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Subordination Problems of Robertson Functions

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Abstract. In the present paper, we are concerned with subordination problems related to λ-Robertson function. The radii of λ-spirallikeness and starlikeness of λ-Robertson function are also determined.

1. Introduction and main results

Let $\mathbb{D}_r = \{z \in \mathbb{C} : |z| < r\}$ for $0 < r \leq 1$ and $\mathbb{D} = \mathbb{D}_1$ be the unit disc. Let $\mathcal{A}$ be the family of functions f analytic in $\mathbb{D}$, and $\mathcal{A}_1$ be the subset of $\mathcal{A}$ consisting of functions f which are normalized by $f(0) = f'(0) - 1 = 0$. A function $f \in \mathcal{A}$ is said to be subordinate to a function $F \in \mathcal{A}$ in $\mathbb{D}$ (in symbols $f \prec F$ or $f(z) \prec F(z)$) if there exists an analytic function $\omega(z)$ on $\mathbb{D}$ with $|\omega(z)| < 1$ and $\omega(0) = 0$, such that

$$f(z) = F(\omega(z))$$

in $\mathbb{D}$. When F is a univalent function, the condition $f \prec F$ is equivalent to $f(\mathbb{D}) \subseteq F(\mathbb{D})$ and $f(0) = F(0)$. Let

$$\mathcal{P}_\lambda = \left\{p \in \mathcal{A} : p(0) = 1, \ \Re e^{-i\lambda} p(z) > 0\right\}.$$

Here and hereafter we always suppose $-\pi/2 < \lambda < \pi/2$. Note that $\mathcal{P}_\lambda$ is a convex and compact subset of $\mathcal{A}$ which is equipped with the topology of uniform convergence on compact subsets of $\mathbb{D}$. Since $\mathcal{P}_0$ is the well-known Carathéodory class, we call $\mathcal{P}_\lambda$ *the tilted Carathéodory class by angel* λ. Some characterizations and estimates of elements in $\mathcal{P}_\lambda$ are known (for a short survey, see [11]).

For a function $f \in \mathcal{A}$, let

$$Q_f(z) = \frac{zf'(z)}{f(z)} \tag{1}$$

and

$$P_f(z) = 1 + \frac{zf''(z)}{f'(z)}.$$

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It is worthwhile to note that

$$Q_f(z) + \frac{zQ'_f(z)}{Q_f(z)} = P_f(z).$$

These quantities are important for investigation of geometric properties of analytic functions. Next we will define two subclasses of analytic functions related to these two quantities.

A function $f \in \mathcal{A}_1$ is said to be a λ-*spirallike function* (denoted by $f \in \mathcal{SP}_\lambda$) if

$$Q_f \in \mathcal{P}_\lambda.$$

Note that $\mathcal{SP}_0$ is precisely the set of starlike functions normally denoted by $\mathcal{S}^*$. Spirallike functions were introduced and proved to be univalent by Špaček [10] in 1932. For general references about spirallike functions, see e.g. [2] or [1].

A function $f \in \mathcal{A}_1$ is said to be a λ-*Robertson function* if $zf'(z) \in \mathcal{SP}_\lambda$, i.e.

$$P_f \in \mathcal{P}_\lambda.$$

Let $\mathcal{R}_\lambda$ denote the set of these functions. Note that $\mathcal{R}_0$ is precisely the set of convex functions sometimes denoted by $\mathcal{K}$. Convex functions have been the subject of numerous investigations, among which the following result was proved by MacGregor [6] in 1975.

Theorem A. *Let $f \in \mathcal{K}$. Then the subordination relation*

$$\frac{zf'(z)}{f(z)} \prec \frac{zf'_0(z)}{f_0(z)}$$

holds, where $f_0(z) = z/(1-z)$.

We are interested in more general subordination problems related to λ-Robertson functions. For this purpose, we first introduce some specific functions for convenience.

A distinguished member of $\mathcal{R}_\lambda$ is

$$f_\lambda(z) = \frac{(1-z)^{1-2e^{i\lambda}\cos\lambda} - 1}{2e^{i\lambda}\cos\lambda - 1}.$$

A simple calculation yields

(2) $$Q_\lambda(z) = \frac{zf'_\lambda(z)}{f_\lambda(z)} = \frac{e^{2i\lambda}z}{1-z-(1-z)^{1+e^{2i\lambda}}},$$

$$P_\lambda(z) = 1 + \frac{zf''_\lambda(z)}{f'_\lambda(z)} = \frac{1+e^{2i\lambda}z}{1-z}$$

and

$$Q_\lambda(z) + \frac{zQ'_\lambda(z)}{Q_\lambda(z)} = P_\lambda(z). \tag{3}$$

In [4], Kim and Srivastava posed the open problem which is an extension of Theorem A whether

$$\frac{zf'(z)}{f(z)} \prec \frac{zf'_\lambda(z)}{f_\lambda(z)}$$

holds for $f \in \mathcal{R}_\lambda$ with general λ. In other words, if

$$q(z) + \frac{zq'(z)}{q(z)} \prec P_\lambda(z) \tag{4}$$

in $\mathbb{D}$, then whether

$$q \prec Q_\lambda$$

holds in $\mathbb{D}$ for general λ? Relation (4) is a kind of Briot-Bouquet differential subordinations which have a surprising number of important applications in the theory of univalent functions. Many sources and references are given in [7].

