z, W) \\ & + R(\xi, Y, QZ, W) + R(\xi, Y, Z, QW) = 0. \end{aligned}$$

Moreover, by the use of (2.3) in (5.3), we have

$$(5.4) \quad \begin{aligned} & 2n\kappa^2[g(Y, Z)\eta(W) - g(Y, W)\eta(Z)] \\ & + \kappa[S(Y, Z)\eta(W) - S(Y, W)\eta(Z)] \\ & + \kappa[S(Y, Z)\eta(W) - 2n\kappa g(Y, W)\eta(Z)] \\ & + \kappa[2n\kappa g(Y, Z)\eta(W) - S(Y, W)\eta(Z)] = 0. \end{aligned}$$

Then putting $W = \xi$ in (5.4) we have

$$2\kappa[-S(Y, Z) - 2n\kappa g(Y, Z) + 4n\kappa\eta(Y)\eta(Z)] = 0,$$

which gives us either $\kappa = 0$ or the condition

$$(5.5) \quad S(Y, Z) - 2n\kappa[-g(Y, Z) + 2\eta(Y)\eta(Z)] = 0$$

holds on M . In the case of $\kappa = 0$, M is locally isometric to $E^{n+1} \times S^n(4)$ for $n > 1$ [6]. If the condition (5.5) holds on M , then contracting the equation (5.5) over Y and Z , we get $r = 2n\kappa(1 - 2n)$. For a non-Sasakian $N(\kappa)$ -contact metric manifold, since $r = 2n(2n - 2 + \kappa)$ [8], we obtain $\kappa = \frac{1}{n} - 1$. In this case, if $n = 1$, then $\kappa = 0$, hence M is flat [6]. But this case can not occur because of the non-flatness assumption. Thus the manifold is locally isometric to the Example 2.1. \square

6. Examples

Example 6.1. In [12] J. Milnor gave a complete classification of three dimensional manifolds admitting the Lie algebra structure

$$(6.1) \quad [e_2, e_3] = c_1e_1, \quad [e_3, e_1] = c_2e_2, \quad [e_1, e_2] = c_3e_3.$$

As in the case of the given example of [7], let us consider η be the dual 1-form to the vector field e_1 . Using (6.1) we get

$$d\eta(e_2, e_3) = -d\eta(e_3, e_2) = \frac{c_1}{2} \neq 0$$

and $d\eta(e_i, e_j) = 0$ for $(i, j) \neq (2, 3), (3, 2)$. It is easy to check that η is a contact form and e_1 is the characteristic vector field. Defining a Riemannian metric g by $g(e_i, e_j) = \delta_{ij}$, then, because we must have $d\eta(e_i, e_j) = g(e_i, \phi e_j)$, ϕ has the same metric as $d\eta$ with respect to the basis e_i . Moreover, for g to be an associated metric, we must have $\phi^2 = -I + \eta \otimes e_1$. So for (ϕ, e_1, η, g) to be a contact metric structure

we must have $c_1 = 2$. The unique Riemannian connection ∇ corresponding to g is given by

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z]) \\ &\quad - g(Y, [X, Z]) + g(Z, [X, Y]). \end{aligned}$$

So using $c_1 = 2$ we easily get

$$\nabla_{e_1} e_1 = 0, \quad \nabla_{e_2} e_2 = 0, \quad \nabla_{e_3} e_3 = 0,$$

$$\nabla_{e_1} e_2 = \frac{1}{2}(c_2 + c_3 - 2)e_3, \quad \nabla_{e_2} e_1 = \frac{1}{2}(c_2 - c_3 - 2)e_3,$$

$$\nabla_{e_1} e_3 = -\frac{1}{2}(c_2 + c_3 - 2)e_2, \quad \nabla_{e_3} e_1 = \frac{1}{2}(2 + c_2 - c_3)e_2.$$