In the present paper, we solve the above problem in a restricted disc and obtain the radii of spirallikeness and starlikeness for Robertson funtions as well.

Theorem 1. *Let $q \in \mathcal{A}$ with $q(0) = 1$ satisfy the differential subordination*

$$q(z) + \frac{zq'(z)}{q(z)} \prec P_\lambda(z) \tag{5}$$

and $R_1(\lambda)$ be defined by

$$R_1(\lambda) = \sup\{r < 1 : Q_\lambda(rz) \prec P_\lambda(rz) \text{ in } \mathbb{D}\}. \tag{6}$$

Then

$$q(z) \prec Q_\lambda(z)$$

in $|z| < R_1(\lambda)$.

By the discussion in Section 1, we can deduce the following corollary immediately from Theorem 1.

Corollary 1. *Let $f \in \mathcal{R}_\lambda$. Then*

$$\frac{zf'(z)}{f(z)} \prec \frac{zf'_\lambda(z)}{f_\lambda(z)}$$

in $|z| < R_1(\lambda)$, where $R_1(\lambda)$ is given in (6).

Remark 1. *The radius of λ-spirallikeness of λ-Robertson functions is at least $R_1(\lambda)$, since*

$$\frac{zf'(z)}{f(z)} \prec \frac{zf'_\lambda(z)}{f_\lambda(z)} \prec P_\lambda(z)$$

in $\mathbb{D}_{R_1(\lambda)}$ *for any* $f \in \mathcal{R}_\lambda$.

Remark 2. *For* $0 \le r < 1$, *let*

$$\begin{aligned}\psi_\lambda(r) &= \max_{|z|=1} |P_\lambda^{-1}(Q_\lambda(rz))/r| \\ &= \max_{|z|=1} \left| \frac{1}{r} \frac{mrz - 1 + (1-rz)^m}{1-(1-rz)^m} \right|\end{aligned}$$

where $m = 1 + e^{2i\lambda}$. *We see that* $\psi_\lambda(r)$ *is an increasing function defined on* $[0,1)$ *with* $\psi_\lambda(0) = 0$. *By the definition of subordination,* $R_1(\lambda)$ *defined in (*6*) could be expressed in terms of* ψ_λ*:*

$$R_1(\lambda) = \sup\{r < 1 \,:\, \psi_\lambda(r) < 1\} = \psi_\lambda^{-1}(1).$$

Theorem 2. *Let* $q \in \mathcal{A}$ *with* $q(0) = 1$ *satisfy the differential subordination*

$$q(z) + \frac{zq'(z)}{q(z)} \prec P_\lambda(z). \tag{7}$$

Then

$$q(R_2 z) \prec P_0(z)$$

in $\mathbb{D}$, *where*

$$R_2 := R_2(\lambda) = \frac{2}{\sqrt{4 + 2\sqrt{3}|\sin(2\lambda)|}}. \tag{8}$$

Corollary 2. *The radius of starlikeness of* λ*-Robertson functions is at least* $R_2(\lambda)$ *given in* (8).

Note that $R_1(0) = R_2(0) = 1$, thus both Corollary 1 and Corollary 2 imply Theorem A. Note also that in [1], Ahuja and Silverman posed the problem to find the radius of starlikeness for all λ-Robertson functions. Corollary 2 implies that this radius is at least

$$R_2 = \min\{R_2(\lambda) :\, -\pi/2 < \lambda < \pi/2\} = R_2(\pi/4) = \sqrt{3} - 1 \approx 0.732.$$

Libera and Ziegler in [5] have shown that the radius of close-to-convexity for all λ-Robertson functions is approximately 0.99097524 and the radius of convexity is $\sqrt{2}/2$.

2. Proofs of resluts

In order to obtain our main results, the following lemmas are required.

Lemma 1([11]). *Let* $p \in \mathcal{P}_\lambda$. *Then we have*

$$|p(z) - A(r)| \le B(r),$$

where

$$A(r) = \frac{1 + r^2 e^{2i\lambda}}{1 - r^2}, \; B(r) = \frac{2r\cos\lambda}{1 - r^2}$$

and $r = |z| < 1$. *Equality holds if and only if* $p(z) = P_\lambda(xz)$ *with* $|x| = 1$.

Lemma 2([7, Lemma 2.2d])**.** *Let* $g(z)$ *and* $h(z)$ *be in* $\mathcal{A}$ *with* $g(0) = h(0)$. *If* $g \not\prec h$ *in* $\mathbb{D}$, *then there exist two points* z_0 *with* $|z_0| < 1$ *and* η_0 *with* $|\eta_0| = 1$ *and* $s \ge 1$ *such that*

$$g(\mathbb{D}_{|z_0|}) \subset h(\mathbb{D}),$$
$$g(z_0) = h(\eta_0)$$

and

$$z_0 g'(z_0) = s\eta_0 h'(\eta_0).$$

The next lemma is due to Nunokawa [8]. We only quote the relevant part.

Lemma 3([8]). *Let* $p(z) \in \mathcal{A}$ *satisfy* $p(0) = 1$ *and* $p(z) \ne 0$ *in* $\mathbb{D}$. *If there exsits a point* $z_0 \in \mathbb{D}$ *such that* $\Re p(z) > 0$ *in* $|z| < |z_0|$ *and* $p(z_0) = ia$ *where* $a \in \mathbb{R} \setminus \{0\}$, *then*

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik$$

where $k \ge (a + 1/a)/2$ *if* $a > 0$ *and* $k \le -(a + 1/a)/2$ *if* $a < 0$.

Proof of Theorem 1. For simplicity, we let $R = R_1(\lambda)$ and $p(z) = q(z) + zq'(z)/q(z)$, thus $p \in \mathcal{P}_\lambda$. If $q(z) \not\prec Q_\lambda(z)$ in $|z| < R$, then Lemma 2 implies the existences of z_0 with $|z_0| < R$, η_0 with $|\eta_0| = R$ and $s \ge 1$ such that

$$\begin{aligned} q(\mathbb{D}_{|z_0|}) &\subset Q_\lambda(\mathbb{D}), \\ q(z_0) &= Q_\lambda(\eta_0), \\ z_0 q'(z_0) &= s\eta_0 Q'_\lambda(\eta_0). \end{aligned} \tag{9}$$

Thus in view of (3), (5) and (9), we have

$$\begin{aligned} p(z_0) &= q(z_0) + \frac{z_0 q'(z_0)}{q(z_0)} \\ &= Q_\lambda(\eta_0) + s\frac{\eta_0 Q'_\lambda(\eta_0)}{Q_\lambda(\eta_0)} \\ &= sP_\lambda(\eta_0) + (1 - s)Q_\lambda(\eta_0). \end{aligned} \tag{10}$$

Therefore (10) and Lemma 1 show that

$$\begin{aligned}|p(z_0)-A(R)| &= |s(P_\lambda(\eta_0)-A(R))-(1-s)(Q_\lambda(\eta_0)-A(R))| \\ &\geq |s(P_\lambda(\eta_0)-A(R))|-(s-1)|(Q_\lambda(\eta_0)-A(R))| \\ &\geq sB(R)-(s-1)B(R) \\ &= B(R)\end{aligned}$$

which contradicts with $p(z) \in \mathcal{P}_\lambda$. Therefore we get the assertion. □

Proof of Theorem 2. For simplicity, we let $R = R_2(\lambda)$ and $p(z) = q(z)+zq'(z)/q(z)$, thus $p \in \mathcal{P}_\lambda$. If $q(Rz) \not\prec P_0(z)$ in $\mathbb{D}$, it follows from $q(0) = 1$ that there esists a point $z_0 \in \mathbb{D}$ such that $\Re q(Rz) > 0$ for $|z| < |z_0|$ and $q(Rz_0) = ia$ where $a \in \mathbb{R} \setminus \{0\}$, then by Lemma 3, we have

$$\frac{Rz_0q'(Rz_0)}{q(Rz_0)} = ik,$$

where $k \geq (a+1/a)/2$ if $a > 0$ and $k \leq -(a+1/a)/2$ if $a < 0$. Therefore

$$p(Rz_0) = q(Rz_0) + \frac{Rz_0q'(Rz_0)}{q(Rz_0)} = ia + ik,$$

which implies $p(Rz_0) \in \Omega$ since $|a+k| \geq \sqrt{3}$, where $\Omega = \{it, |t| \geq \sqrt{3}\}$. Next we will show that

$$p(\mathbb{D}_R) \cap \Omega = \varnothing,$$

which contradicts the above assertion. Since $p \in \mathcal{P}_\lambda$, it is sufficient to prove for functions $P_\lambda(z)$. Suppose that there is a point $z_1 \in \mathbb{D}$ such that $P_\lambda(z_1) = it_0$ with $|t_0| \geq \sqrt{3}$, then a simple calculation gives that

$$z_1 = \frac{it_0 - 1}{it_0 + e^{2i\lambda}}.$$

Hence

$$|z_1|^2 = \frac{t_0^2+1}{t_0^2+1-2t_0\sin(2\lambda)} \geq \frac{4}{4+2\sqrt{3}|\sin(2\lambda)|} = R$$

since $|t_0| \geq \sqrt{3}$. Therefore $P_\lambda(\mathbb{D}_R) \cap \Omega = \varnothing$. The proof is completed. □

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