But we also know that

$$\nabla_{e_2} e_1 = -\phi e_2 - \phi h e_2.$$

Comparing now those two relations of $\nabla_{e_2} e_1$ and using $\phi e_1 = 0$, $\phi e_3 = -e_2$ we conclude that

$$h e_2 = \frac{c_3 - c_2}{2} e_2.$$

And hence

$$h e_3 = -\frac{c_3 - c_2}{2} e_3.$$

Thus e_i are eigenvectors of h with corresponding eigenvalues $(0, \lambda, -\lambda)$ where $\lambda = \frac{c_3 - c_2}{2} e_2$. Moreover by direct calculation we have

$$R(e_2, e_1)e_1 = \left[1 - \frac{(c_3 - c_2)^2}{4}\right]e_2 + [2 - c_2 - c_3]h e_2,$$

$$R(e_3, e_1)e_1 = \left[1 - \frac{(c_3 - c_2)^2}{4}\right]e_3 + [2 - c_2 - c_3]h e_3.$$

$$R(e_2, e_3)e_1 = 0.$$

Putting $k = 1 - \frac{(c_3 - c_2)^2}{4}$ and $\mu = 2 - c_2 - c_3$ we conclude, from these relations that e_1 belongs to the (k, μ) -nullity distribution, for any c_2, c_3 .

Now putting $c_2 = c_3 = 1$, we get the manifold is a $N(\kappa)$ -contact metric manifold with $\kappa = 1$. Thus the manifold is a Sasakian manifold. Also from the expressions of the curvature tensor, for $c_2 = c_3 = 1$, we get that the manifold is a manifold

of constant curvature. Thus the Theorem 4.2 is verified.

Example 6.2([5]). Every Sasakian space form $M^{2n+1}(c)$ is pseudo-symmetric. Also the Sasakian space form $M^{2n+1}(-3)$ is pseudo-symmetric but not semi-symmetric.

Acknowledgements The authors are thankful to the referee for his valuable comments and suggestions towards the improvement of the paper.

References

- [1] K. Arslan, R. Deszcz, R. Ezentas and M. Hotlos, *On A certain class of conformally flat manifolds*, Bull. Inst. Math.Acad. Sinica, **26**(1998), 183-199.
- [2] K. Arslan, R. Deszcz and R. Ezentas, *On a certain class of hypersurfaces in semi-Euclidean spaces*, Soochow J. Math., **25**(1999), 223-236.
- [3] N. Hashimoto and M. Sekizawa, *Three dimensional conformally flat pseudo-symmetric spaces of constant type*, Arch. Math.(Brno), **36**(2000), 279-286.
- [4] C. Baikoussis and T. Koufogiorgos, *On a type of contact manifolds*, J. Geom., **46**(1-2)(1993), 1-9.
- [5] M. Belkhef, R. Deszcz and L. Verstraelen, *Symmetry Properties of Sasakian Space Forms*, Soochow J. of Math., **31**(4)(2005), 611-616.
- [6] D. E. Blair, *Riemannian geometry of contact and symplectic manifolds*, Progress in Mathematics, 203. Birkhauser Boston, Inc., Boston, MA, USA, 2002.
- [7] D. E. Blair, J. S. Kim and M. M. Tripathi, *On the concircular curvature tensor of a contact metric manifold*, J. Korean Math. Soc., **42**(5)(2005), 883-892.
- [8] D. E. Blair, T. Koufogiorgos and B. J. Papantoniou, *Contact metric manifolds satisfying a nullity condition*, Israel J. Math., **91**(1-3)(1995), 189-214.
- [9] E. Boeckx, *A full classification of contact metric (κ, μ) -spaces*, Illinois J. of Math., **44**(2000), 212-219.
- [10] M.C.Chaki and M. Tarafdar, *On a type of Sasakian manifold*, Soochow J. Math., **16**(1)(1990), 23-28.
- [11] R. Deszcz, *On pseudosymmetric spaces*, Bull. Belgian Math. Soc., Ser. A, **44**(1992), 1-34.
- [12] J. Milnor, *Curvature of left invariant metrics on Lie groups*, Advances in Mathematics, **21**(1976), 293-311.
- [13] D. Perrone, *Contact Riemannian manifolds satisfying $R(X, \xi) \cdot R = 0$* , Yokohama Math. J., **39**(2)(1992), 141-149.
- [14] Z. I. Szabo, *Structure theorems on Riemannian spaces satisfying $R(X, Y) \cdot R = 0$, The local version*, J. Diff. Geom., **17**(1982), 531-582.
- [15] S. Tanno, *Isometric immersions of Sasakian manifolds in spheres*, Kodai Math. J., **21**(1969), 448-458.

- [16] K. Yano and M. Kon, Structures on manifolds, Series in Pure Mathematics, 3. World Scientific Publishing Co., Singapore, 1984.